Vector Calculus (MATH 20E) Fall 2014
Midterm 1 Solutions (Version A)

1. (10 points) Let $f(x,y) = e^{xy}$.

(a) Find the tangent plane to the surface given by the graph of the function $z = f(x,y)$ at the point $(1,0,1)$.

Solution Since $\frac{\partial f}{\partial x} = ye^{xy}$ and $\frac{\partial f}{\partial y} = xe^{xy}$, so $\frac{\partial f}{\partial x}(1,0) = 0$ and $\frac{\partial f}{\partial y}(1,0) = 1$. Therefore, the equation of the tangent plane is:

$$z = f(1,0) + \left(\frac{\partial f}{\partial x}(1,0)\right)(x-1) + \left(\frac{\partial f}{\partial y}(1,0)\right)(y-0) = 1 + y$$

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(b) Determine the second order Taylor polynomial of the function at this point.

Solution From the derivatives we have in part a), we compute:

$$\begin{aligned} f_{xx}(1,0) &= \frac{\partial}{\partial x}(ye^{xy})\Big|_{(1,0)} = y^2e^{xy}\Big|_{(1,0)} = 0 \\ f_{yy}(1,0) &= \frac{\partial}{\partial x}(xe^{xy})\Big|_{(1,0)} = x^2e^{xy}\Big|_{(1,0)} = 1 \\ f_{yx}(1,0) &= f_{xy}(1,0) = \frac{\partial}{\partial x}(xe^{xy})\Big|_{(1,0)} = (e^{xy} + xye^{xy})\Big|_{(1,0)} = 1 \end{aligned}$$

Therefore, combined with the result from a), the second order Taylor polynomial of f is:

$$\begin{aligned} g(x,y) &= 1 + y + \frac{1}{2}\left[f_{xx}(1,0)(x-1)^2 + f_{yy}(1,0)(y-0)^2 + 2f_{xy}(1,0)(x-1)(y-0)\right] \\ &= 1 + y + \frac{1}{2}\left[(x-1)^2 + 1\cdot(y-0)^2 + 2\cdot 1\cdot(x-1)(y-0)\right] \\ &= 1 + y + (\frac{y^2}{2} + xy - y) \\ &= 1 + xy + \frac{y^2}{2} \end{aligned}$$

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2. (10 points) Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function $g(u,v) = (e^u \cos v, e^u \sin v)$.

(a) Compute $Dg(u,v)$ for any given point $(u,v) \in \mathbb{R}^2$.

Solution Let $g_1(u,v) = e^u \cos v$ and $g_2(u,v) = e^u \sin v$. We compute the total derivative:

$$Dg(u,v) = \begin{pmatrix} \dfrac{\partial g_1}{\partial u} & \dfrac{\partial g_1}{\partial v} \\ \dfrac{\partial g_2}{\partial u} & \dfrac{\partial g_2}{\partial v} \end{pmatrix} = \begin{pmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{pmatrix}$$

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(b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, $f = f(x,y)$. Express the partial derivatives $\dfrac{\partial}{\partial u}(f \circ g)$ and $\dfrac{\partial}{\partial v}(f \circ g)$ as linear combinations of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution Note that we have a function $f \circ g\colon \mathbb{R}^2 \to \mathbb{R}$. By the chain rule,

$$\begin{aligned} D(f\circ g)(u,v) &= Df(g(u,v))Dg(u,v) \\ \begin{pmatrix} \dfrac{\partial}{\partial u}(f\circ g) & \dfrac{\partial}{\partial v}(f\circ g) \end{pmatrix} &= \begin{pmatrix} \left.\dfrac{\partial f}{\partial x}\right|_{g(u,v)} & \left.\dfrac{\partial f}{\partial y}\right|_{g(u,v)} \end{pmatrix} \begin{pmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{pmatrix} \\ &= \begin{pmatrix} e^u \cos v \dfrac{\partial f}{\partial x} + e^u \sin v \dfrac{\partial f}{\partial y} & -e^u \sin v \dfrac{\partial f}{\partial x} + e^u \cos v \dfrac{\partial f}{\partial y} \end{pmatrix} \end{aligned}$$

