Lecture 15: May 4, 2011

Incidence Algebras

Let P be a finite poset. As last time, let $P^{(2)}$ denote the subset of P^2 consisting of all pairs (π,σ) s.t. $\pi \leq \sigma$.

The <u>incidence algebra</u> of P is an object alalogous to the group algebra of a group. It is defined as the set of all functions on $P^{(2)}$:

$$\mathbb{C}P^{(2)} = \{ F : P^{(2)} \to \mathbb{C} \}.$$

The vector-space structure is the obvious one. The product is defined as a kind of convolution:

$$F * G\ (\pi,\sigma) = \sum_{\pi \leq \tau \leq \sigma} F(\pi,\tau) G(\tau,\sigma).$$

It is easy to verify that this product is associative and distributive — this will also follow from a repre-sentation of $\mathbb{C}P^{(2)}$ we will construct shortly.

Note that this algebra has a unit: set $\delta_P \in \mathbb{C}P^{(2)}$ to be

$$\delta_P(\pi,\sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{if } \pi < \sigma \end{cases}.$$

Then clearly $F * \delta_P = \delta_P * F = F$ for all F.

The vector space $\mathbb{C}P = \{ f : P \to \mathbb{C} \}$ is naturally a (right- or left-) $\mathbb{C}P^{(2)}$-module; the canonical choice is right-module, with

$$f * F(\pi) = \sum_{\tau \leq \pi} f(\tau) F(\tau,\pi).$$

In fact, both of these actions are just matrix multi-plication.

<u>Prop</u>: Given P, there is a vector space isomorphism $V : \mathbb{C}P \to \mathbb{C}^N$, and a vector-space injective homomorphism $M : \mathbb{C}P^{(2)} \to \mathrm{Mat}_N(\mathbb{C})$ s.t. $M(F*G) = M(F)M(G)$ and $V(f*F) = V(f)M(F)$. [Here $N = \#P$.]

Pf. The key is to list P in a way that respects its poset structure. So, list $P = \{\pi_1, \pi_2, \ldots, \pi_N\}$ such that it never happens that $i<j$ but $\pi_i > \pi_j$. (It is an easy <u>Exercise</u> to show this is always possible.) Thus, we have that if $i<j$ then either $\pi_i < \pi_j$ or π_i, π_j are incomparable.

Having chosen this ordering, define (for $F \in \mathbb{C}P^{(2)}$) the matrix

$$[M(F)]_{ij} = \begin{cases} 0, & \text{if } i>j \\ F(\pi_i,\pi_i) & \text{if } i=j \\ F(\pi_i,\pi_j) & \text{if } i<j \ \& \ \pi_i<\pi_j \\ 0, & \text{if } i<j \ \& \ \pi_i,\pi_j \text{ are incomparable} \end{cases}$$

Note that $M(F)$ is upper-triangular. So $M(F)M(G)$ is too.

For $i \leq j$, $\quad [M(F)M(G)]_{ij} = \sum_{k=1}^{N} [M(F)]_{ik}[M(G)]_{kj}$

$$= \sum_{k=i}^{j} \underbrace{[M(F)]_{ik}[M(G)]_{kj}}_{\neq 0 \text{ only if } \pi_i \leq \pi_k \leq \pi_j}$$

$$= \sum_{\substack{i \leq k \leq j \\ \pi_i \leq \pi_k \leq \pi_j}} F(\pi_i,\pi_k)G(\pi_k,\pi_j)$$

and this is, of course, equal to

$$F*G(\pi_i,\pi_j) = [M(F*G)]_{ij}.$$

Similarly, set $v(f) = [f(\pi_1), \ldots, f(\pi_N)]$; the proof that $v(f*F) = v(f)M(F)$ is analogous. ///

This representation shows that the product/action is distributive and associative: $(F*G)*H = F*(G*H)$ and $(f*F)*G = f*(F*G)$.

Note that $M(\delta)$ is the identity matrix in Mat_N. Since the matrices $M(F)$ are all upper-triangular, this makes it easy to characterize the invertible elements in $\mathbb{C}P^{(2)}$.

<u>Prop</u>: F is invertible in $\mathbb{C}P^{(2)}$ iff $F(\pi,\pi) \neq 0 \ \forall \pi \in P$.

