Lecture 13: April 29, 2011

Factorization of Intervals in $NC(n)$

If $\pi \leq \tau$ in $NC(n)$, the lattice $[\pi, \tau]$ is actually a product of $NC(k)$'s.

Def: Let L, M be lattices. The product lattice $L \times M$ is the Cartesian product as sets, where

$$(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2) \iff \lambda_1 \leq \lambda_2 \in L \ \& \ \mu_1 \leq \mu_2 \in M.$$

It is easily verified that $L \times M$ is a lattice with

$$(\lambda_1, \mu_1) \wedge (\lambda_2, \mu_2) = (\lambda_1 \wedge \lambda_2, \mu_1 \wedge \mu_2)$$
$$(\lambda_1, \mu_1) \vee (\lambda_2, \mu_2) = (\lambda_1 \vee \lambda_2, \mu_1 \vee \mu_2)$$

and therefore $1_{L \times M} = (1_L, 1_M)$ and $0_{L \times M} = (0_L, 0_M)$.

Theorem: If $\pi \leq \tau$ in $NC(n)$, there exists a sequence $k_1, k_2, \ldots, k_n \in \mathbb{N}$ s.t. the interval $[\pi, \tau]$ is isomorphic (as a lattice) to

$$[\pi, \tau] \cong NC(1)^{k_1} \times \cdots \times NC(n)^{k_n}.$$

For the proof, we need one more bit of notation.

Def: Let $V \subseteq [n]$. If $\pi \in \mathcal{P}(n)$, then $\pi|_V \in \mathcal{P}(V)$ is the partition $\pi|_V = \{B \cap V : B \in \pi\}$. As usual, we can think of $\pi|_V \in \mathcal{P}(|V|)$. Note that if $\pi \in NC(n)$, then $\pi|_V \in NC(V)$.

Pf of Theorem: Step 1: let $\tau = \{V_1, \ldots, V_r\}$. Since any $\alpha \in [\pi, \tau]$ is $\leq \tau$, each block of α is contained in a V_j. In particular, $\alpha|_{V_j}$ consists of those blocks in α that were contained in V_j. This sets up a natural identification (as lattices)

$$[\pi, \tau] \cong \prod_{j=1}^{r} [\pi|_{V_j}, \tau|_{V_j}].$$

But $\tau|_{V_j} = \{V_j\}$; treating this as $\in NC(|V_j|)$, it is $1_{|V_j|}$.

Thus, $[\pi,\tau]$ is isomorphic to a product of intervals each of the form $[\theta, 1_m]$ for some $m \leq n$. So we need only treat such intervals in the foregoing.

Step 2: Using the Kreweras complement K_m on $NC(m)$, we have

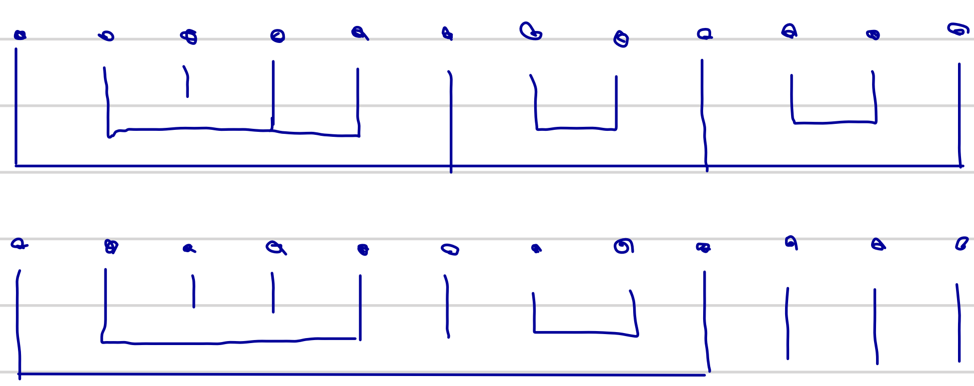
$$[\theta, 1_m] \underset{\text{anti-isomorphic}}{\overset{\uparrow}{\approx}} [K_m(1_m), K_m(\theta)] = [0_m, K_m(\theta)].$$

Now we return to step 1, and factor

$$[0_m, K_m(\theta)] \cong \prod_{V \in K_m(\theta)} [\underbrace{0_m|_V}_{\text{equal to } 0_{|V|}}, \underbrace{K_m(\theta)|_V}_{\text{identified as } 1_{|V|}}] \cong \prod_{V \in K_m(\theta)} NC(|V|).$$