where each partial derivatives in the last line are calculated at $g(u,v)$. Therefore,

$$\begin{aligned} \frac{\partial}{\partial u}(f\circ g) &= e^u \cos v \frac{\partial f}{\partial x} + e^u \sin v \frac{\partial f}{\partial y} \\ \frac{\partial}{\partial v}(f\circ g) &= -e^u \sin v \frac{\partial f}{\partial x} + e^u \cos v \frac{\partial f}{\partial y} \end{aligned}$$

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3. (10 points) Consider the level surface given by $x^2y^2 + y^2z^2 + z^2x^2 - 3xyz = 0$; this is called *Steiner's surface*. The point $(1, 1, 1)$ is on this surface. Show that there is a neighborhood of this point where Steiner's surface is the graph of any one of the three variables x, y, or z as a function of the other two.

Solution Let $F(x, y, z) = x^2y^2 + y^2z^2 + z^2x^2 - 3xyz$.
We want to show that x is a function of y and z near $(1, 1, 1)$, that is, $x = x(y, z)$. By the Implicit Function Theorem, we need to show that $\frac{\partial F}{\partial x}(1, 1, 1) \neq 0$. Indeed, we have

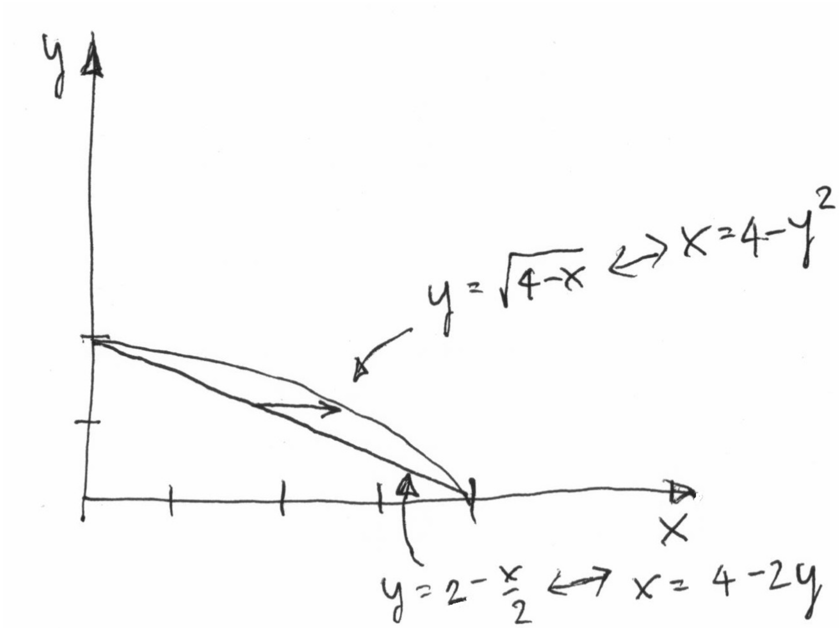
$$\frac{\partial F}{\partial x}(1,1,1) = 2xy^2 + 2xz^2 - 3yz\Big|_{(1,1,1)} = 1 \neq 0$$

Since F and the point $(1, 1, 1)$ are symmetric in x, y and z, analogous conclusions hold for $y = y(x, z)$ and $z = z(x, y)$. ◀

4. (10 points) Consider the iterated integral

$$\int_0^4 \int_{2-\frac{x}{2}}^{\sqrt{4-x}} \frac{\sqrt{y}}{2-y}\, dy\, dx.$$

Sketch the region of integration. Use Fubini's theorem to rewrite the integral in the other order, and evaluate the integral.



