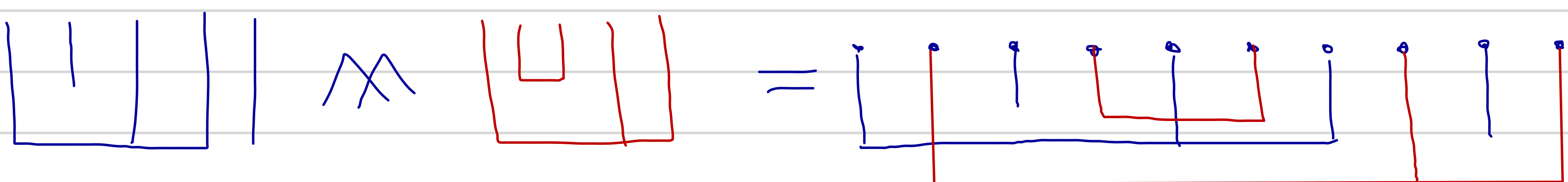
Lecture 12: April 27, 2011

An Important Function $NC(n) \to NC(n)$

Thus far we have talked explicitly about partitions on the sets $[n]$; of course, all that matters is the order on the set. If I is an ordered set with $\#I = n$, then $NC(I)$ is isomorphic to $NC(n)$ in the obvious way.

Consider the ordered sets $2[n] = \{2, 4, \ldots, 2n\}$
and $2[n]-1 = \{1, 3, \ldots, 2n-1\}$.
Both have the same size as $[n]$. So, if $\pi, \lambda \in NC(n)$, we can think of them living in $NC(2[n]-1)$ or $NC(2[n])$: $2\pi - 1 \in NC(2n-1)$, $2\lambda \in NC(2[n])$. Now <u>interleave</u> them:

$$\mathcal{P}(2n) \ni \pi \pitchfork \lambda \equiv (2\pi - 1) \cup (2\lambda).$$

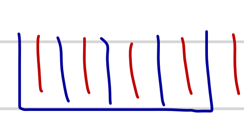


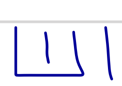
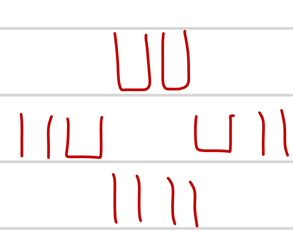
As this example shows, if $\pi, \lambda \in NC(n)$, it is not generally true that $\pi \pitchfork \lambda \in NC(2n)$. This brings up the question of when this <u>does</u> happen.

<u>Def</u>. Fix $\pi \in NC(n)$. Define $\mathcal{K}_\pi \subseteq NC(n)$ by

$$\mathcal{K}_\pi = \{\lambda \in NC(n) : \pi \pitchfork \lambda \in NC(2n)\}.$$

<u>Note</u>: 0_n cannot cross any blocks in an interleaving, so $0_n \in \mathcal{K}_\pi \ \forall \pi$. Here are a couple of examples.