Steps 1 & 2 show that $NC(n) \approx NC(1)^{k_1} \times \cdots \times NC(n)^{k_n}$ for some $k_1, \ldots, k_n \in \mathbb{N}$. The map $K_1^{k_1} \times \cdots \times K_n^{k_n}$ is an anti-automorphism of this product-lattice, and so $NC(n)$ is also isomorphic to it. ///

Eg. $\tau = \{\{1,6,9,12\}, \{2,4,5\}, \{3\}, \{7,8\}, \{10,11\}\}$
$\pi = \{\{1,9\}, \{2,5\}, \{3\}, \{4\}, \{6\}, \{7,8\}, \{10\}, \{11\}, \{12\}\}$



τ

π

To decompose $[\pi,\tau]$, we begin by breaking π up over the 5 blocks in τ:

$$[\pi,\tau] \cong \prod_{V \in \tau} [\pi|_V, 1_V] = [\{\{1,9\},\{6\},\{12\}\}, 1_{\{1,9,6,12\}}] \times [\{\{2,5\},\{4\}\}, 1_{\{2,5,4\}}]$$
$$\times [\{\{3\}\}, 1_{\{3\}}] \times [\{\{7,8\}\}, 1_{\{7,8\}}] \times [\{\{10\},\{11\}\}, 1_{\{10,11\}}]$$

$$\cong [\sqcup\!\sqcup|, \sqcup\!\sqcup\!\sqcup] \times [\sqcup, \sqcup\!\sqcup] \times [1,1] \times [\sqcup,\sqcup] \times [||, \sqcup].$$

The final 3 factors give us $NC(1)^2 \times NC(2)$. For the first 2, we use K.

- $[\underline{111}\,1,\ \underline{1111}] \overset{K_4}{\cong} [1111,\ K_4(\underline{111}\,1)] = [1111,\ \underline{11}\,\underline{11}]$ ↓ decompose over the blocks of $\underline{11}\,\underline{11}$
$\cong [11, \underline{11}] \times [11, \underline{11}]$
$= NC(2)^2$.

- $[\underline{11}\,,\ \underline{111}] \overset{K_3}{\cong} [111,\ K_3(\underline{11})] = [111,\ \underline{11}\,1] \cong [11, \underline{11}] \times [1,1] = NC(2) \times NC(1)$.

Altogether, we have $NC(1)^3 \times NC(2)^4$. (In particular, $\#[\pi,\lambda] = 16$.)

<u>Remarks</u> (1) One might expect, on first glance, that $[\underline{111}\,1,\ \underline{1111}] \cong [111, \underline{111}] = NC(3)$. The reason this does not happen is that the chain $\underline{111}\,1 < \underline{1\,1}\,\underline{1\,1} < \underline{1111}$ is not in $NC(4)$. This is another demonstration of structural difference between NC and $\mathcal{P}$; the interested reader may like to verify that, as lattices, $[\underline{111}\,1, \underline{1111}]_{\mathcal{P}(4)} \cong \mathcal{P}(3) = NC(3)$.

(2) Had we followed the K-procedure with all 5 factors (rather than immediately discarding the trivial third and fourth factors) we would have found $[1_1, 1_1] \cong NC(1)$ but $[1_2, 1_2] \cong [11, 11] \cong NC(1)^2$. So our answer would be $NC(1)^4 \times NC(2)^4$. This highlights the fact that the exponent k_1 isn't meaningful, since $NC(1)$ is trivial.

(3) One may wonder, though, if the exponents $k_2, \ldots, k_n$ are well-defined. They have canonical choices given by the above algorithm, but it is (a priori) possible that there exist examples with $(k_2, \ldots, k_n) \neq (\ell_2, \ldots, \ell_n)$ but

$$[\pi,\lambda] \cong NC(2)^{k_2} \times \cdots \times NC(n)^{k_n} \cong NC(2)^{\ell_2} \times \cdots \times NC(n)^{\ell_n}.$$

It so happens this is <u>not</u> possible. To see why, we must delve into several interesting enumeration questions in the lattices $NC(n)$. That is our next topic.

