Lecture 16: May 6, 2011

Suppose L is a finite lattice. The following is a special form of Equations $(\Sigma_1), (\Sigma_2)$ that holds in this setting.

Prop. (Partial Möbius Inversion) Let $f, g \in \mathbb{C}L$ satisfy

$$f(\tau) = \sum_{\pi \leq \tau} g(\pi).$$

Then for any $(\omega, \tau) \in P^{(2)}$,

$$\sum_{\sigma \in [\omega, \tau]} f(\sigma)\mu_L(\sigma, \tau) = \sum_{\substack{\pi \in L \\ \pi \vee \omega = \tau}} g(\pi).$$

Remark: taking $\omega = 0_L$ gives the usual equation $\sum_{\sigma \leq \tau} f(\sigma)\mu_L(\sigma,\tau) = g(\tau)$, since $\pi \vee 0_L = \pi$ and $\therefore$ $\pi = \tau$ is the only term in the sum. On the other hand, taking $\omega = \tau$ makes the LHS $= f(\tau)\mu_L(\tau,\tau) = f(\tau)$, and the RHS $= \sum_{\pi \leq \tau} g(\pi)$ since $\pi \vee \tau = \tau$ iff $\pi \leq \tau$. Thus, the result is a family of intermediate equations.

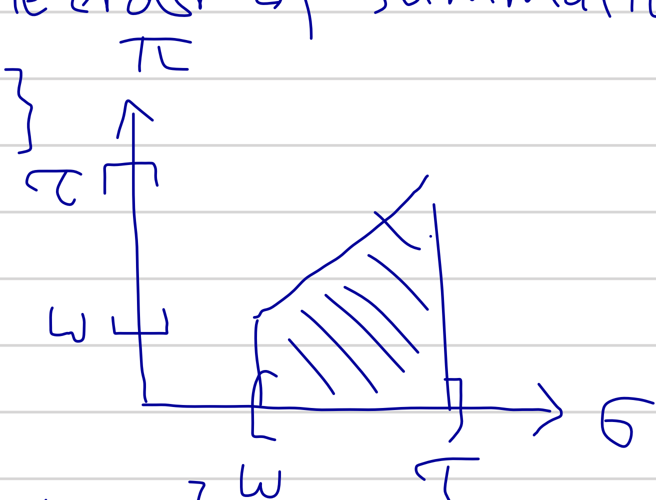
Pf. Begin by expanding

$$\sum_{\sigma \in [\omega,\tau]} f(\sigma)\mu_L(\sigma,\tau) = \sum_{\sigma \in [\omega,\tau]} \sum_{\pi \leq \sigma} g(\pi)\mu_L(\sigma,\tau).$$

We now switch the order of summation:

$\{(\sigma, \pi) : \omega \leq \sigma \leq \tau \ \& \ \pi \leq \sigma\}$

$\pi \leq \tau$ & $\sigma \leq \tau$
& $\sigma \geq \omega$
& $\sigma \geq \pi$



$= \{(\sigma, \pi) : \pi \leq \tau \ \& \ \pi \vee \omega \leq \sigma \leq \tau\}$.

$$\therefore \text{sum} = \sum_{\pi \leq \tau} \sum_{\pi \vee \omega \leq \sigma \leq \tau} \mu_L(\sigma,\tau) g(\pi).$$

Again, the sum is $\sum_{\pi \leq \tau} \sum_{\pi \vee \omega \leq \sigma \leq \tau} \mu_L(\sigma,\tau) g(\pi)$.

Now fix π in the outer sum. For the internal sum, there are two cases: $\pi \vee \omega = \tau$ or $\pi \vee \omega < \tau$. In the former case, the only term is $\sigma = \pi \vee \omega = \tau$, which gives contribution

$$\mu_L(\tau,\tau) g(\pi) = g(\pi).$$

In the second case, we have

$$\sum_{\pi \vee \omega \leq \sigma \leq \tau} \mu_L(\sigma,\tau) \cdot g(\pi) = 0 \cdot g(\pi) \quad \text{(by Eq. } (\Sigma_2), \text{ since } \pi \vee \omega \neq \tau)$$

Hence, the full sum over π reduces to

$$\sum_{\substack{\pi \leq \tau \\ \pi \vee \omega = \tau}} g(\pi).$$ But $\pi \vee \omega = \tau \Rightarrow \pi \leq \tau$, so this proves the claim. ///

An immediate corollary is a direct improvement of Equations Σ_1, Σ_2.

