Today: §2.1: Matrix Operations

Next: Midterm 1 Review

Reminders:

MyMathLab Homework #3: Due TONIGHT.

Midterm 1: THIS Wed, Jan 31, 8-10pm.

- ↳ Covers §1.1-1.5, 1.7-1.9
- ↳ practice midterms posted on webpage.
- ↳ seat assignment posted on TritonEd.

§ 2.1 : Matrix Operations

We can treat matrices (of the same size!) like vectors, and add them and multiply them by scalars.

Eg. $A = \begin{bmatrix} 1 & -1 & 7 \\ 3 & 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 2 & 1 \\ 4 & 2 & 8 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$

$$2A - B = \begin{bmatrix} 2 & -2 & 14 \\ 6 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 1 \\ 4 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 13 \\ 2 & 2 & -6 \end{bmatrix}$$

$\rightarrow$ ~~$\begin{bmatrix} 1 & -2 & 13/2 \\ 2 & 2 & -6 \end{bmatrix}$~~

$B + 2C$ NOT DEFINED!

Addition and scalar multiplication of matrices behave nicely together, just like for real numbers:

$$A + B = B + A$$

$$(A+B) + C = A + (B+C)$$

$$A + 0 = A$$

↑ zero matrix of appropriate dimensions

$$r(A+B) = rA + rB$$

$$(r+s)A = rA + sA$$

$$r(sA) = (rs)A$$

Warning:

(1) All only when sizes match!

(2) Beware: not every property from real algebra works for matrices...

Matrix addition and scalar multiplication correspond to addition and scalar multiplication of linear transformations.

Eg. $A = \begin{bmatrix} 1 & -1 & 7 \\ 3 & 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 2 & 1 \\ 4 & 2 & 8 \end{bmatrix}$

$T, S: \mathbb{R}^3 \to \mathbb{R}^2$

$T(\underline{x}) = A\underline{x}$ $S(\underline{x}) = B\underline{x}$ $\underline{x} \in \mathbb{R}^3$

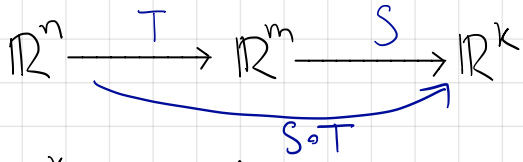
$(T+S)(\underline{x}) = T(\underline{x}) + S(\underline{x}) = A\underline{x} + B\underline{x} = (A+B)\underline{x}$

$(rT)(\underline{x}) = r \cdot T(\underline{x}) = rA\underline{x} = (rA)\underline{x}$

Matrix Multiplication

If $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^m \to \mathbb{R}^k$

we can compose them



$(S \circ T)(\underline{x}) = S(T(\underline{x}))$

$S \circ T: \mathbb{R}^n \to \mathbb{R}^k$. Notice: $\underline{x}, \underline{y} \in \mathbb{R}^m$

$$S \circ T(\underline{u} + \underline{v}) = S(T(\underline{u} + \underline{v})) = S(\underbrace{T(\underline{u})}_{\underline{x}} + \underbrace{T(\underline{v})}_{\underline{y}})$$

$$= S(\underline{x} + \underline{y}) = S(\underline{x}) + S(\underline{y})$$

$$= S(T(\underline{u})) + S(T(\underline{v}))$$

$$= (S \circ T)(\underline{u}) + (S \circ T)(\underline{v})$$

$S \circ T(r\underline{v}) = r(S \circ T)(\underline{v})$

$S \circ T$ respects addition & scalar multiplication

↖ a linear transformation!

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\qquad$ $S : \mathbb{R}^m \longrightarrow \mathbb{R}^k$

$T(\underline{x}) = A\underline{x}$ $\qquad$ $S(\underline{y}) = B\underline{y}$

A is $m \times n$, $\underline{x}$ is $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\qquad$ B is $k \times m$

$$A = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{bmatrix}$$

$$S \circ T(\underline{x}) = S(T(\underline{x})) = S(A\underline{x}) = S(x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n)$$

$$= x_1 S(\underline{a}_1) + x_2 S(\underline{a}_2) + \cdots + x_n S(\underline{a}_n)$$

$$= x_1 B\underline{a}_1 + x_2 B\underline{a}_2 + \cdots + x_n B\underline{a}_n$$

$$= \begin{bmatrix} B\underline{a}_1 & B\underline{a}_2 & \cdots & B\underline{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\Rightarrow$ $\begin{bmatrix} B\underline{a}_1 & B\underline{a}_2 & \cdots & B\underline{a}_n \end{bmatrix}$ is the standard matrix of $S \circ T$.

Definition: If A is $m \times n$ and B is $k \times m$ the product BA is defined by

$$BA = B\left[\underline{a}_1 \; \underline{a}_2 \; \cdots \; \underline{a}_n\right] := \left[B\underline{a}_1 \; B\underline{a}_2 \; \cdots \; B\underline{a}_n\right].$$

Eg. $\underset{3\times 2}{\underset{B}{\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}}} \underset{2\times 2}{\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}}} = \left[B\begin{bmatrix}1\\2\end{bmatrix} \quad B\begin{bmatrix}3\\-1\end{bmatrix}\right] = \begin{bmatrix} 1(4)+2(-2) & ; & 3(4)-1(-2) \\ 1(-3)+2(0) & ; & 3(-3)-1(0) \\ 1(3)+2(5) & ; & 3(3)-1(5) \end{bmatrix}$

$$= \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix} = BA$$

Eg. $\underset{2\times 2}{\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}}} \; \not\times \; \underset{3\times 2}{\underset{B}{\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}}}$

AB does not even make sense.

Properties of Matrix Multiplication

$$A(BC) = (AB)C$$

$$A(B+C) = AB + AC$$

$$(B+C)A = BA + CA$$

$$(rA)B = r(AB) = A(rB)$$

If A is $m \times n$, $\underset{m\times m}{I_m}\,\underset{m\times n}{A} = A = \underset{m\times n}{A}\,\underset{n\times n}{I_n}$

$n \times n$ identity matrix [MATLAB: eye(n)]

$$AB \neq BA \text{ in general.}$$

Even square matrices don't generally commute.

$$A = \begin{bmatrix} 4 & 7 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 9 \\ 8 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 7 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 9 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 64 & 57 \\ 38 & 39 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 9 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 35 & \cdot \\ \cdot & \cdot \end{bmatrix}$$

Transpose

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

$m \times n$ $\quad$ $n \times m$

Eg. $\begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -5 & 2 & 0 \\ 1 & -3 & 1 \\ 0 & 4 & 3 \end{bmatrix}$; $\begin{bmatrix} 1 & 7 & 6 \\ 7 & 3 & -1 \\ 6 & -1 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 7 & 6 \\ 7 & 3 & -1 \\ 6 & -1 & 4 \end{bmatrix}$

Symmetric.

Properties

If A is $m \times n$, then A^T is $n \times m$

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

$(rA)^T = rA^T$