Solution

When drawing the picture, we notice that both graphs intersect x-axis and y-axis at $(4,0)$ and $(0,2)$ respectively. The equation $y=\sqrt{4-x}$ is equivalent to $x=4-y^2, y>0$, which is an upside down parabola. So we expect it to lie above the line $y=2-\dfrac{x}{2}$. Note that the equation $y=2-\dfrac{x}{2}$ is equivalent to $x=4-2y$.

If we fix any y, where $0\le y\le 2$, then the x-value of any point in the region will run from $4-2y$ to $4-y^2$. By applying Fubini's theorem, the double integral is equal to:

$$\begin{aligned}
\int_0^2 \int_{4-2y}^{4-y^2} \frac{\sqrt{y}}{2-y}\, dx\, dy &= \int_0^2 \frac{x\sqrt{y}}{2-y}\bigg|_{x=4-2y}^{4-y^2} dy \\
&= \int_0^2 \frac{\sqrt{y}}{2-y}\left((4-y^2)-(4-2y)\right)\, dy \\
&= \int_0^2 \frac{\sqrt{y}}{2-y}(2y-y^2)\, dy \\
&= \int_0^2 \frac{\sqrt{y}}{2-y}[y(2-y)]\, dy \\
&= \int_0^2 y\sqrt{y}\, dy = \frac{2}{5}y^{5/2}\bigg|_{y=0}^{2} = \frac{8\sqrt{2}}{5}
\end{aligned}$$

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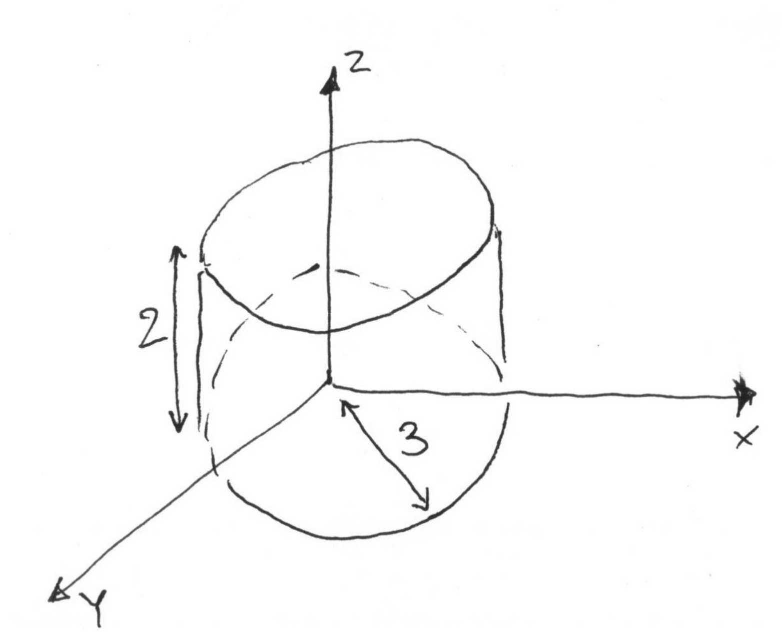
5. (10 points) Let T be the cylindrical region in $\mathbb{R}^3$ given by $0 \le z \le 2$ and $x^2 + y^2 \le 9$. Suppose that the mass density of the region at the point (x, y, z) is equal to the distance between this point and the original $(0, 0, 0)$. Write down (but *do not evaluate*) a triple integral the computes the mass of T.



Solution

Let $D(x, y, z)$ be the density function. From the assumption that this is equal to the distance between (x, y, z) and $(0, 0, 0)$,

$$D(x,y,z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

For each fixed z, the level set is a circle of radius 3. This gives us a double integral over each level set $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \ldots \, dy \, dx$. Since z varies from 0 to 2, we obtain the triple integral:

$$\begin{aligned} \text{Mass} &= \int\int\int_T D(x,y,z) dV \\ &= \int_0^2 \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{x^2+y^2+z^2} \, dy \, dx \, dz \end{aligned}$$

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