<u>Corollary</u>: If L is a finite lattice, and $\omega \neq 0_L$, then

$$\sum_{\pi \vee \omega = 1_L} \mu_L(0_L, \pi) = 0.$$

Pf. Set $g \in \mathbb{C}L$ equal to $g(\pi) = \mu_L(0_L, \pi)$. Then letting f be the partial sum of g,

$$f(\sigma) = \sum_{\pi \leq \sigma} g(\pi) = \sum_{0_L \leq \pi \leq \sigma} \mu_L(0_L,\pi) \zeta_L(\pi,\sigma) = \mu_L * \zeta_L(0_L,\sigma)$$

which, by definition of μ_L, is equal to 0 unless $\sigma = 0_L$ (in which case $f(0_L) = 1$).

Now apply the preceeding proposition:

$$\sum_{\pi \vee \omega = 1_L} g(\pi) = \sum_{\omega \leq \sigma \leq 1_L} f(\sigma) \mu_L(\sigma, 1_L) = 0 \quad \text{b/c } \omega > 0_L.$$ ///

Partial Möbius inversion will be an important tool for us. For its first application, we compute $\mu_{NC(n)}$.

Recall that, by the nice properties of μ_L (respecting restriction, isomorphism, and product), we have that if $[\pi,\tau] \cong NC(1)^{k_1} \times \cdots \times NC(n)^{k_n}$ then

$$\mu_{NC(n)}(\pi,\tau) = s_1^{k_1} \cdots s_n^{k_n}$$

where $s_n = \mu_{NC(n)}(0_n, 1_n)$.

<u>Prop</u>: For $n \geq 1$, $s_n = (-1)^{n-1} C_{n-1}$.

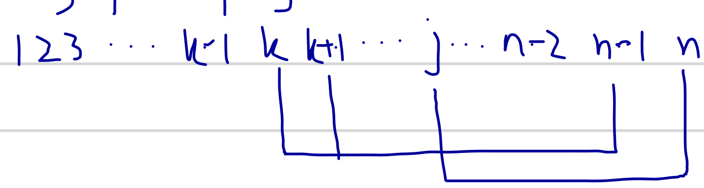
Pf. We have already calculated that $\mu_{NC(1)}(0_1,1_1) = \mu_{NC(1)}(1,1) = 1$, $\mu_{NC(2)}(11,\sqcup) = -1$ (b/c $\sqcup$ covers 11), and inductively that $\mu_{NC(3)}(111,\sqcup\!\sqcup) = 2$; these all match the above formula. So take $n \geq 4$. We use the above Corollary to partial Möbius inversion in the lattice $NC(n)$:

$$\sum_{\pi \wedge \omega = 1_n} \mu_{NC(n)}(0_n, \pi) = 0.$$

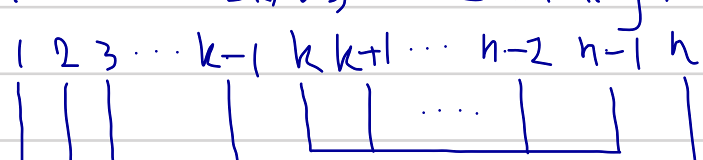
Here take $\omega = 111\cdots1\sqcup = \{\{1\},\{2\},\ldots\{n-2\},\{n-1,n\}\}$.
So we need to compute $\{\pi : \pi \wedge \omega = 1_n\}$.

<u>Claim</u>: if π has a block contained in $[n-2]$ then $\pi \wedge \omega \neq 1_n$.
<u>Pf</u>. If π has such a block B, it is easy to check that $\{B, [n]-B\}$ is $\geq \pi$ and $\geq \omega$, so $\pi \wedge \omega < 1_n$.