E.g. if $\pi = 1_n$, then $\mathcal{K}_{1_n} = \{0_n\}$ 

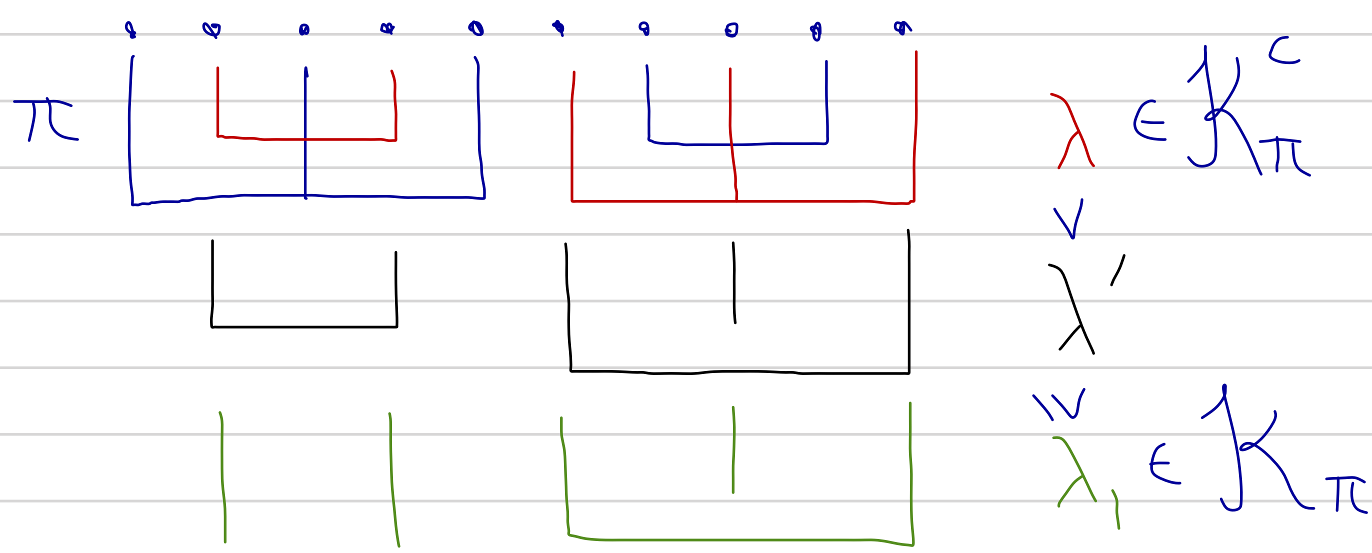
E.g. if $\pi =$ , $\mathcal{K}_\pi = \{$  $\}$

In fact, $\mathcal{K}_\pi$ is always a lattice.

<u>Lemma</u>: Fix $\pi \in NC(n)$. Let $\mathcal{K}_\pi = \{\lambda \in NC(n) : \pi \bowtie \lambda \in NC(2n)\}$.
Then $\mathcal{K}_\pi$ is a non-empty, downward-closed sub-lattice of $NC(n)$.

Pf. First, since all blocks of 0_n are singletons, $\pi \bowtie 0_n \in NC(2n)$ $\forall \pi \in NC(n)$, so $0_n \in \mathcal{K}_\pi$. Similarly, if $\lambda_1 \in \mathcal{K}_\pi$ and $\lambda_0 \leq \lambda_1$, this means any block of λ_0 is contained in a block of λ_1. Since λ_1's blocks do not cross π, it follows that λ_0's blocks do not cross π, and so $\lambda_0 \in \mathcal{K}_\pi$. This shows $\mathcal{K}_\pi$ is downward-closed. In particular, if $\lambda_1, \lambda_2 \in \mathcal{K}_\pi$, since $\lambda_1 \wedge \lambda_2 \leq \lambda_1$, it follows that $\lambda_1 \wedge \lambda_2 \in \mathcal{K}_\pi$, so $\mathcal{K}_\pi$ is closed under $\wedge$.

It remains to show that $\mathcal{K}_\pi$ is closed under $\vee$. Recall that $\lambda_1 \vee \lambda_2 = \inf D(\lambda_1, \lambda_2)$, where $D(\lambda_1,\lambda_2) = \{\lambda \in NC(n) : \lambda \geq \lambda_1 \ \& \ \lambda \geq \lambda_2\}$.
Let $\lambda_1, \lambda_2 \in \mathcal{K}_\pi$, and suppose that $\lambda \in D(\lambda_1,\lambda_2)$ is such that $\pi \bowtie \lambda$ has a crossing. Let $B \in \lambda$ be a block that crosses π.
Let λ' be a refinement of λ achived by breaking $B = B_1 \cup B_2$ into two blocks, removing this particular crossing. Then $\lambda_1, \lambda_2 \leq \lambda'$ too — since $\lambda_1, \lambda_2 \in \mathcal{K}_\pi$, they do not cross π, and so any block of λ_1 or λ_2 that is contained in B must in fact be contained in either B_1 or B_2.



Thus, if $\lambda \in D(\lambda_1,\lambda_2)$ is not in $\mathcal{K}_\pi$, then $\exists \lambda' \in D(\lambda_1,\lambda_2)$ s.t. $\lambda' < \lambda$.
It follows that $\lambda_1 \vee \lambda_2 = \inf D(\lambda_1,\lambda_2) \notin \mathcal{K}_\pi^c$, so $\lambda_1 \vee \lambda_2 \in \mathcal{K}_\pi$.

