Lecture 17: May 9, 2011

Let $f_n : NC(n) \to \mathbb{C}$ be a multiplicative family. As we noted last time, this means the functions f_n are completely determined by the sequence $\alpha_n = f_n(1_n)$ of complex numbers.

Conversely, if $(\alpha_n)_{n=1}^{\infty}$ is any sequence in $\mathbb{C}$, it defines a (unique) multiplicative family: if $\pi = \{B_1, \ldots, B_r\}$ is in $NC(n)$, then $f_n(\pi) = \alpha_{|B_1|} \cdots \alpha_{|B_r|}$.

Then $f_n(1_n) = \alpha_{|1_n|} = \alpha_n$, as claimed. Thus, for any sequence (α_n), we may talk about α_π for $\pi \in \bigsqcup_n NC(n)$, where $\alpha_\pi = f_n(\pi)$ if $\pi \in NC(n)$.

The example relevant to us is as follows:

Let $(\mathcal{A}, \varphi)$ be a NCPS, and let $a \in \mathcal{A}$. Let $\varphi_n^a = \varphi(a^n)$. This is a sequence in $\mathbb{C}$. It extends uniquely to a multiplicative family $\{\varphi_\pi^a : \pi \in \bigsqcup_n NC(n)\}$.

E.g. $\varphi^a_{\sqcup\!\sqcup\,|\,\sqcup} = \varphi(a^3)\varphi(a)\varphi(a^2)$.

<u>Def</u>: The <u>free cumulants</u> of $a \in (\mathcal{A}, \varphi)$ are the complex numbers κ_n^a, given by

$$\kappa_n^a = \sum_{\pi \in NC(n)} \varphi_\pi^a \, \mu_n(\pi, 1_n).$$

So, κ_n is a polynomial in the moments (up to degree n) of a.

E.g. $\kappa_1^a = \sum_{\pi \in NC(1)} \varphi_\pi^a \mu_1(\pi, 1_1) = \varphi_1^a \mu_n(1,1) = \varphi(a) \cdot 1 = \varphi(a)$.

E.g. $\kappa_2^a = \varphi^a_{||} \mu_2(||, \sqcup) + \varphi^a_{\sqcup} \mu_2(\sqcup, \sqcup) = \varphi(a)^2(-1) + \varphi(a^2)(1)$
$= \varphi(a^2) - \varphi(a)^2 = \mathrm{Var}(a)$.

E.g. $\kappa_3^a = \varphi^a_{|||} \cdot 2 - \varphi^a_{|\sqcup} - \varphi^a_{\sqcup|} - \varphi^a_{\sqcup\,|} + \varphi^a_{\sqcup\!\sqcup} = \varphi(a^3) - 3\varphi(a^2)\varphi(a) + 2\varphi(a)^3$

(Statisticians have a name for this, too — it's the <u>skewness</u> of a.)

So, what's the point of these κ_n? Well, suppose we extend them to functions on $\bigsqcup_n NC(n)$. There are two ostensibly distinct ways to do this:

(1) Make the multiplicative family $\kappa^a_\pi = \kappa^a_{|B_1|} \cdots \kappa^a_{|B_r|}$ for $\pi = \{B_1, \ldots, B_r\}$.

(2) Take <u>partial sums</u> in the definition:

$$\kappa^a(\pi) = \sum_{\sigma \leq \pi} \varphi^a_\sigma \cdot \mu_n(\sigma, \pi) \quad \text{for } \pi \in NC(n)$$

$$\text{i.e. } \kappa^a = \varphi^a * \mu_n \quad \text{on } NC(n).$$

But as we showed last time, the definition (2) IS multiplicative (since φ^a is). So, both $\pi \mapsto \kappa^a_\pi$ and κ^a are multiplicative, so are determined by their values $\kappa^a(1_n)$, $\kappa^a_{1_n}$ — and these are equal. Thus

$$\kappa^a_\pi = \kappa^a(\pi) \quad \forall \pi.$$

Now, looking at definition (2) of this object, we have $\kappa^a = \varphi^a * \mu_n$. So, by definition, $\varphi^a = \kappa^a * \zeta_n$. I.e.

$$\varphi^a_\pi = \sum_{\sigma \leq \pi} \kappa^a_\sigma \zeta_n(\sigma, \pi) = \sum_{\sigma \leq \pi} \kappa^a_\sigma .$$

Specializing to the case $\pi = 1_n$, we have:

$$\boxed{\varphi(a^n) = \sum_{\pi \in NC(n)} \kappa^a_\pi .}$$

The moment–cumulant formula.

Sometimes one can pick off the free cumulants of a by reading this "reverse" formula directly.

E.g. Let s be semicircular. Then, as we have calculated, $\varphi(s^n) = C_{n/2}$ if n is even, and $= 0$ if n is odd. I.e.

$$\varphi(s^n) = \sum_{\pi \in NC(n)} \mathbb{1}_{\{\pi \in NC_2(n)\}} \stackrel{\vdots}{=} \sum_{\pi \in NC(n)} \kappa^s_\pi .$$

One obvious solution is $\kappa^s_\pi = \mathbb{1}\{\pi \in NC_2(n)\}$. Since we know (from the machinery of multiplicative families) that there is only one solution, we have found the free cumulants. In particular,

$$\kappa^s_n \equiv \kappa^s_{1_n} = \mathbb{1}_{\{1_n \in NC_2(n)\}} = \delta_{n2}.$$

So, while the moments $\varphi(s^n)$ are quite complicated, the free cumulants are exceedingly simple: only $\kappa^s_2 = 1$ is non-zero. (As we will see, this is why s is the "central distribution" – it has the simplest possible non-trivial free cumulants.)

