Lecture 11: April 25, 2011

We saw last time that we can express the moments of the semicircle law as

$$\mathbb{E}(X^n) = \sum_{\pi \in NC(n)} \kappa(\pi)$$

in this case, the function $\kappa(\pi) = \begin{cases} 1, & \pi \in NC_2(n) \\ 0, & \text{otherwise} \end{cases}$.

As we will see, we can get a lot of mileage out of expressing moments (in fact joint moments) as sums over $NC(n)$.

The first step in understanding this is to fully understand the combinatorial structure of the sets $NC(n)$.

<u>Def</u>: A <u>(finite) lattice</u> is a (finite) partially-ordered set (poset) with the following additional properties.

* Given $\pi, \sigma \in L$, the set $\{\tau \in L : \tau \geq \pi \ \& \ \tau \geq \sigma\} \neq \emptyset$ and has a unique minimal element, denoted $\pi \vee \sigma$. It is called the <u>join</u> or <u>sup</u> of π, σ.

* Given $\pi, \sigma \in L$, the set $\{\tau \in L : \tau \leq \pi \ \& \ \tau \leq \sigma\} \neq \emptyset$ and has a unique maximal element, denoted $\pi \wedge \sigma$. It is called the <u>meet</u> or <u>inf</u> of π, σ.

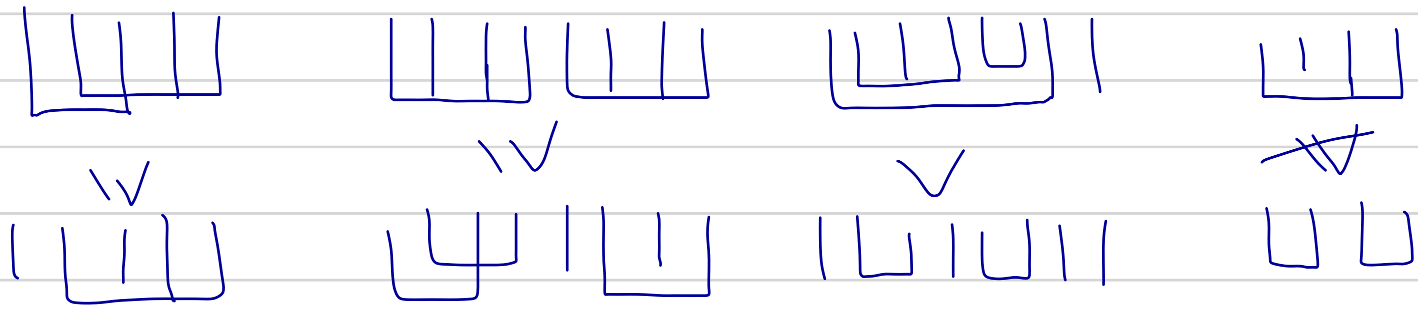
It is easily verified by induction that any finite collection $\pi_1, \ldots, \pi_n$ in a lattice has a meet $\pi_1 \wedge \cdots \wedge \pi_n$ and a join $\pi_1 \vee \cdots \vee \pi_n$, and that the operations $\wedge$ and $\vee$ are commutative and associative.

In particular, in a finite lattice $L = \{\pi_1, \ldots, \pi_n\}$, there is a unique maximal element $\max L = \pi_1 \vee \cdots \vee \pi_n$, which is denoted 1_L, and a unique minimal element $\min L = \pi_1 \wedge \cdots \wedge \pi_n$, which is denoted 0_L.

<u>Def</u>: On the set $\mathcal{P}(n)$ of all partitions of $[n]$, define an order relation by (reverse) <u>refinement</u>: $\pi \leq \sigma$ iff each block of π is contained in a block of σ.

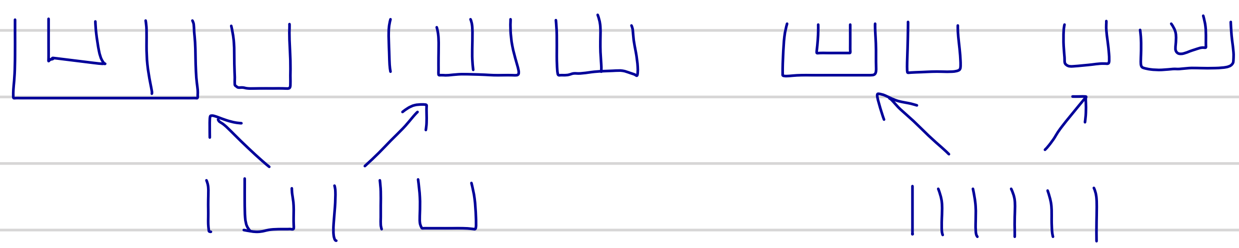
It is a trivial exercise to verify that this defines a partial order.

