University of California, San Diego
Department of Mathematics

SOLUTIONS

1. (10 points) *Consider the parametrization*

$$\Phi(u,v) = \begin{pmatrix} u^2+v^2 \\ u+v \\ u-v \end{pmatrix}.$$

(a) Show that Φ parametrizes the surface $2x = y^2 + z^2$.

At any point (x, y, z) in the image of Φ, we have $2x = 2(u^2 + v^2)$, and we also have $y^2+z^2 = (u+v)^2+(u-v)^2 = u^2+2uv+v^2+u^2-2uv+v^2 = 2u^2+2v^2 = 2(u^2+v^2) = 2x$. So Φ parametrizes (a part of) the surface $2x = y^2 + z^2$.

(b) Find an equation for the tangent plane at the point $(2, 2, 0)$.

The tangent plane to the surface at the point $\Phi(u, v)$ has normal vector $\mathbf{T}_u \times \mathbf{T}_v$. So we need to find which parameters (u, v) give us the point $(2, 2, 0)$. This means we need to solve the equations

$$\begin{aligned} u^2+v^2 &= 2 \\ u+v &= 2 \\ u-v &= 0 \end{aligned}$$

The last two equations give us a unique solution for (u, v): the last equation says $v = u$, and subbing this into the second equation gives us $u + u = 2$, so $u = 1$, and therefore $v = 1$. We must double check that this gives us a point on the surface, meaning the first equation is satisfied: indeed $1^2 + 1^2 = 2$.
Now, we calculate the derivative $D\Phi(u, v)$ to ge the tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$:

$$D\Phi(u,v) = \begin{bmatrix} 2u & 2v \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = [\mathbf{T}_u \ \mathbf{T}_v].$$

Substituting in the point in question $(u, v) = (1, 1)$, we have

$$\mathbf{T}_u = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{T}_v = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and therefore} \quad \mathbf{T}_u \times \mathbf{T}_v = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}.$$

Hence, the tangent plane at the point $(2, 2, 0)$ is

$$-2(x-2)+4(y-2)+0(z-0) = 0, \quad \text{or, simplified} \quad -x+2y-2 = 0.$$

(This exam is worth 45 points.)

2. (15 points) *The plane $x + 2y - z = 0$ intersects the cylinder $x^2 + y^2 = 4$ in a curve* $\mathbf{c}$.

   (a) *Parametrize* $\mathbf{c}$ *so that its projection into the (x, y)-plane is traversed counterclockwise.*

   The projection of the cylinder into the (x, y)-plane is a circle, which we can parametrize counterclockwise in the usual way $(x, y) = (2\cos t, 2\sin t)$ for $0 \leq t \leq 2\pi$. Now the actual intersection is also in the surface $x + 2y - z = 0$, which means that the z coordinate satisfies $z = x + 2y = 2\cos t + 4\sin t$. Thus, the a parametrization that is counterclockwise relative to the (x, y)-plane is

   $$\mathbf{c}(t) = (2\cos t, 2\sin t, 2\cos t + 4\sin t).$$

   (b) *Compute the line integral* $\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, *where* $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$.

   Using the parametrization from part (a), we have

   $$\dot{\mathbf{c}}(t) = (-2\sin t, 2\cos t, -2\sin t + 4\cos t).$$

   Hence, appealing to the definition of line integrals, we have

   $$\begin{aligned}\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(2\cos t, 2\sin t, 2\cos t + 4\sin t) \cdot (-2\sin t, 2\cos t, -2\sin t + 4\cos t)\, dt \\ &= \int_0^{2\pi} (-2\sin t, 2\cos t, 1) \cdot (-2\sin t, 2\cos t, -2\sin t + 4\cos t)\, dt \\ &= \int_0^{2\pi} 4\sin^2 t + 4\cos^2 t - 2\sin t + 4\cos t\, dt.\end{aligned}$$

   The integrant simplifies to $4 - 2\sin t + 4\cos t$. Since we are integrating over a full period, the $\sin$ and $\cos$ terms integrate to 0, and we are simply left with

   $$\int_0^{2\pi} 4\, dt = 8\pi.$$

   (c) *Is the vector field* $\mathbf{F}$ *from part (b) the gradient of a function? Explain why or why not.*

   No, it is not the gradient of any function. Indeed, since $\mathbf{c}$ is a closed curve, the Fundamental Theorem of Calculus tells us that, for any smooth function f,

   $$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = 0.$$

   But, as calculated in (b), in this case the line integral is not 0. So there cannot be any function f with $\mathbf{F} = \nabla f$.

