Lecture 14: May 2, 2011

One of the big benefits of the bijection $\Lambda : NC(n) \to Luk(n)$ we developed last time is that it nicely encodes the block structure. Indeed: for $k \in [n]$,

$$\#\{B \in \pi : \#B = k\} = \#\{\lambda \in \Lambda(\pi) : \lambda = k-1\}$$

In particular, since Λ is a bijection, this means we can do the following translation of "block-profile" enumeration.

Fix $r_1, r_2, \ldots, r_n \in \mathbb{N}$. When does $\exists \pi \in NC(n)$ with r_k blocks of size k for $k \in [n]$? The requirement is that

$$r_1 + 2r_2 + \cdots + nr_n = n.$$

If $r_1, \ldots, r_n$ satisfy this equation, then

$$\#\{\pi \in NC(n) : \forall k \in [n], \pi \text{ has } r_k \text{ blocks of size } k\}$$
$$= \#\{\lambda \in Luk(n) : \forall k \in [n], \lambda \text{ has } r_k \text{ steps of height } k-1\}.$$

This is something we can actually enumerate, through the following rotation trick.

Raney's Lemma: there is a bijection

$$R : Luk(n) \times [n+1] \longrightarrow \{\lambda \in \{-1,0,1,2,\ldots\}^{n+1} \text{ s.t. } \lambda_1 + \cdots + \lambda_{n+1} = -1\}$$

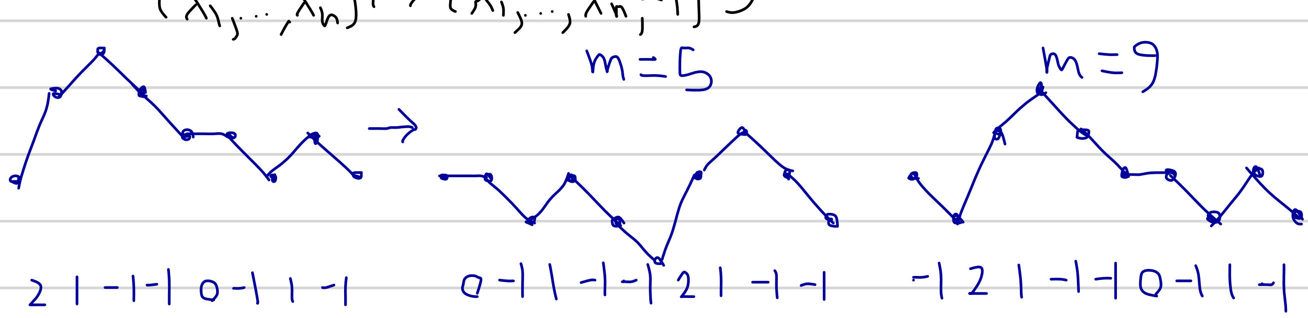
almost-rising paths $(0,0) \to (n+1,-1)$.

Pf. The map is to adjoin -1 to the end of λ, then rotate:

$$(\lambda, m) \xmapsto{R} (\lambda_m, \ldots, \lambda_n, -1, \lambda_1, \ldots, \lambda_{m-1})$$

$$(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1, \ldots, \lambda_n, -1)$$

Eg.



The statement that this map is a bijection can be equivalently stated as: given any almost-rising path $(0,0)\to(n+1,-1)$, there is a unique cyclic rotation whose first n steps form a $Luk(n)$ path. (This is the precise statement of Raney's lemma.)

We begin with uniqueness (ie injectivity). The examples show that rotations can drop below -1, but one thing rotations preserve is the location of the lowest height.