Pf. First, an upper-triangular matrix $M \in \mathrm{Mat}_N(\mathbb{C})$ is invertible iff all its diagonal entries are non-zero. So, if $F(\pi_i,\pi_i)=0$, then $[M(F)]_{ii}=0$ and $M(F)$ is not invertible, $\therefore$ F cannot be invertible (since M is injective). Conversely, if $F(\pi,\pi)\neq 0\ \forall \pi$, then $[M(F)]_{ii}$ is non-zero for all $i\in[N]$, and so $M(F)$ is invertible in $M_N(\mathbb{C})$: $\exists M \in M_N(\mathbb{C})$ s.t. $M(F)M = M\cdot M(F) = I$. We need to show $\exists G \in \mathbb{C}P^{(2)}$ s.t. $M = M(G)$. This will follow from the following

Lemma: Let $\mathcal{M}$ be a set of upper-triangular matrices containing all diagonal matrices, and closed under addition, scalar multiplication, and matrix multiplication. If $M \in \mathcal{M}$ is an invertible matrix, then $M^{-1}\in\mathcal{M}$.

Pf. Let $M = D - T$ where D is diagonal and $[D]_{ii}=[M]_{ii}$. So T is strictly upper triangular, and thus nilpotent: $T^N=0$. By assumption, $T\in\mathcal{M}$. Since M is invertible and upper-triangular, D has no 0 entries so is invertible. Thus $M = D(I-D^{-1}T)$ and so $M^{-1} = (I-D^{-1}T)^{-1}D^{-1}$. Since D^{-1} is diagonal,

$$[D^{-1}T]_{ii} = \sum_{k=1}^{N}[D^{-1}]_{ik}[T]_{ki} = [D]_{ii}^{-1}[T]_{ii} = 0$$

So $D^{-1}T$ is also strictly upper-triangular, $(D^{-1}T)^N=0$. Hence, the power-series converges

$$(I-D^{-1}T)^{-1} = I + D^{-1}T + (D^{-1}T)^2 + \cdots + (D^{-1}T)^{n-1}.$$

This matrix is in $\mathcal{M}$, since $\mathcal{M}$ contains D^{-1}, T, and is closed under sum and product. Thus

$$M^{-1} = (I-D^{-1}T)^{-1}D^{-1} \in \mathcal{M}. \qquad ///$$

The image of our map M is a set of upper-triangular matrices, closed under sum and product by the fact that M is a representation. Any diagonal matrix D is in its image: $D = M(F)$ where $F(\pi_i,\pi_j) = \delta_{ij}[D]_{ii}$. ///

Thus, there is a quick check for invertibility in the incidence algebra! But why would we want to invert anyway? Here's why.

Suppose $f, g : P \to \mathbb{C}$ are functions related by

$$f(\pi) = \sum_{\sigma \leq \pi} g(\sigma).$$

(This kind of relationship comes up a lot in combinatorics; as we will soon see, we have already encountered it computing moments in the free CLT.) Knowing f, can we recover g?

Def: The Zeta function ζ_P of a lattice P is the "constant 1" function in $\mathbb{C}P^{(2)}$:

$$\zeta_P(\pi, \tau) = 1 \quad \forall \pi \leq \tau.$$

Note, then, that $\sum_{\sigma \leq \pi} g(\sigma) = \sum_{\sigma \leq \pi} g(\sigma) \zeta_P(\sigma, \pi) = g * \zeta_P(\pi)$.

So the above relationship is simply $f = g * \zeta_P$. Since $\zeta_P \neq 0$ anywhere (including the 'diagonal'), it does have an inverse in $\mathbb{C}P^{(2)}$, and so

$$g = f * \zeta_P^{-1}; \text{ i.e. } g(\pi) = \sum_{\sigma \leq \pi} f(\sigma) \zeta_P^{-1}(\sigma, \pi).$$

Def: The Möbius function of a poset, $\mu_P \in \mathbb{C}P^{(2)}$, is the inverse of ζ_P in the incidence algebra. I.e. it is the function s.t.

$$f(\pi) = \sum_{\sigma \leq \pi} g(\sigma) \iff g(\pi) = \sum_{\sigma \leq \pi} f(\sigma) \mu_P(\sigma, \pi).$$

This process of reversing a summation over a poset is sometimes called Möbius inversion. It is a common unifying framework in modern combinatorics.