3. (10 points) *The glass dome of a futuristic greenhouse is the surface $z = 8 - 2x^2 - 2y^2$; the greenhouse has a flat floor at height $z = 0$. The temperature T throughout the greenhouse is given by the function*

$$T(x, y, z) = x^2 + y^2 + 2(z-2)^2.$$

This gives rise to a temperature gradient $\mathbf{F} = -\nabla T$.

(a) *Parametrize the glass surface (not including the glass floor). Make sure the normal vector is* outward *pointing.*

Since the greenhouse surface is the graph of a function, we will use x and y as parameters. That is, the parametrization will be

$$\Phi(x, y) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 8 - 2x^2 - 2y^2 \end{bmatrix}.$$

We must determine the parameter range: this is given by the fact that the surface ends at the floor $z = 0$. This means the only points in the surface are where $z \geq 0$, which means that $8 - 2x^2 - 2y^2 \geq 0$, which simplifies to $x^2 + y^2 \leq 4$, the disk of radius 2. Thatis our parameter range.

Finally, we need to check if we indeed chose a parametrization with *outward* pointing normal vector. We compute the normal vector by calculating the derivative $D\Phi(x, y)$ of the paramtrization to get the tangent vectors:

$$D\Phi(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -4x & -4y \end{bmatrix} = [\mathbf{T}_x \ \mathbf{T}_y].$$

Now we take the cross product to find the normal vector:

$$\mathbf{T}_x \times \mathbf{T}_y = \begin{bmatrix} 4x \\ 4y \\ 1 \end{bmatrix}.$$

Notice that the z-component of this normal vector is positive, which means it points *up*. That is indeed the outward pointing direction from the concave surface, so we chose the right orientation for the parametrization.

(b) *Calculate the total heat flux out of the greenhouse.*

The temperature gradient is $\mathbf{F} = -\nabla T = -[2x, 2y, 4(z-2)]$, and so the heat flux is

$$\begin{aligned} \iint \mathbf{F} \cdot \mathbf{dS} &= \iint_{x^2+y^2 \leq 4} \mathbf{F}(\Phi(x, y)) \cdot \mathbf{T}_x \times \mathbf{T}_y \, dxdy \\ &= \iint_{x^2+y^2 \leq 4} \begin{bmatrix} -2x \\ -2y \\ -4(6 - 2x^2 - 2y^2) \end{bmatrix} \cdot \begin{bmatrix} 4x \\ 4y \\ 1 \end{bmatrix} dxdy \\ &= \iint_{x^2+y^2 \leq 4} (-8x^2 - 8y^2 - 24 + 8x^2 + 8y^2) \, dxdy = \iint_{x^2+y^2 \leq 4} -24 \, dxdy. \end{aligned}$$

This just gives -24 times the area of the disk of radius 2, so the answer is $-24 \cdot \pi(2)^2 = -96\pi$.

4. (10 points) *Let S be the piece of the sphere of radius 1, centered at $\mathbf{0}$, for which $x \geq 0$, $z \geq 0$, and $y \geq x$. Compute the scalar surface integral $\iint_S \frac{1}{1+z}\, dS$.*

We use the spherical coordinate parametrization of the sphere

$$\Phi(\phi, \theta) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

The conditions $y \geq x \geq 0$ mean that $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$. The condition $z \geq 0$ means that $0 \leq \phi \leq \frac{\pi}{2}$. Finally, in spherical coordinates on the unit sphere, we have $dS = \sin\phi\, d\phi d\theta$. So the surface integral in question is

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\phi} \sin\phi\, d\phi d\theta.$$

The integrand does not vary with θ, so we integrate this out right away to get

$$\frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\phi} \sin\phi\, d\phi.$$

Now we make the change of variables $u = \cos\phi$, so $du = -\sin\phi\, d\phi$. Changing the limits of integration accordingly, we get

$$\frac{\pi}{4} \int_1^0 \frac{1}{1+u}(-du) = \frac{\pi}{4} \ln|1+u||_0^1 = \frac{\pi}{4} \ln 2.$$