Let $(p_k)_{k=1}^{n+1}$ denote the partial sums in $(\lambda,-1)$ - $p_k=\lambda_1+\cdots+\lambda_k$, and $p_{n+1}=-1$; since λ is in $Luk(n)$, $p_k\geq 0$ for $k\leq n$ and $p_n=0$. Now, let $(q_k)_{k=1}^{n+1}$ be the partial sums of $R(\lambda,m)$; then

$$q_k=\begin{cases} p_{k+m-1}-p_{m-1}, & m+k-1\leq n+1 \\ p_{k+m-1-(n+1)}-p_{m-1}-1, & m+k-1>n+1\end{cases}.$$

So, for $k<n+2-m$, $q_k\geq 0-p_{m-1}$. At $k=n+2-m$, we have $q_{n+2-m}=p_{n+1}-p_{m-1}=-1-p_{m-1}$. For $k>n+2-m$, $q_k\geq 0-p_{m-1}-1$.

Ie. the lowest height reached by $R(\lambda,m)$ is $-p_{m-1}-1$, and the first time this occurs is $n+2-m$. So given an almost-rising path $(0,0)\to(n+1,-1)$ that is constructed as $R(\lambda,m)$ for some $\lambda\in Luk(n)$, we can identify m: if k is the index where the lowest height is first reached, then $m=n+2-k$. Thus we rotate this amount back, and find λ. So R is one-to-one.

To show R is onto, we begin with an arbitrary almost-rising path $\mu:(0,0)\to(n+1,-1)$. Let k be the position of the first occurrence of the lowest height in μ. By the uniqueness argument, we know that the only rotation of μ that could be of the form $(\lambda,-1)$ for a $\lambda\in Luk(n)$ must start with μ_{k+1}. So, define

$$\tilde{\mu}=(\mu_{k+1},\dots,\mu_{n+1},\mu_1,\dots,\mu_k)$$

Note: if the first n terms form a $Luk(n)$ path, their sum is 0. The sum of $\tilde{\mu}$ is the sum of μ which is -1 by assumption, so $\mu_k=-1$.

So it suffices to show that $(\mu_{k+1}, \dots, \mu_{n+1}, \mu_1, \dots, \mu_{k-1}) \in Luk(n)$.
Suppose, for a contradiction, that there is a partial sum < 0. If $\mu_{k+1} + \dots + \mu_p < 0$ for $p \leq n+1$, then $\mu_1 + \dots + \mu_k + \mu_{k+1} + \dots + \mu_p$ $< \mu_1 + \dots + \mu_k$ ← but this is the minimal height, so that is not possible. If $\mu_k + \dots + \mu_{n+1} + \mu_1 + \dots + \mu_p < 0$ for $p \in [k-1]$, then since (as just shown) $\mu_{k+1} + \dots + \mu_{n+1} \geq 0$, it follows that $\mu_1 + \dots + \mu_p < 0$, again contradicting the minimality of the height at k.

So we have shown this path never falls below the x-axis. Also, if the sum were > 0, as $\mu_k \in \{-1, 0, 1, \dots\}$ it would follow that $\sum \mu = \sum \tilde{\mu} \geq 0$, but we know $\sum \mu = -1$. Thus, $\tilde{\mu} = (\lambda, -1)$ for a $\lambda \in Luk(n)$, and so $\mu = R(\lambda, m)$ where $m = n+2-k$.
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Exercise: Show (directly) that

$$\#\{\text{almost-rising paths } (0,0) \to (n+1, -1)\} = \binom{2n}{n}.$$

Hence, the bijection of Raney's lemma gives another proof of the count

$$\# NC(n) = \# Luk(n) = \frac{1}{n+1} \binom{2n}{n} = C_n.$$

This is the only example I know of where C_n occurs combinatorially as the obvious ratio.

Now, the bijection obviously preserves the number of steps of size k in the path, except when $k = -1$ (in which case it increases by 1 the number of steps).

In other words, if $r_1, \dots, r_n \in \mathbb{N}$ satisfy $r_1 + 2r_2 + \dots + nr_n = n$, then

$$\{\lambda \in Luk(n) : \forall k \in [n]\ \#\{j : \lambda_j = k-1\} = r_k\} \times [n+1]$$

$$\xrightarrow{R} \{\mu : (0,0) \to (n+1, -1) \text{ almost-rising s.t. } \forall k \in [n]\ \#\{j : \mu_j = k-1\} = r_k\}$$

Hence,

$$\#\{\pi \in NC(n) : \forall k \in [n]\ \#\{B \in \pi : \#B = k\} = r_k\} \times (n+1)$$

$$= \#\{\text{almost rising } \mu : (0,0) \to (n+1, -1) \text{ s.t. } \forall k \in [n]\ \#\{j : \mu_j = k-1\} = r_k\}.$$

Note, for any such path, the number of $\mu_j = -1$ is $(n+1) - (r_1 + \dots + r_n)$.