Computing μ_P. By definition, $\zeta_P * \mu_P = \delta_P = \mu_P * \zeta_P$. Evaluating these identities @ any $\pi \le \sigma$ gives the two systems

$$(\Sigma_1) \qquad \zeta_P * \mu_P = \delta_P \Leftrightarrow \sum_{\pi \le \tau \le \sigma} \mu_P(\tau, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{if } \pi < \sigma \end{cases} \qquad (\forall \pi \le \sigma)$$

$$(\Sigma_2) \qquad \mu_P * \zeta_P = \delta_P \Leftrightarrow \sum_{\pi \le \tau \le \sigma} \mu_P(\pi, \tau) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{if } \pi < \sigma \end{cases}.$$

Either of these systems completely determines the Möbius function. Some easy special cases are:

* $\mu_P(\pi, \pi) = 1$ (the sum over the trivial interval)
* if σ covers π, then $0 = \sum_{\tau \in [\pi, \sigma]} \mu_P(\pi, \tau) = \mu_P(\pi, \pi) + \mu_P(\pi, \sigma)$

  and so $\mu_P(\pi, \sigma) = -1$.

Other values can be built up inductively.

Eg. Let $P = NC(3)$. Since $|||$ is covered by $|\sqcup$, $\sqcup$, and $\sqcup|$, and each of these is covered by $\sqcup\!\sqcup$, we already know $\mu_{NC(3)}$ except on $(|||, \sqcup\!\sqcup)$. To calculate this final value, use (Σ_1):

$$0 = \sum_{||| \le \tau \le \sqcup\!\sqcup} \mu_{NC(3)}(|||, \tau) = \mu(|||, |||) + \mu(|||, |\sqcup) + \mu(|||, \sqcup) + \mu(|||, \sqcup|) + \mu(|||, \sqcup\!\sqcup)$$

$$= 1 + (-1) + (-1) + (-1) + \mu(|||, \sqcup\!\sqcup).$$

Thus $\mu_{NC(3)}(|||, \sqcup\!\sqcup) = 2$.

We could continue this way inductively computing $\mu_{NC(n)}$; the following theorem makes this much easier.

Theorem: Let P be a poset.

(a) If $\pi \le \tau \in P$, then the poset $[\pi, \tau]$ satisfies $\mu_{[\pi, \tau]} = \mu_P|_{[\pi, \tau]}$.

(b) If $\Phi : P \to Q$ is an isomorphism, $\mu_Q(\Phi(\pi), \Phi(\sigma)) = \mu_P(\pi, \sigma)$.

(c) If P' is another poset, then $\mu_{P \times P'} = \mu_P \otimes \mu_{P'}$, i.e.

$$\mu_{P \times P'}((\pi, \pi'), (\sigma, \sigma')) = \mu_P(\pi, \sigma) \cdot \mu_{P'}(\pi', \sigma').$$

Pf. Exercise. [Just stare at Equations (Σ_1), (Σ_2).]

This dramatically reduces the work in computing $\mu_{NC(n)}$. Appealing to the factorization of intervals: if $\pi \leq \sigma \in NC(n)$, then $\exists\, k_1, \ldots, k_n \in \mathbb{N}$ s.t.

$$[\pi, \sigma] \cong NC(1)^{k_1} \times \cdots \times NC(n)^{k_n}.$$

Well, $[\pi,\sigma]$ is a sub-poset, and so $\mu_{NC(n)}(\pi,\sigma) = \mu_{[\pi,\sigma]}(\pi,\sigma)$. By the isomorphism, then, we have

$$\mu_{NC(n)}(\pi,\sigma) = \mu_{NC(1)}(0_1, 1_1)^{k_1} \cdots \mu_{NC(n)}(0_n, 1_n)^{k_n}$$

since the isomorphism carries $\pi \mapsto (0_1^{,k_1}, 0_2^{,k_2}, \ldots, 0_n^{,k_n})$
& $\sigma \mapsto (1_1^{,k_1}, 1_2^{,k_2}, \ldots, 1_n^{,k_n})$.

We have a(n easy, fast) algorithm to calculate $k_1, \ldots, k_n$. Hence, to calculate $\mu_{NC(n)}$, it suffices to calculate just the values

$$\mu_{NC(m)}(0_m, 1_m), \quad 1 \leq m \leq n.$$

<u>Remark</u>: While we know k_1 is not uniquely-determined, this is no bother since $0_1 = 1_1$ so $\mu_{NC(1)} \equiv 1$.

To easily calculate these values, we return to the general setup. It turns out that, in the special case that P is a <u>lattice</u>, there is a handy improvement on Equations (Σ_1) and (Σ_2), known as <u>partial Möbius inversion</u>.