So, which non-crossing partitions have no block completely contained in $[n-2]$? Well, this is the set of $NC(n)$ partitions in which every block contains either $n-1$ or n. If these two are in the <u>same</u> block, that means $\pi = 1_n$. Otherwise, π must have two blocks. Let k be smallest s.t. $k \sim_\pi n-1$; then if $k < j \leq n-2$, we must have $j \sim_\pi n-1$, for if $j \sim_\pi n$ there would be a crossing:



Thus, the only $\pi \in NC(n)$ with two blocks, one containing $n-1$ and the other containing n, are

$(1 \leq k \leq n-1)$ $\pi_k =$ 

It is easy to check that, indeed, $\pi_k \wedge \omega = 1_n$ for all k. So

$$\{\pi \in NC(n) : \pi \wedge \omega = 1_n\} = \{1_n, \pi_1, \ldots, \pi_{n-1}\}.$$

Thus $\underbrace{\mu_{NC(n)}(0_n, 1_n)}_{s_n} + \sum_{k=1}^{n-1} \mu_{NC(n)}(0_n, \pi_k) = 0$.

look @ the interval decomposition of $[0_n, \pi_k]$: π_k has two blocks, one of size k, one of size $n-k$. We don't need the Kreweras complement here since the bottom is 0_n:

$$[0_n, \pi_k] \cong NC(k) \times NC(n-k).$$

$$\therefore \mu_{NC(n)}(0_n, \pi_k) = s_k \cdot s_{n-k}.$$

$$\boxed{\therefore s_n + \sum_{k=1}^{n-1} s_k s_{n-k} = 0, \quad n \geq 4.}$$

Now let $c_n = (-1)^n s_{n+1}$. Then $s_k s_{n-k} = (-1)^{k-1} c_{k-1} (-1)^{n-k-1} c_{n-k-1}$

$$= (-1)^n c_{k-1} c_{n-k-1} \quad \therefore$$

while $s_n = (-1)^{n-1} c_{n-1}$

$$\therefore c_{n-1} - \sum_{k=1}^{n-1} c_{k-1} c_{n-k-1} = 0, \quad n \geq 4.$$

This is the Catalan recurrence: replacing $n-1$ with n

$$c_n = \sum_{k=1}^{n} c_{k-1} c_{n-k}, \quad n \geq 3.$$

Note, for $n = 0, 1, 2$, we already showed that $c_n = (-1)^n s_{n+1} = C_n$, and so $c_n = C_n$. Thus $s_n = (-1)^{n-1} c_{n-1} = (-1)^n C_{n-1}$ as claimed. ///

Actually, the most important feature of $\mu_{NC(n)}$ is that it respects the multiplicative structure in $NC(n)$ — or, more precisely, the family of functions

$$\mu_n = \mu_{NC(n)}$$

respect the multiplicative structure of the lattices $NC(n)$.

Multiplicative Functions on $NC(n)$

<u>Def</u>: A family of functions $\{f_n : NC(n) \to \mathbb{C}\}$ is called <u>multiplicative</u> if, given any $n \geq 1$ and any $\pi \in NC(n)$, if $\pi = \{B_1, \dots, B_r\}$ then

$$f_n(\pi) = f_{|B_1|}(1_{|B_1|}) \cdots f_{|B_r|}(1_{|B_r|}).$$

<u>Note</u>: a multiplicative family is completely determined by the sequence of complex numbers $\alpha_n = f_n(1_n)$.

E.g. Let $f_n(\pi) = \mu_n(0_n, \pi)$. The decomposition of the interval $[0_n, \pi]$ over the blocks $\{B_1, \dots, B_r\}$ of π is

$$[0_n, \pi] \cong [0_{|B_1|}, 1_{|B_1|}] \times \cdots \times [0_{|B_r|}, 1_{|B_r|}]$$

as we saw in calculating μ_n, the properties of the Möbius function then yield that

$$f_n(\pi) = \mu_n(0_n, \pi) = \mu_{|B_1|}(0_{|B_1|}, 1_{|B_1|}) \cdots \mu_{|B_r|}(0_{|B_r|}, 1_{|B_r|})$$
$$= f_{|B_1|}(1_{|B_1|}) \cdots f_{|B_r|}(1_{|B_r|}).$$

So this is a multiplicative family.

The Möbius function is actually "more multiplicative" than this. Without formally defining a multiplicative family of functions on $NC(n)^{(2)}$, the important point is the following.

<u>Thm</u>: Suppose g_n are functions $NC(n) \to \mathbb{C}$, and define $f_n : NC(n) \to \mathbb{C}$ by

$$f_n(\pi) = \sum_{\sigma \leq \pi} g_n(\sigma);$$

i.e. $f_n = g_n * \zeta_n$, or equivalently $g_n = f_n * \mu_n$. (Here we use the short-hand $\zeta_n = \zeta_{NC(n)}$.)

