Today: § 1.4: The Matrix Equation $A\underline{x} = \underline{b}$

& § 1.5: Solution Sets

Next: § 1.7: Linear Independence

Reminders:

MyMathLab Homework #1 & #2: Due Mon, Jan 22

MATLAB Homework #1: Due TONIGHT!

Theorem: Let A be an $m \times n$ matrix.
The following four statements are equivalent.

(a) $A\underline{x} = \underline{b}$ can be solved for $\underline{x}$, for <u>any</u> $\underline{b} \in \mathbb{R}^m$.

(b) Every vector $\underline{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot in each row.

Eg. Let $A = \begin{bmatrix} 3 & 5 & -4 & 1 \\ -3 & -2 & 4 & 2 \\ 6 & 1 & -8 & -7 \end{bmatrix}$. Do the columns of A span $\mathbb{R}^3$?

$$\Big\downarrow \text{ rref (MATLAB)}$$

$$\begin{bmatrix} 1 & 0 & -4/3 & -4/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question: Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in the span of the columns of A?

$$\left[\begin{array}{cccc:c} 3 & 5 & -4 & 1 & 0 \\ -3 & -2 & 4 & 2 & 0 \\ 6 & 1 & -8 & -7 & 1 \end{array}\right]$$

1.5 Solution Sets of Linear Systems

Def: A system of equations is **homogeneous** if it has the form $A\underline{x} = \underline{0}$ ← zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\boxed{\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \end{aligned}}$$

system of equations ⟺ matrix equation

vector equation ⟺ augmented matrix

Key observation: there is always at least one solution: $\underline{x} = \underline{0}$.
But there may be more.

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right]$$

parametric solution

Inhomogeneous Systems are linear systems that are not homogeneous.

i.e. $A\underline{x} = \underline{b}$ where $\underline{b} \neq \underline{0}$.

E.g. $\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 4 \\ 1 & 0 & 1 & 2 & 3 \end{array}\right]$

— compare to

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 4 \\ 1 & 0 & 1 & 2 & 3 \end{array}\right] \rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

E.g. $\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Uparrow$$

We saw $\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

We also saw

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Theorem. Let A be a matrix. Denote the (parametric) solution set of the homogeneous equation

$$A\underline{x} = \underline{0}$$

as $\mathbf{V_h}$.

(a) Even though $A\underline{x} = \underline{0}$ is always consistent ($\underline{x} = \underline{0}$ is always in $\mathbf{V_h}$), for given $\underline{b} \neq \underline{0}$, the equation $A\underline{x} = \underline{b}$ may be inconsistent.

(b) If $A\underline{x} = \underline{b}$ is consistent (i.e. $\underline{b}$ is in the span of the columns of A) and if $\underline{x} = \underline{p}$ is any particular solution, then the general solution is $\underline{x} = \underline{p} + \mathbf{V_h}$.