But this is just counted by the multinomial coefficient.

<u>Theorem</u>: If $r_1, \dots, r_n \in [n]$ s.t. $r_1 + 2r_2 + \cdots + nr_n = n$, then

$$\#\{\pi \in NC(n) : \forall k \in [n]\ \#\{B \in \pi : \#B = k\} = r_k\} \times (n+1)$$
$$= \frac{(n+1)!}{r_1! \cdots r_n!\,((n+1) - (r_1 + \cdots + r_n))!}$$

I.e.

$$\#\{\pi \in NC(n) \text{ with } r_k \text{ blocks of size } k,\ k \in [n]\} = \frac{n!}{r_1! \cdots r_n!\,((n+1) - r_1 - \cdots - r_n)!}$$

We will use this formula many times when calculating moments.

<u>Exercise</u>: By similar methods, show that for $k \in [n]$

$$\#\{\pi \in NC(n) : \pi \text{ has } k \text{ blocks}\} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

These are the <u>Narayana Numbers</u>. Note they are invariant under $k \mapsto n-k+1$, which is the vertical symmetry in the Hasse-diagram of $NC(n)$, made possible by the existence of an involution.

If $\pi \in NC(n)$, the set of $\sigma \in NC(n)$ with $|\sigma| = |\pi|$ is the <u>maximal anti-chain</u> containing π. (I.e. the largest set of mutually-incomparable elements.) So we have enumerated the maximal anti-chains.

Multi-chain Enumeration

$$\# NC(n) = C_n, \text{ where } C_0 = 1,\ C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k} = \sum_{\substack{i,j \geq 0 \\ i+j = n-1}} C_i C_j.$$

Of course, elements of $NC(n)$ can be viewed as multi-chains of "length 0".

<u>Def</u>: If P is a poset, a multichain $\pi_0 \leq \pi_1 \leq \cdots \leq \pi_{m-1}$ has "length" $m-1$. The set of all multichains of length $m-1$ in P is denoted $P^{(m)}$. (So $P^{(1)} = P$.)

Theorem: Denote $\#NC(n)^{(m)} = C_n^{(m)}$ (So $C_n^{(1)}$ are the Catalan numbers.) Then, denoting $C_0^{(m)} = 1$ $\forall m$,

$$C_n^{(m)} = \sum_{\substack{i_1,\dots,i_{m+1}\geq 0 \\ i_1+\dots+i_{m+1}=n-1}} C_{i_1}^{(m)} \cdots C_{i_{m+1}}^{(m)}.$$

This can be proved with a (trickier) recursive decomposition over the largest element of the block containing 1 (at the top of the multichain, then on down). The interested reader is encouraged to work this out as an Exercise. We will prove this a different way in the second combinatorial part of the course (dealing with incidence algebras of posets, and associated formal power-series).

It is (tedious but) easy to show by induction that

$$C_n^{(m)} = \frac{1}{mn+1}\binom{(m+1)n}{n}.$$

/ Personal remark: a careful understanding of the asymptotics of these numbers, and how they arise in free probability, has resulted in my two best (certainly most famous) papers, along with two others. /

For our present purposes, we can use these numbers to show that the interval decomposition in $NC(n)$ is unique. The key is the following fact.

Lemma: Let $r,s\geq 1$ and $n_1,\dots,n_r,k_1,\dots,k_s\geq 2$. If

$$\forall m\geq 1 \quad C_{n_1}^{(m)}\cdots C_{n_r}^{(m)} = C_{k_1}^{(m)}\cdots C_{k_s}^{(m)},$$

then $r=s$, and $(n_1,\dots,n_r)$ is a permutation of $(k_1,\dots,k_s)$.