Then $(f_n)_{n \geq 1}$ is multiplicative iff $(g_n)_{n \geq 1}$ is multiplicative.

The proof only uses the fact that the functions $F_n = \zeta_n, \mu_n$ have the following multiplicative properties:

(M) If $\tau \le \pi \in NC(n)$, where $\pi = \{B_1, \dots, B_r\}$, and we use the bijection $\sigma \mapsto (\sigma|_{B_1}, \dots, \sigma|_{B_r})$ to construct the isomorphism
$$[\tau, \pi] \cong [\tau_1, 1_{|B_1|}] \times \cdots \times [\tau_r, 1_{|B_r|}]$$
(where τ_k is the image of $\tau|_{B_k}$) then
$$F_n(\tau, \pi) = F_{|B_1|}(\pi_1, 1_{|B_1|}) \cdots F_{|B_r|}(\pi_r, 1_{|B_r|}).$$

This property of μ_n is what we used to calculate it; that it holds for ζ_n is a triviality (since $\zeta_n \equiv 1$ on $NC(n)^{(2)}$).

So what we're really proving is:

Let $F_n : NC(n)^{(2)} \to \mathbb{C}$ have property (M). If $f_n : NC(n) \to \mathbb{C}$ form a multiplicative family and $g_n = f_n * F_n$, then g_n form a multiplicative family.

Pf. Let $\beta_n = g_n(1_n) = f_n * F_n(1_n) = \sum_{\tau \le 1_n} f_n(\tau) F_n(\tau, 1_n)$

i.e. $\tau \in NC(n)$.

Now, fix $\pi \in NC(n)$, $\pi = \{B_1, \dots, B_r\}$. Then, as in def$^{\underline{n}}$ (M), for any $\tau \le \pi$ we have
$$F_n(\tau, \pi) = F_{|B_1|}(\tau_1, 1_{|B_1|}) \cdots F_{|B_r|}(\tau_r, 1_{|B_r|})$$
Similarly, since f_n is multiplicative, for any $\tau \le \pi$
$$f_n(\tau) = f_{|B_1|}(\tau_1) \cdots f_{|B_r|}(\tau_r)$$

each one of these factors into blocks of τ since $\tau \le \pi$; all the factors are disjoint, giving the usual decomposition of $f_n(\tau)$.

So
$$g_n(\pi) = \sum_{\tau \le \pi} f_n(\tau) F_n(\tau, \pi)$$
$$= \sum_{\tau_1, \dots, \tau_r} \left(f_{|B_1|}(\tau_1) \cdots f_{|B_r|}(\tau_r) \right) F_{|B_1|}(\tau_1, 1_{|B_1|}) \cdots F_{|B_r|}(\tau_r, 1_{|B_r|}).$$

The sum is over all $\tau_1 \in NC(|B_1|), \ldots, \tau_r \in NC(|B_r|)$ independently; this is because the map $\tau \mapsto (\tau_1, \ldots, \tau_r)$ is an isomorphism $[0_n, \pi] \to NC(|B_1|) \times \cdots \times NC(|B_r|)$.

Finally, we see this product factors as

$$\prod_{k=1}^{r} \sum_{\tau_k \in NC(|B_k|)} f_{|B_k|}(\tau_k) \, F_{|B_k|}(\tau_k, 1_{|B_k|}) = \prod_{k=1}^{r} \beta_{|B_k|}$$

as desired. ///

Eg. We saw that $f_n = \mu_n(0_n, \cdot)$ is multiplicative. Induction on the theorem shows that convolution powers

$$f_n^{(k)}(\pi) = \underbrace{\mu_n * \cdots * \mu_n}_{k \text{ times}}(0_n, \pi)$$

form a multiplicative family for any $k \in \mathbb{Z}$ (where $\mu_n^{*(-k)} = \zeta_n^{*(k)}$).

Exercise: Let L be a lattice. Show that

$$\underbrace{(\zeta_L * \zeta_L * \cdots * \zeta_L)}_{k+1 \text{ times}}(0_L, 1_L) = \#\text{multichains in } L \text{ of length } k-1.$$