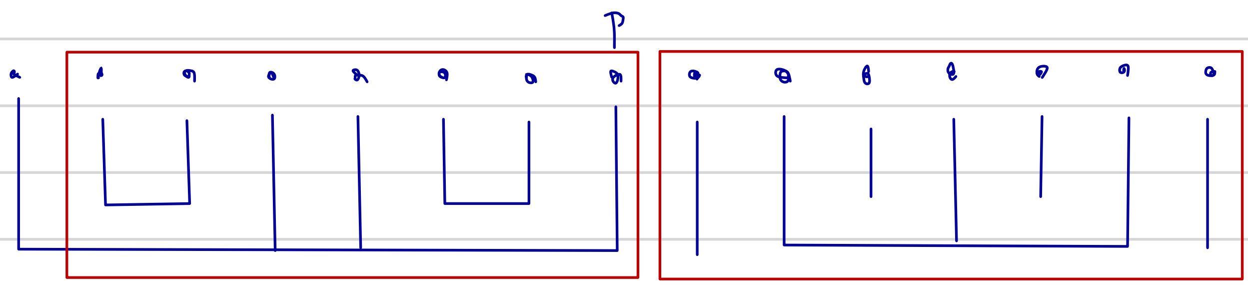
<u>Enumeration in $NC(n)$</u>

<u>Theorem</u>: $\#NC(n) = C_n$, the Catalan number.

We will outline three proofs of this.

Pf. #1: Similar to the recursive enumeration of $NC_2(2n)$. Let $NC^{(p)}(n)$ denote those partitions in $NC(n)$ s.t. the block containing 1 has largest element p.

$$\pi = \{\{1,\dots,p\},\dots\}.$$



identified with a(n arbitrary) $\in NC(\{2,\dots,p\}) \cong NC(p-1)$

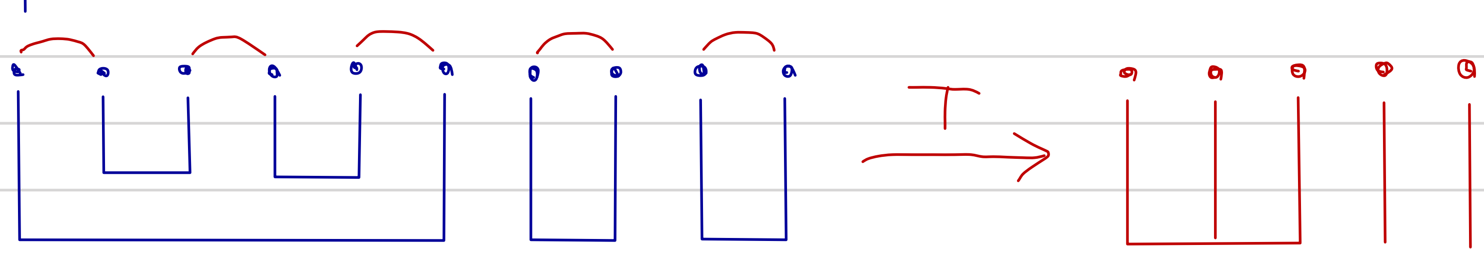
$\in NC(\{p+1,\dots,n\}) \cong NC(n-p)$

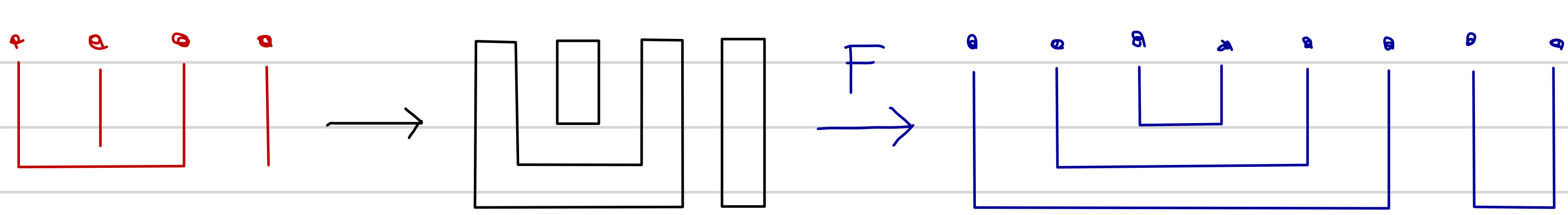
$\longrightarrow$ thus $\#NC^{(p)}(n) = \#NC(p-1)\cdot\#NC(n-p)$.

Thus $\#NC(n) = \sum_{p=1}^{n} \#NC(p-1)\cdot\#NC(n-p)$.

Again we find the Catalan recurrence. Since $\#NC(1)=1$, this concludes the proof.

Pf. #2: We already showed $\#NC_2(2n) = C_n$. There is a nice bijection $NC(n) \longleftrightarrow NC_2(2n)$, as the following picture shows:





Exercise: Formally define the functions $NC_2(2n) \underset{F}{\overset{T}{\rightleftarrows}} NC(n)$ and show they are well-defined inverses.

Pf. #3: We will give an explicit bijection between $NC(n)$ and a set of lattice paths (of length n) that generalize Dyck paths, and are counted by Catalan numbers. We do this not just for another proof, but because the lattice paths will encode other NC-statistics for easy counting.

