This week:

homework #2 (due Friday Oct, 11 11:59pm)
E.g. (from last lecture)

An urn has 4 red and 7 blue balls. Choose two balls.

\[ A = \{ \text{1st ball is red} \} \]
\[ B = \{ \text{2nd ball is blue} \} \]

1) choose balls with replacement

\[ P(A) = \frac{4}{11} \]  
\[ P(B) = \frac{7}{11} \]
\[ P(A \cap B) = \frac{4 \cdot 7}{11 \cdot 11} = P(A)P(B) \]

A and B independent

2) choose balls without replacement

\[ P(A) = \frac{4 \cdot 10}{11 \cdot 10} = \frac{4}{11} \]
\[ P(B) = \frac{10 \cdot 7}{11 \cdot 10} = \frac{7}{11} \]
\[ P(A \cap B) = \frac{4 \cdot 7}{11 \cdot 10} \neq \frac{4 \cdot 7}{11 \cdot 11} \]

A and B are not independent
A and B independent \iff A and B^c independent

**Proof.** (\implies) Suppose that A and B are independent.

\[
P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B))
\]

\[
= P(A)P(B^c)
\]

\[
A = (A \cap B) \cup (A \cap B^c)
\]

\[
\text{\& disjoint}
\]

\[
P(A) = P(A \cap B) + P(A \cap B^c)
\]
More than two events?

**Def.** A collection $A_1,\ldots,A_n$ of events is mutually independent if

for any subcollection $A_{i_1}, A_{i_2},\ldots, A_{i_k}$

$$(1 \leq i_1 < i_2 < \cdots < i_k \leq n)$$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

**E.g.** When $n=3$, this means that we must have

$$P(A_1 \cap A_2) = P(A_1) P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1) P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2) P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$$
**Important example**

Toss a coin three times

\[ A = \{ \text{there is exactly 1 Tails in the first two} \} \]
\[ B = \{ \text{there is exactly 1 Tails in the last two} \} \]
\[ C = \{ \text{there is exactly 1 Tails in first and last toss} \} \]

\[ A = \{ (H,T,*), (T,H,*), (T,T,H) \} \]
\[ B = \{ (*,H,T), (*,T,H) \} \]
\[ C = \{ (H,*), (T,*) \} \]

\[ P(A) = \frac{4}{8} = \frac{1}{2} = P(B) = P(C) \]
\[ P(A \cap B) = \frac{2}{8} = \frac{1}{4} = P(A \cap C) = P(B \cap C) \]

\[ A \cap B \cap C = \emptyset \]
\[ P(A \cap B \cap C) = 0 \]

\( \Rightarrow \) \( A, B, C \) are pairwise independent.
**Random variables**

$(\Omega, \mathcal{F}, P)$-probability space

**Definition.** A (measurable) function $X: \Omega \to \mathbb{R}$ is called a **random variable**.

For any $B \subseteq \mathbb{R}$ we can compute $P(X \in B)$.
Def. Let $X$ be a random variable (rv). The probability distribution of $X$ is the collection of probabilities $P(X \in B)$ for all $B \subset \mathbb{R}$.

Remark. Strictly speaking, $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ Borel sets

Examples
1) Coin toss: $\Omega = \{H, T\}$, $X(H) = 0$, $X(T) = 1$
   \[ P(X = 0) = P(\{H\}) = \frac{1}{2} = P(X = 1) \] (fair coin)

2) Roll a die: $\Omega = \{1, 2, \ldots, 6\}$, $X(\omega) = \omega$
   For any $1 \leq i \leq 6$ \[ P(X = i) = \frac{1}{6} \]
3) Roll a die twice: \( \Omega = \{(i,j): i,j \in \{1, \ldots, 6\}\} \)

\[ X_1((i,j)) = i \) (first number) \( X_2((i,j)) = j \) (second number) \]

for \( 1 \leq i \leq 6 \) \( P(X_1 = i) = \frac{1}{6} \) \( P(X_2 = i) = \frac{1}{6} \)

\[ S = X_1 + X_2 \]

\[
\begin{align*}
P(S = 2) &= \frac{1}{36} \\
P(S = 3) &= \frac{2}{36} \\
P(S = 4) &= \frac{3}{36} \\
P(S = 5) &= \frac{4}{36} \\
P(S = 6) &= \frac{5}{36} \\
P(S = 7) &= \frac{6}{36} \\
P(S = 8) &= \frac{5}{36} \\
P(S = 9) &= \frac{4}{36} \\
P(S = 10) &= \frac{3}{36} \\
P(S = 11) &= \frac{2}{36} \\
P(S = 12) &= \frac{1}{36}
\end{align*}
\]
4) Choosing a point from unit disk \( \Omega \) at random.

\[
\Omega = \{ \mathbf{w} \in \mathbb{R}^2 : \text{dist}(\mathbf{w}, \mathbf{0}) \leq 1 \}
\]

\( X(\mathbf{w}) = \text{dist}(\mathbf{w}, \mathbf{0}) \)

For any \( r < 0 \), \( P(X \leq r) = 0 \)

For any \( r > 1 \), \( P(X \leq r) = 1 \)

For any \( r \in [0, 1] \), \( P(X \leq r) = \frac{\text{size } D_r}{\text{size } D_1} = \frac{\pi r^2}{\pi} = r^2 \)

\( \{ X \leq r \} = \{ X \in (-\infty, r] \} \)

\text{missing in class}
Def. Random variable \( X \) is a discrete rv is there exists a finite or infinite countable collection of points \( \{a_i : i \in \mathbb{N}\} \subset \mathbb{R} \) such that \( \sum_i P(X = a_i) = 1 \)

Example (lecture 3). Toss a coin until first \( T \).

\( X \) = total number of tosses.

(Already computed before) for any \( i = 1, 2, ... \)

\[ P(X = i) = \frac{1}{2^i} \]

\[ \sum_i P(X = i) = \sum_i \frac{1}{2^i} = 1 \quad (\text{geometric series}) \]
Discrete rv $X$ is completely described by its probability mass function (pmf) $p_X$ given by

$$p_X(k) = P(X = k)$$

for all possible values of $X$.

Example. $S =$ sum of two dice

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</thead>
<tbody>
<tr>
<td>$p_X(k)$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{2}{36}$</td>
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What if for every $x \in \mathbb{R}$ $P(X = x) = 0$?
**Probability density function**

**Def.** Let $X$ be a rv. If function $f: \mathbb{R} \to \mathbb{R}$ satisfies

$$P(X \leq b) = \int_{-\infty}^{b} f(x) \, dx$$

then $f$ is a probability density function of $X$

**Remark.** Definition implies that for $B \subset \mathbb{R}$

$$P(X \in B) = \int_{B} f_X(x) \, dx$$
E.g. Distance to 0 from a random point in a disk

\[ \int_{-\infty}^{\infty} f_X(x) \, dx = P(X \leq r) = \begin{cases} 0, & r > 0 \\ r^2, & 0 \leq r \leq 1 \\ 1, & r > 1 \end{cases} \]
**Important example:** uniform distribution on \([a, b]\) in \(\mathbb{R}\)

**Definition** Let \(a, b \in \mathbb{R}\), \(a \leq b\).  

\(X\) has uniform distribution on \([a, b]\)  
(\text{denoted } X \sim \text{Uniform}([a, b]))

if its pdf is given by  
\[
    f_X(x) = \begin{cases} 
        0, & x < a \\
        \frac{1}{b-a}, & a \leq x \leq b \\
        0, & x > b 
    \end{cases}
\]