Pf. We will show that $\max(n_1,\dots,n_r) = \max(k_1,\dots,k_s)$; the result then follows by induction on $r+s$.

$\therefore$ For a contradiction, assume $n = \max(n_1, \ldots, n_r) > \max(k_1, \ldots, k_s)$. As $k_1 \geq 2$, $n \geq 3$.

<u>A theorem of Dirichlet</u>: If a, b are relatively-prime, the arithmetic progression $\{al+b : l \geq 1\}$ contains infinitely many primes.

In particular, there is an $\ell \geq 1$ s.t. $n\ell + n - 1 = p$ is prime. Now, for this choice of ℓ,

$$C_n^{(\ell)} = \frac{1}{\ell n+1}\binom{(\ell+1)n}{n} = \frac{(\ell n+n)\overset{p}{(\ell n+n-1)}\cdots(\ell n+2)}{n!}$$

$p > n$ so $n!$ is not divisible by p.

Hence $C_n^{(\ell)}$ is divisible by p.

Thus, p divides $C_{n_1}^{(\ell)} \cdots C_{n_r}^{(\ell)} = C_{k_1}^{(\ell)} \cdots C_{k_s}^{(\ell)}$.

On the other hand, note that $k_i < n$ for all $i \in [s]$. But

$$C_{k_i}^{(\ell)} = \frac{(\ell k_i + k_i)\cdots(\ell k_i + 2)}{k_i!}$$

$p = \ell n + n - 1 = (\ell+1)(n - \frac{1}{\ell+1}) > \ell k_i + k_i$

So $p >$ each factor in the numerator

So $p \nmid C_{k_i}^{\ell}$ for any i, and so cannot divide the product. This is a contradiction. ///

This helps us prove the uniqueness of the interval decomposition in $NC(n)$ because of the following (obvious) fact:

If $P_1, \ldots, P_n$ are posets and $m \geq 1$, then

$$(P_1 \times \cdots \times P_n)^{(m)} \cong P_1^{(m)} \times \cdots \times P_n^{(m)}.$$

<u>Corollary</u>: if $n_1, \ldots, n_r, k_1, \ldots, k_s \geq 2$, and if

$$NC(n_1) \times \cdots \times NC(n_r) \cong NC(k_1) \times \cdots \times NC(k_s)$$

then $r = s$ and $(n_1, \ldots, n_r)$ is a permutation of $(k_1, \ldots, k_s)$.

Pf. By the preceeding remark, we have for $m \geq 1$

$$\begin{array}{ccc} (NC(n_1)\times\cdots\times NC(n_r))^{(m)} & \cong & (NC(k_1)\times\cdots\times NC(k_s))^{(m)} \\ \wr\| & & \wr\| \\ NC(n_1)^{(m)}\times\cdots\times NC(n_r)^{(m)} & & NC(k_1)^{(m)}\times\cdots\times NC(k_s)^{(m)} \end{array}$$

Thus, taking cardinalities, for each $m \geq 1$ we have

$$C_{n_1}^{(m)}\cdots C_{n_r}^{(m)} = C_{k_1}^{(m)}\cdots C_{k_s}^{(m)}.$$

The result follows now from the lemma. ///

Thus, in the canonical factorization of an interval $[\pi,\tau]$ in $NC(n)$ via Kreweras complement

$$[\pi,\tau] \cong NC(1)^{k_1}\times\cdots\times NC(n)^{k_n}$$

the exponents $k_2,\ldots,k_n$ are, in fact, uniquely-determined. (This fact will not be needed in the rest of this course, but it does enrich our understanding of the lattice structure of $NC(n)$.

Remark: Note that the set of intervals $[\pi,\tau]$ in a poset P is canonically identified with the set $P^{(2)}$ of multichains of length 1 in P. We will spend the next few lectures studying this set for a general poset/lattice, before returning to $NC(n)$.