Def: an <u>almost-rising path</u> of length n is a sequence $(\lambda_1,\dots,\lambda_n)$ in $\{-1,0,1,2,3,\dots\}^n$. We identify it as a lattice path with piecewise linear segments joining the points

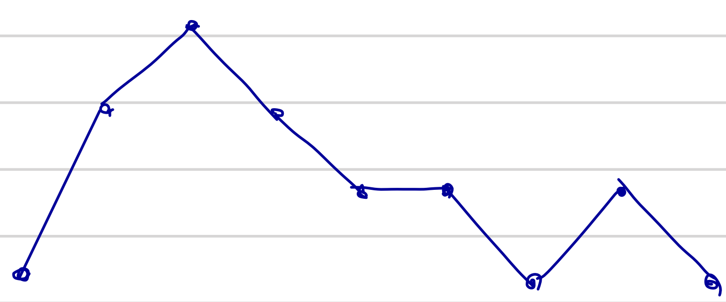
$$(0,0),\ (1,\lambda_1),\ (2,\lambda_1+\lambda_2),\ \dots,\ (n,\lambda_1+\cdots+\lambda_n).$$

A <u>Lukasiewicz path</u> is an almost-rising path that ends on the x-axis and never falls below it. That is: $(\lambda_1,\dots,\lambda_n)$ where $\lambda_j \ge -1$ in $\mathbb{Z}$, $\lambda_1+\cdots+\lambda_k \ge 0$ for $1\le k\le n$, and $\lambda_1+\cdots+\lambda_n = 0$.

The set of Lukasiewicz paths of length n is denoted $\mathrm{Luk}(n)$.

For example, a Dyck path of length $2n$ is in $\mathrm{Luk}(2n)$. But they form a (n exponentially) small subset.

E.g. $(2,1,-1,-1,0,-1,1,-1) \in \mathrm{Luk}(8)$



Def: Let $\pi \in NC(n)$. For $1\le j\le n$, define λ_j as follows:
* if $\exists B\in\pi$ s.t. $j=\min B$, set $\lambda_j = \#B-1$.
* otherwise, set $\lambda_j = -1$.

Denote $(\lambda_1,\dots,\lambda_n) = \Lambda(\pi)$. Evidently, $\Lambda(\pi)$ is an almost-rising path.

E.g. If $\pi =$  $\in NC(8)$, then $\Lambda(\pi)$ is the Lukasiewicz path pictured above.

Thm: The map $\Lambda: NC(n) \to \{\text{almost-rising paths of length } n\}$ is a bijection $NC(n)\to \mathrm{Luk}(n)$.

Pf. First, note that $\lambda_1+\cdots+\lambda_k = \sum_{B\in\pi}\sum_{j\in B\cap[k]} \lambda_j$.

For fixed $B\in\pi$, if $\min B \ge k$, $\{\lambda_j : j\in B\cap[k]\} = \{\#B-1, \underbrace{-1,\dots,-1}_{\max \#B-1\ -1\text{'s}}\}$ so the sum is ≥ 0. If $\min B < k$, $B\cap[k]=\emptyset$ and the sum is $=0$.

This shows the partial sums are all ≥ 0. When $k=n$, the sums are

$$\sum_{j\in B} \lambda_j = \#B-1+(\underbrace{-1-1\cdots-1}_{\#B-1}) = 0.$$

Thus $\Lambda(\pi) \in Luk(n)$. [Note, this is true for <u>any</u> $\pi \in \mathcal{P}(n)$; $\Lambda: \mathcal{P}(n) \to Luk(n)$.]

To show Λ is a bijection $NC(n) \to Luk(n)$, we will find its inverse $\Pi: Luk(n) \to NC(n)$. We define it in two steps.

<u>Step 1</u>: Fist, from the path $\lambda \in Luk(n)$, construct the following list

$$L(\lambda) = \left\{ (a,b) : \begin{array}{l} a \text{ is the position of a non-falling} \\ \text{segment in } \lambda,\ b = 1 + \text{the height @ } a \end{array} \right\}$$

E.g. $\underset{1}{(3},\underset{2}{-1},\underset{3}{0},\underset{4}{-1},\underset{5}{0},\underset{6}{1},\underset{7}{-1},\underset{8}{-1},\underset{9}{0})$ $\underset{a}{b-1}$ $\xrightarrow{L} \left\{ \begin{array}{l} (1,4),(3,1),(5,1), \\ (6,2),(9,1) \end{array} \right\}$

Now, from $L(\lambda)$, recursively construct $\Pi(\lambda)$ as follows. $\Pi(\lambda)$ has $\#L(\lambda)$ blocks. As above, list the elements of $L(\lambda)$ in increasing a-order $\{(a_1,b_1),\ldots,(a_r,b_r)\}$.