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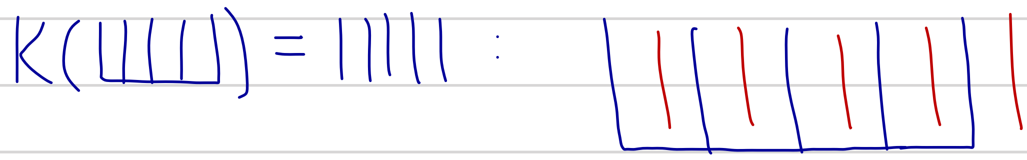
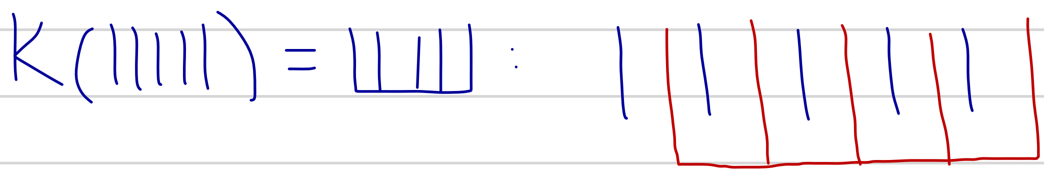
So, in particular, $\mathcal{K}_\pi$ has a unique maximal element $\sup \mathcal{K}_\pi \in \mathcal{K}_\pi$.

<u>Def</u>: For $\pi \in NC(n)$, the <u>Kreweras complement</u> $K(\pi)$ is defined as $K(\pi) = \sup \mathcal{K}_\pi$; i.e. $K(\pi)$ is the largest NC partition with $\pi \bowtie K(\pi)$ NC.

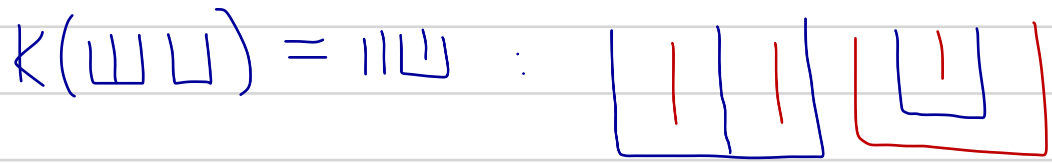
So $K : NC(n) \longrightarrow NC(n)$ is a function.

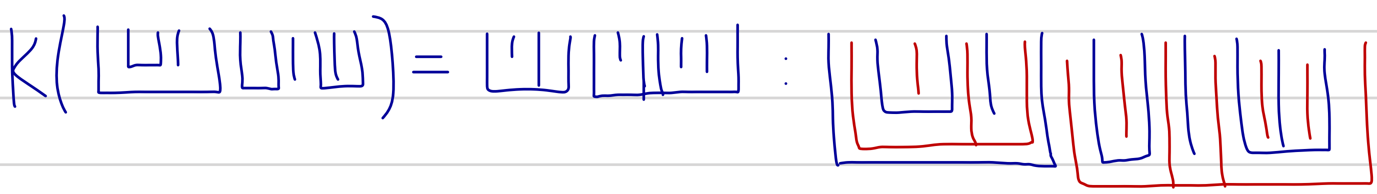
Eg's





In general $K(1_n)=0_n$ and $K(0_n)=1_n$.





In the next pages, we will develop the main properties of K. When appropriate, we refer to the Kreweras Complement map on $NC(n)$ as K_n.

We are interested in the function $K_n : NC(n) \to NC(n)$ defined by $K_n(\pi) = \max\{\lambda \in NC(n) : \pi \bowtie \lambda \in NC(2n)\}$. To establish its main properties, a little more language is useful.

<u>Def</u>. Let L be a finite lattice, and let $\pi \le \lambda$ in L. The <u>interval</u> $[\pi,\lambda]$ is the set of all $\alpha \in L$ s.t. $\pi \le \alpha \le \lambda$. Note that $[\pi,\lambda]$ is a sub-lattice of L, with $0_{[\pi,\lambda]} = \pi$ and $1_{[\pi,\lambda]} = \lambda$.

If $\pi < \lambda$ and $[\pi,\lambda] = \{\pi,\lambda\}$, we say <u>$\lambda$ covers π</u>.