Eg's:



So $\mathcal{P}(n)$ is a poset. Any subset of a poset inherits the poset structure, so $NC(n) \subseteq \mathcal{P}(n)$ is also a poset. (For that matter, the subsets $NC_2(n) \subseteq \mathcal{P}_2(n)$ are also posets, but they are trivial: all elements are incomparable.)

<u>Prop</u>: If $\pi, \sigma \in NC(n)$, then $\pi \wedge \sigma$ exists in $NC(n)$, and is equal to the <u>common refinement</u> of π and σ: if $\pi = \{B_1, \ldots, B_r\}$ and $\sigma = \{C_1, \ldots, C_s\}$, the the blocks of $\pi \wedge \sigma$ are the disjoint subsets $\{B_i \cap C_j : 1 \le i \le r, 1 \le j \le s, B_i \cap C_j \neq \emptyset\}$.

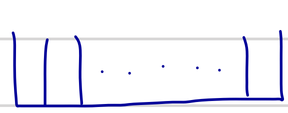


Pf. Since the common refinement of π, σ is a refinement, it follows by definition that it is $\le \pi$ and $\le \sigma$. Now, suppose $\tau \le \pi$ and $\tau \le \sigma$. If $D \in \tau$ then $\exists B \in \pi$ s.t. $D \subseteq B$, and $\exists C \in \sigma$ s.t. $D \subseteq C$. Thus $D \subseteq B \cap C \in$ the common refinement. Thus, τ is a refinement of the common refinement and so $\tau \le$ the common refinement. Thus $\pi \wedge \sigma =$ the common refinement. ///

It follows that $NC(n)$ has a minimal element, and it is easy to see that

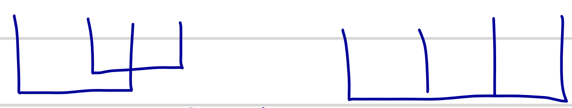
$$0_{NC(n)} = \mid\mid\mid\mid \cdots \mid\mid\mid .$$

We denote this as 0_n. As for suprema, they do exist, but are a little harder to describe. It is easy to see that $NC(n)$ has a unique maximal element

$1_{NC(n)} =$  $\equiv 1_n$.

Exercise: Let P be a finite poset that has meets, and possesses a unique maximal element. Show that P is a lattice. [Hint: if $\pi, \sigma \in P$, the set $\{\tau \in P : \pi \leq \tau \ \& \ \sigma \leq \tau\}$ is not empty – e.g. it contains 1_P. Since it is finite, it has a meet = inf. Show this is the join of π and σ.]

Thus, $NC(n)$ is a lattice. It is worth mentioning that $\mathcal{P}(n)$ also has the same maximal and minimal elements $1_n, 0_n$, and that $\wedge$ in $\mathcal{P}(n)$ is also given by common refinement. Thus $\mathcal{P}(n)$ is also a lattice. However, the join $\vee$ in $\mathcal{P}(n)$ is not the same as in $NC(n)$ in general. For example,

