Today: § 1.3: Vector Equations

& § 1.4: The Matrix Equation $A\underline{x} = \underline{b}$

Next: § 1.5: Solution Sets

Reminders:

MyMathLab Homework #1 & #2: Due Mon, Jan 22

MATLAB Homework #1: Due THIS Fri Jan 19

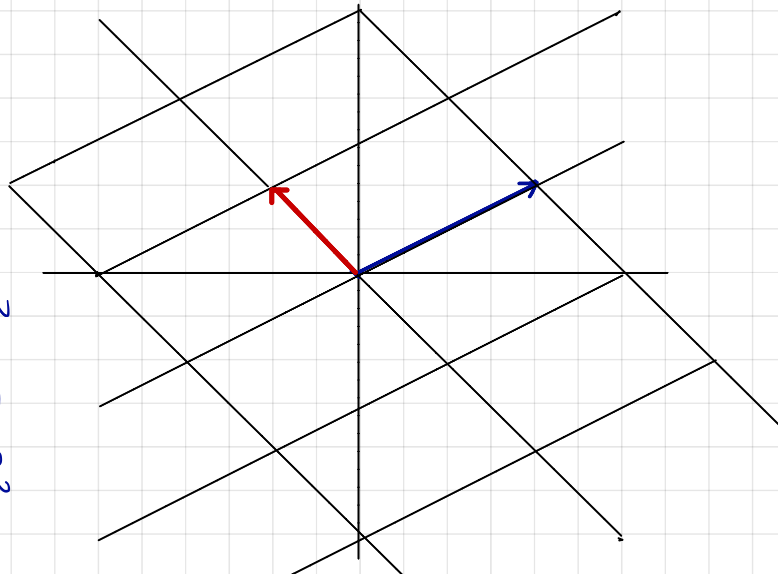
Linear Combinations

Given vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ a linear combination is a vector of the form

$$x_1 \mathbf{u}_1 + x_2 \mathbf{u}_2 + \cdots + x_n \mathbf{u}_n$$

for some scalars $x_1, x_2, \dots, x_n$.

Eg. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$



Span

The span of a collection of vectors is the set of all linear combinations of those vectors.

E.g. Is $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ in the span generated by $\left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \right\}$?

The question "Is $\underline{w}$ in the span of $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$?"
is the same as the question
"Is the system whose augmented matrix
is $\left[\begin{array}{cccc|c} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n & \underline{w} \end{array}\right]$?"

To answer this question, we use
to carry this augmented matrix to ,
and check to see if

§1.4 The matrix equation $A\underline{x} = \underline{b}$

Matrix multiplication

Let $A = [\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n]$ be an $m \times n$ matrix.

Let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ be a column vector, whose number n of entries matches the number of in A.

$$A\underline{x} = [\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} :=$$

E.g. $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix}$

E.g. Compute

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \\ 28 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 32 \end{bmatrix}$$

E.g. If $\underline{u}_1, \underline{u}_2, \underline{u}_3$ are three column vectors, write the linear combination $2\underline{u}_1 - 3\underline{u}_2 + 8\underline{u}_3$ as a matrix product.

Matrix multiplication is a concise way of representing ________.

The most basic algebra equation is $ax = b$.

If A is an $m \times n$ matrix, and $\underline{b} \in \mathbb{R}^m$ is a column vector, can we solve the matrix equation $A\underline{x} = \underline{b}$ for $\underline{x} \in \mathbb{R}^n$?

E.g. $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & 6 \\ -3 & 2 & 7 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Theorem: $A\underline{x} = \underline{b}$ can be solved if and only if

$\underline{b}$ is a of the columns of A

iff $\underline{b} \in$ of the columns of A.

E.g. $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & 6 \\ -3 & 2 & 7 \end{bmatrix}$

Question: Can I solve $A\underline{x} = \underline{b}$ for any choice of $\underline{b}$?

I.e. Is every vector $\underline{b} \in \mathbb{R}^3$ in ?

Theorem: Let A be an $m \times n$ matrix.
The following four statements are equivalent.

(a) $A\underline{x} = \underline{b}$ can be solved for $\underline{x}$, for <u>any</u> $\underline{b} \in \mathbb{R}^m$.

(b) Every vector $\underline{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot in each row.

Eg. Let $A = \begin{bmatrix} 3 & 5 & -4 & 1 \\ -3 & -2 & 4 & 2 \\ 6 & 1 & -8 & -7 \end{bmatrix}$. Do the columns of A span $\mathbb{R}^3$?

$$\Big\downarrow \text{ rref (MATLAB)}$$

$$\begin{bmatrix} 1 & 0 & -4/3 & -4/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question: Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in the span of the columns of A?

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 0 & 0 \\ -4 & 2 & -6 & 4 & 0 \\ 3 & -2 & -7 & 2 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & -4/3 & -4/3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$