A sequence $\pi_1 \le \pi_2 \le \cdots \le \pi_n$ in L is called a <u>multichain</u>. If indeed $\pi_1 < \pi_2 < \cdots < \pi_n$, it is a <u>chain</u>. If, for $1 < k \le n$ π_k covers π_{k-1}, it is called a <u>saturated chain</u>.

One of our goals this week is to characterize the structure of all interval lattices in $NC(n)$. The map K_n is a powerful tool in this quest.

Let λ cover π in $NC(n)$. What does this mean? Since $\pi < \lambda$, each block of π is contained in a block of λ; the strict inequality means there must be at least one block $B \in \pi$ s.t. $\exists C \in \lambda$ with $B \subsetneq C$. If there were another block $C' \ne C$ in λ and a $B' \in \pi$ with $B' \subsetneq C'$, then λ could not cover π, for:

$$\pi = \{\cdots B \cdots B' \cdots\} < \{\cdots B \cdots C' \cdots\} < \{\cdots C \cdots C' \cdots\} = \lambda.$$

So, when λ covers π, only <u>one</u> block in λ can be broken apart in π. A similar argument shows this one block must be broken into <u>two</u> pieces. I.e.

<u>Prop</u>: Let $\lambda = \{C_1, \ldots, C_r\} \in NC(n)$. Then λ covers $\pi \in NC(n)$ iff $\exists i \in \{1, \ldots, r\}$ s.t. $\pi = \{C_1, \ldots, C_{i-1}, C_i^1, C_i^2, C_{i+1}, \ldots, C_r\}$ where $C_i^1 \sqcup C_i^2 = C_i$ (and $C_i^1, C_i^2 \ne \emptyset$).

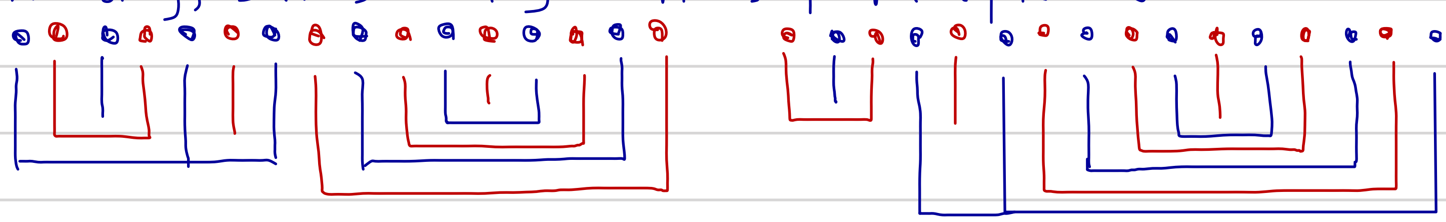
In particular, this means there are at most $|\lambda|$ partitions covered by λ, and if λ covers π then $|\pi| = |\lambda| + 1$. (So $NC(n)$ is a graded lattice, with levels given by common block size.)

Lemma: Let K_n denote the Kreweras complement on $NC(n)$. Then K_n is an anti-homomorphism of posets: If $\pi \le \lambda$, then $K_n(\lambda) \le K_n(\pi)$.

Pf. By definition $\lambda \ltimes K_n(\lambda)$ is non-crossing. Since $\pi \le \lambda$ and no block of λ cross the blocks of $K_n(\lambda)$, it follows that the refinement π also does not cross $K_n(\lambda)$ — so $\pi \ltimes K_n(\lambda) \in NC(2n)$. Thus $K_n(\lambda) \in \mathcal{K}_\pi$. In particular, it follows that $K_n(\lambda) \le \max \mathcal{K}_\pi = K_n(\pi)$. ///