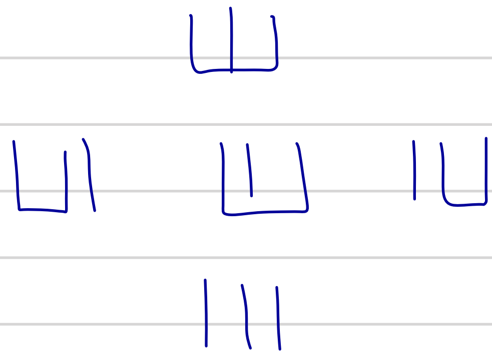


$\pi \vee \sigma \in \mathcal{P}(4)$ $\qquad$ $\pi \vee \sigma \in NC(4)$

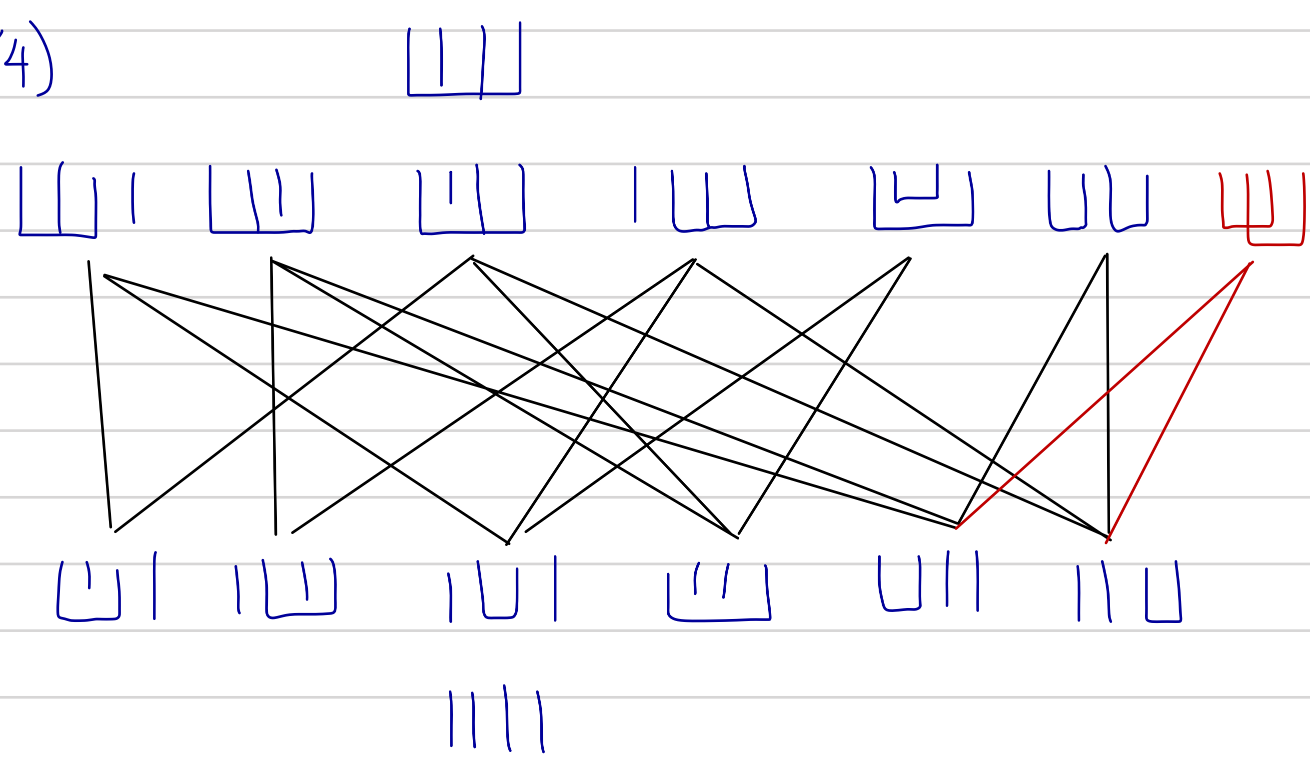
π $\qquad$ σ

Eg's $NC(1) = \mathcal{P}(1)$, $NC(2) = \mathcal{P}(2)$ 

$NC(3) = \mathcal{P}(3)$



$NC(4) \neq \mathcal{P}(4)$



Operations on partitions

Let $\sigma \in S_n$ be a permutation of $[n]$. Then σ acts on $\mathcal{P}(n)$ as follows:

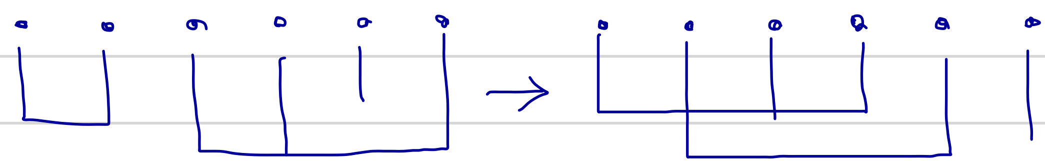
if $\pi \in \mathcal{P}(n)$, $\pi = \{B_1, \ldots, B_r\}$, then $\sigma \cdot \pi = \{\sigma \cdot B_1, \ldots, \sigma \cdot B_r\}$ where

$$\sigma \cdot \{k_1, \ldots, k_\ell\} = \{\sigma^{-1}(k_1), \ldots, \sigma^{-1}(k_\ell)\}.$$

Since σ is a bijection of $[n]$ to itself, the sets $\sigma \cdot B_1, \ldots, \sigma \cdot B_r$ form a new partition of $[n]$.