<u>Claim</u>: For $1\leq k\leq r$, $a_k+b_k+b_{k+1}+\cdots+b_r \leq n+1$.

Pf. In the path up to (but not including) segment a_k, the partial sum is

$$(b_1-1)+\cdots+(b_{k-1}-1) - \#\{-1\text{s up to } a_k\}.$$

Among these a_k-1 terms, there are $k-1$ non-falling steps, so this sum $= (b_1-1)+\cdots+(b_{k-1}-1) - ((a_k-1)-(k-1))$

$$= b_1+\cdots+b_{k-1} - a_k + 1.$$

Since the path is in $Luk(n)$, this sum is ≥ 0, so $b_1+\cdots+b_{k-1} \geq a_k - 1$. On the other hand, the total sum of the path is

$(b_1-1)+\cdots+(b_r-1) - \underbrace{\#\{-1 \text{ slopes}\}}_{=n-r} = b_1+\cdots+b_r - n \underset{\text{b/c } Luk(n)}{=} 0.$

Thus $b_1 + \cdots + b_{k-1} = n - (b_k + \cdots + b_r)$. Whence

$$n - (b_k + \cdots + b_r) \geq a_k - 1 \Rightarrow a_k + b_k + \cdots + b_r \leq n+1.$$

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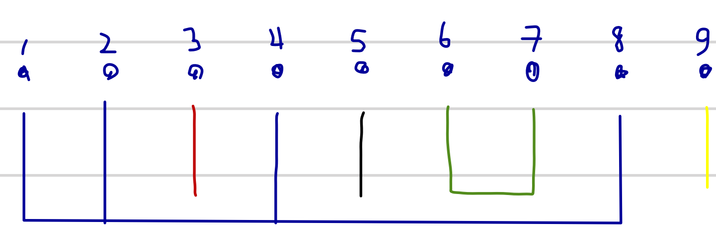
This allows us to proceed with

<u>Step 2</u>: Now that we have the list $L(\lambda) = \{(a_1,b_1), \dots, (a_r,b_r)\}$, we form a partition $\Pi(\lambda)$ as follows.

* $\Pi(\lambda) \ni \{a_r, a_r+1, \dots, a_r+b_r-1\}$. ← possible b/c $a_r + b_r - 1 \leq n$.
* Now remove these indices from $[n]$, and denote the remaining ordered set $[n]_{r-1}$. Then $\Pi(\lambda)|[n]_{r-1}$ contains an interval block with b_{r-1} elements, with least element a_{r-1}.
  ↳ Since we removed b_r indices from $[n]$ to form $[n]_{r-1}$, the largest index in this block is $a_{r-1} + b_{r-1} - 1 + b_r$, and again by the claim this is $\leq n$, so this is possible.
* Continue reducing this way adding an interval block of length b_k with least element a_k, among the reduced indices, until all indices are used up. (There will be no indices left after the r^{th} step, as $a_1 + b_1 - 1 + b_2 + \cdots + b_r = n$, as the proof of the claim shows.)

E.g. with $L(\lambda) = \{(1,4), (3,1), (5,1), (6,2), (9,1)\}$,

$\Pi(\lambda) =$ 

<u>Exercise</u>: Show that $NC(n)$ can be characterized by the recursive description of the map Π: $\pi \in NC(n)$ iff $\exists$ interval block in π and, wlog taking the right-most block B, $\pi|_{[n]-B} \in NC([n]-B)$.

It is clear from the construction that $\Lambda\Pi(\lambda) = \lambda$ for $\lambda \in Luk(n)$; the exercise shows that $\Pi\Lambda(\pi) = \pi$ for $\pi \in NC(n)$. Thus, $NC(n) \cong Luk(n)$. ///

We've already seen that $\# NC(n) = C_n$ so $\# Luk(n) = C_n$. But the real use of this bijection is in more delicate enumerations.

We'll come to those in the next lecture.

Exercises (for fun)

(1) Describe the action of the Kreweras complement on Lukasiewicz paths; i.e. describe $\Lambda K \Pi$ on $Luk(n)$.

(2) Compose the bijections

$$Dyck(2n) \to NC_2(2n) \to NC(n) \to Luk(n)$$

to form a bijection between Dyck paths of length $2n$ and Lukasiewicz paths of length n. Describe this bijection.