Math 180A: Intro to Probability
(for Data Science)

www.math.ucsd.edu/~tkemp/180A

Midterm Exam TONIGHT!
8pm PCYNH 109

* Assigned seats (see TritonEd)
* Bring Student ID
* 1 Double-Sided 8.5" x 11" sheet of hand-written notes
* no electronic devices
* eat a proper dinner before (but not right before)
* try to relax; have fun!

Regrade requests for Lab 2: Thursday, 10/24 8am - 11pm
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(a) Suppose that $A, B \in \mathcal{F}$ satisfy
\[ \mathbb{P}(A) + \mathbb{P}(B) > 1. \]
Making no further assumptions on $A$ and $B$, prove that $A \cap B \neq \emptyset$.

(b) Prove that $A$ is independent from itself if and only if $\mathbb{P}(A) \in \{0, 1\}$.

\[ \Rightarrow \] Suppose $A$ is independent from $A$. 

\[ \frac{\Rightarrow}{\begin{array}{c} \text{I.e.} \\ \mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A) \\ \frac{\Rightarrow}{\begin{array}{c} \mathbb{P}(A) = p \\ \Rightarrow \mathbb{P}(A) \mathbb{P}(A)^2 \\ \Rightarrow p = p^2 \\ \Rightarrow p - p^2 = 0 \\ \Rightarrow p(1-p) = 0 \\ \Rightarrow p = 0 \text{ or } 1-p = 0 \\ \Rightarrow p = 0 \text{ or } p = 1. \end{array}} \]

\[ \iff \] Suppose $\mathbb{P}(A) \in \{0, 1\}$.

\[ \iff \begin{array}{c} \text{Then} \\ \mathbb{P}(A) = \mathbb{P}(A)^2 \\ \Rightarrow \mathbb{P}(A \cap A) \quad \text{i.e. A is independent from A.} \end{array} \]
2. Roll two fair dice repeatedly. If the sum is $\geq 10$, then you win.

(a) What is the probability that you start by winning 3 times in a row?

\[
\text{Single roll, } P(\text{success}) = \frac{1}{36} = \frac{1}{6}
\]

\[
\therefore P(\text{win, win, win}) = P(\text{win})P(\text{win})P(\text{win}) = \left(\frac{1}{6}\right)^3
\]

\[
\text{Bin}(3, \frac{1}{6})
\]

(b) What is the probability that after rolling the pair of dice 5 times you win exactly 3 times?

\[
X = \# \text{ of wins in 5 trials} \quad \therefore X \sim \text{Bin}(5, \frac{1}{6})
\]

\[
\therefore P(X = 3) = \binom{5}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2
\]

(c) What is the probability that the first time you win is before the tenth roll (of the pair), but after the fifth?

\[
N = \text{time of 1st success}, \quad N \sim \text{Geom} \left(\frac{1}{6}\right)
\]

\[
P(5 < N < 10) = P(6 \leq N \leq 9)
\]

\[
\sqrt{\Rightarrow} = \binom{5}{1} \left(\frac{5}{6}\right)^5 + \binom{6}{1} \left(\frac{5}{6}\right)^6 + \binom{7}{1} \left(\frac{5}{6}\right)^7 + \binom{8}{1} \left(\frac{5}{6}\right)^8 = \left(\frac{5}{6}\right)^5 - \left(\frac{5}{6}\right)^9
\]
1. At a political meeting there are 7 progressives and 7 conservatives. We choose five people uniformly at random to form a committee (president, vice-president and 3 regular members).

(a) Let $A$ be the event that we end up with more conservatives than progressives. What is the probability of $A$?

(b) Let $B$ be the event that Ronald, representing conservatives, becomes the president, and Felix, representing liberals, becomes the vice-president. What is the probability of $B$?

\[
\begin{align*}
\text{(a)} & \quad A_j = \{ \text{j conservatives are chosen} \} \\
A &= A_3 \cup A_4 \cup A_5 \quad \mathbb{P}(A) = \mathbb{P}(A_3) + \mathbb{P}(A_4) + \mathbb{P}(A_5) \\
\mathbb{P}(A_j) &= \binom{7}{j} \binom{7}{5-j} \\
\mathbb{P}(A) &= \frac{\binom{7}{3} \binom{7}{2} + \binom{7}{4} \binom{7}{1} + \binom{7}{5} \binom{7}{0}}{\binom{14}{5}} \\
\end{align*}
\]

\[
\begin{align*}
\text{(b)} & \quad \frac{12}{14} = \frac{10}{13.7} = \mathbb{P}(\text{Ronald & Felix are on the committee}) \\
\neq & \quad \mathbb{P}(B) \\
B &= \{(\text{Ronald}, \text{Felix}) \} \\
\mathbb{P}(B) &= \frac{1 \cdot 1 \cdot 12 \cdot 11 \cdot 10}{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10} = \frac{1}{14.13}
\end{align*}
\]
4. Consider a point \( P = (X, Y) \) chosen uniformly at random inside of the unit square \([0, 1]^2 = [0, 1] \times [0, 1] = \{(x, y) : 0 \leq x, y \leq 1\} \). Let \( Z = \min(X, Y) \) be the random variable defined as the minimum of the two coordinates of the point. For example, if \( P = (\frac{1}{2}, \frac{1}{3}) \), then \( Z = \min(\frac{1}{2}, \frac{1}{3}) = \frac{1}{3} \). Determine the cumulative distribution function of \( Z \). Determine if \( Z \) is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of \( Z \). If discrete, determine the probability mass function of \( Z \). If neither, explain why.

(Hint: Draw a picture.)

\[
F_Z(t) = \begin{cases} 
0 & t < 0 \\
1 - (1-t)^2 & 0 \leq t \leq 1 \\
1 & t > 1
\end{cases}
\]

\[
P(\min(X,Y) \leq t) = 1 - P(\min(X,Y) > t)
\]

\[
P(\min(X,Y) \leq t) = 1 - P(X > t, Y > t) = 1 - P((X,Y) \in \emptyset)
\]

\[
S = \{(a,b) : a > t, b > t\}
\]

\[
P((X,Y) \in S^c) = \frac{\text{Area}(S)}{\text{Area}(\square)} = 1 - (1-t)^2
\]

\[
f_Z(t) = \begin{cases} 
0 & t < 0 \\
2(1-t) & 0 \leq t \leq 1 \\
1 & t > 1
\end{cases}
\]

\[
f_X(t) = \lim_{\epsilon \to 0} \frac{P(t-\epsilon \leq X \leq t+\epsilon)}{2\epsilon}
\]