Eg. $\sigma = (1\,4\,3\,6\,5\,2)$
$\sigma^{-1} = (1\,2\,5\,6\,3\,4)$

π $\qquad\qquad$ $\sigma \cdot \pi$



$\{1,2\}, \{3,4,6\}, \{5\} \rightarrow \{1,3,4\}, \{2,5\}, \{6\}$

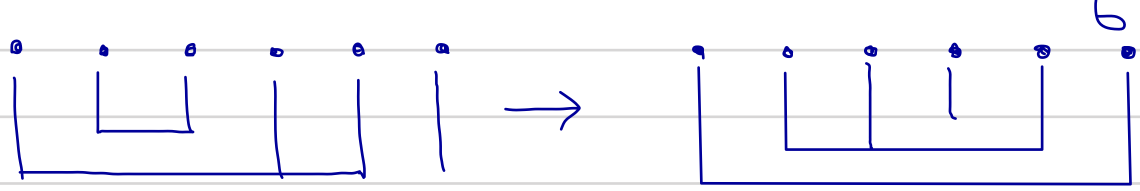
The map $\sigma \cdot : \mathcal{P}(n) \rightarrow \mathcal{P}(n)$ is a bijection (whose inverse is $\sigma^{-1} \cdot$). Moreover, if $B, C \subseteq [n]$ and $\sigma \in S_n$ then $B \subseteq C$ iff $\sigma \cdot B \subseteq \sigma \cdot C$. It follows that, for any $\pi, \tau \in \mathcal{P}(n)$, $\pi \leq \tau$ iff $\sigma \cdot \pi \leq \sigma \cdot \tau$. Thus, $\sigma \cdot$ is a <u>poset automorphism</u> of $\mathcal{P}(n)$.

<u>Exercise</u>: Let L_1, L_2 be lattices, and let $f : L_1 \rightarrow L_2$ be a poset isomorphism. Show that f is a lattice isomorphism: $f(\pi \vee \lambda) = f(\pi) \vee f(\lambda)$ and $f(\pi \wedge \lambda) = f(\pi) \wedge f(\lambda)$.

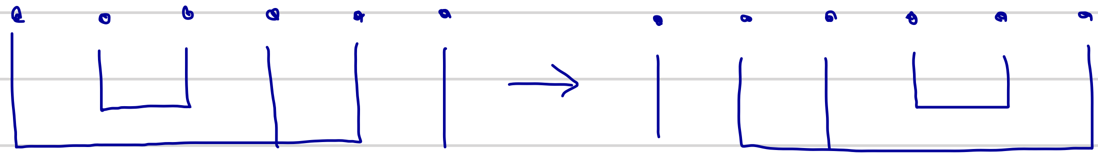
<u>Remark</u>: the same is <u>not</u> true for poset <u>homomorphisms</u>. See <u>Algebra</u> by Birkhoff–MacLane.

Thus, for $\sigma \in S_n$, $\sigma \cdot$ is a lattice automorphism of $\mathcal{P}(n)$. As the above example shows, however, under this action the subposet $NC(n)$ is not preserved by (some of) these automorphisms. There are a few that do work, though.

* if σ is cyclic, $\sigma(NC(n)) \subseteq NC(n)$. Eg. $\sigma = (1\,3\,5)(2\,4\,6)$
$\sigma^{-1} = (1\,5\,3)(2\,6\,4)$



* The reflection permutation $(1\ n)(2\ n-1)\cdots(\lfloor n/2 \rfloor\ \lceil n/2 \rceil)$



<u>Remark</u>: the reflection is its own inverse. The reason to define the action of S_n by inverses is precisely so that a forward cycle of $[n]$ induces a forward rotation of $\mathcal{P}(n)$. This is typical of (left) group actions.

<u>Exercise</u>: Show that rotations and the reflection in S_n indeed induce automorphisms of $NC(n)$. Show they (and their compositions) are the <u>only</u> permutations that are automorphisms of $\mathcal{P}(n)$.

<u>Harder Exercise</u>: Show that all automorphisms of $NC(n)$ are permutations. Thus $\mathrm{Aut}(NC(n)) \cong D_n$ (the dihedral group in S_n, generated by rotations and reflection) when $n \geq 3$.