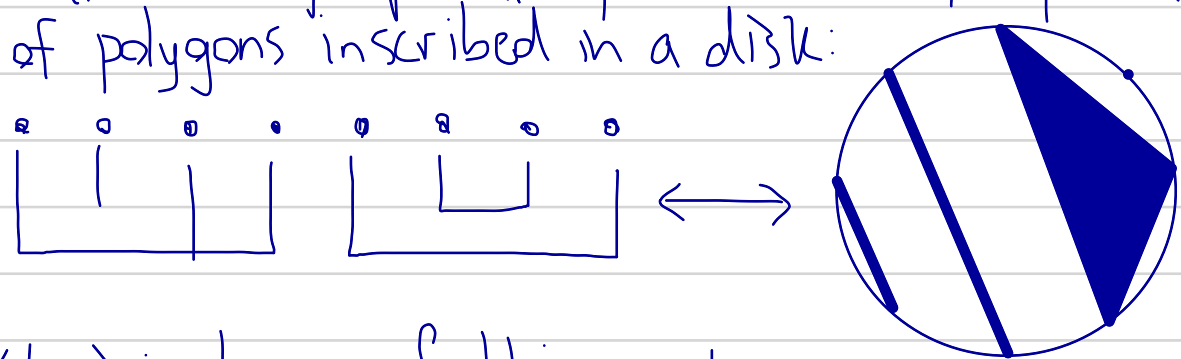
Lemma: Let $\gamma_n \in S_n$ be the cyclic rotation $\gamma_n(k) = k+1 \mod n$. Then $K_n^2 = \gamma_n$.

This is the kind of thing that is a bit of a pain to write out formally, but is totally obvious from a picture.



It is left as an Exercise to those inclined to write out the details. Alternatively,

Exercise: An alternate graphical representation of a partition is a collection of polygons inscribed in a disk:



Describe $K(\pi)$ in terms of this rep'n. Use this to easily prove that $K_n^2 = \gamma_n$.

Corollary: K_n is an anti-automorphism of $NC(n)$.

Pf. Since $(\gamma_n)^n = id \in S_n$, $K_n^{2n} = Id$. Hence K_n is a bijection, and since it is a (poset) anti-homomorphism, it follows that K_n is a (lattice) anti-automorphism. ///

Corollary: If λ covers π, then $K(\pi)$ covers $K(\lambda)$.

Pf. This is true of any anti-automorphism: we know $K(\lambda)\leq K(\pi)$; we also know $K(\lambda)\neq K(\pi)$ since K is one-to-one. Suppose $K(\pi)$ does not cover $K(\lambda)$ – so $\exists\,\alpha$ s.t. $K(\lambda)<\alpha<K(\pi)$. Since K^{-1} is also an anti-automorphism, $\lambda > K^{-1}(\alpha) > \pi$, which contradicts the fact that λ covers π. ///

Corollary: For each $\pi\in NC(n)$, $|\pi|+|K_n(\pi)| = n+1$.

Pf. Since $K_n(0_n)=1_n$, the formula holds for $\pi=0_n$. If $\pi\neq 0_n$, then $\pi>0_n$. Since $NC(n)$ is finite, this means there is a saturated chain $0_n<\pi_1<\cdots<\pi_k=\pi$. At each pair π_i,π_{i+1} in this chain, we have $|\pi_i|=|\pi_{i+1}|+1$ since π_{i+1} covers π_i. By the last corollary, $K(\pi_i)$ covers $K(\pi_{i+1})$, and so $|K(\pi_{i+1})|=|K(\pi_i)|+1$. Thus, for each $i<k$,

$$|\pi_{i+1}|+|K(\pi_{i+1})| = (|\pi_i|-1)+(|K(\pi_i)|+1) = |\pi_i|+|K(\pi_i)|.$$

It follows that $|\pi|+|K(\pi)| = |0_n|+|K(1_n)| = n+1$. ///

Exercise: Let $\rho_n\in S_n$ be the reflection $\rho_n(k)=k-n$. Recall that $\gamma_n(k)=k+1 \bmod n$. Show that (with ρ_n,γ_n also standing for their left-actions on $NC(n)$)

$$K_n\rho_n = \gamma_n\rho_n K_n.$$

Use this to show that $(K_n\rho_n)^2=Id$. Conclude that $K_n\rho_n$ is an <u>involution</u>: a self-inverse anti-automorphism. Show also that $|\pi|+|K_n\rho_n(\pi)|=n+1$.

Most lattices do not possess an involution; such a lattice is called <u>self-dual</u>. If L is a graded lattice, an involution on L must reverse the grading, which gives L up-down symmetry.

Exercise: For $n>3$, show that $\mathcal{P}(n)$ is not self-dual. [Hint: show that $\#\{\pi: |\pi|=k\}\neq\#\{\pi: |\pi|=n+1-k\}$ in general.]

Hard (!) Exercise: Show that $\{K_n,\rho_n,\gamma_n\}$ generate the group of skew-automorphisms of $NC(n)$, which is $\cong D_{2n}$ when $n\geq 3$.