Eg. Can it happen that $\kappa_n = c\cdot\delta_{n1}$ for some $c \in \mathbb{C}$? In this case $\kappa_\pi = \kappa_{|B_1|}\cdots\kappa_{|B_r|} = 0$ unless $|B_j| = 1\ \forall j$, meaning only $\kappa_{0_n} \neq 0$ in $NC(n)$, and $\kappa_{0_n} = c^n$. Thus, if $\exists a$ with these free cumulants, then

$$\varphi(a^n) = \sum_{\pi \in NC(n)} \kappa_\pi = \kappa_{0_n} = c^n, \quad n \geq 1.$$

The unique distribution with these moments is $\mu_a = \delta_c$, a point mass at $c \in \mathbb{C}$.

Eg. Suppose $\mu_a = \frac{1}{2}(\delta_1 + \delta_{-1})$ (symmetric Bernoulli law). The moments are

$$\varphi(a^n) = \tfrac{1}{2}(1 + (-1)^n) = (1,0,1,0,1,0,\ldots)$$

It is not obvious how to recognize the corresponding cumulants. We can compute $\kappa_1 = \varphi(a) = 0$, $\kappa_2 = \mathrm{Var}\, a = 1$, $\kappa_3 = \varphi(a^3) - 3\varphi(a^2)\varphi(a) + 2\varphi(a)^3 = 0, \ldots$ But to compute higher ones, we must compute from the definition...

There's a better way:

generating functions.

Let f_n, g_n be multiplicative families on $NC(n)$, related by $f_n = g_n * \zeta_n$ (or equivalently $g_n = f_n * \mu_n$).

It is easy to check that if $(f_n(1_n))_{n=1}^{\infty}$ grows at most exponentially, the same is true of $(g_n(1_n))_{n=1}^{\infty}$ (b/c μ_n, ζ_n are $\leq$ exponential, and $\# NC(n) \leq$ exponential).

Let $\alpha_n = f_n(1_n)$, $\beta_n = g_n(1_n)$. Then the power-series

$$u(z) = 1 + \sum_{n=1}^{\infty} \alpha_n z^n, \quad v(z) = 1 + \sum_{n=1}^{\infty} \beta_n z^n$$

either both converge only @ $z=0$, or both have non-0 radii of convergence. We assume the latter.

<u>Theorem</u>: The functions u, v are related by

$$v(z u(z)) = u(z), \quad u\left(\frac{z}{v(z)}\right) = v(z)$$

for all sufficiently small z.

<u>Pf</u>. If the first functional equation holds, then set $w = z u(z)$. Note that, for small enough z, $u(z) \neq 0$, so

$$z = w/u(z) = w/v(z \cdot u(z)) = w/v(w).$$

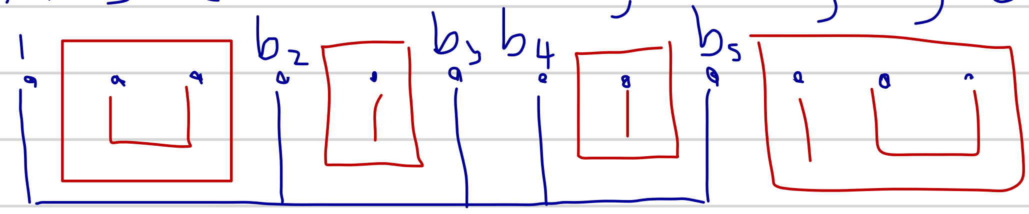
Taking u of both sides yields $u\left(\frac{w}{v(w)}\right) = u(z) = v(w)$ as claimed. So it suffices to prove the first equality.

By definition, $\alpha_n = f_n(1_n) = \sum_{\pi \leq 1_n} g_n(\pi) = \sum_{\pi \in NC(n)} g_n(\pi)$.

In this sum, write $\pi = \{B_1, B_2, \ldots, B_{|\pi|}\}$ where B_1 is the block containing 1. Then we can rewrite the sum as

$$\alpha_n = \sum_{s=1}^{n} \sum_{\substack{1 \in B \subseteq [n] \\ \# B = s}} \sum_{\substack{\pi \in NC(n) \\ B_1(\pi) = B}} g_n(\pi).$$

Now, fix $B\subseteq[n]$ with $1\in B$, $\#B=s$, say $B=\{1=b_1<b_2<\cdots<b_s\}$.



Then for any $\pi\in NC(n)$ with $B_1(\pi)=B$, we must have

$$\pi = B_1\sqcup\pi_1\sqcup\pi_2\sqcup\cdots\sqcup\pi_s$$

where $\pi_j\in NC(\{b_j+1,\dots,b_{j+1}-1\})\cong NC(\overbrace{b_{j+1}-b_j-1}^{=i_j})$.
The multiplicativity of g_n means that

$$g_n(\pi)=\beta_s\cdot g_{i_1}(\pi_1)\cdots g_{i_s}(\pi_s).$$

Now, we can specify B by specifying the gap lengths i_j, so

$$\alpha_n=\sum_{s=1}^{n}\sum_{\substack{i_1,\dots,i_s\in\{0,\dots,n-s\}\\ i_1+\cdots+i_s=n-s}}\ \sum_{\substack{\pi=B\sqcup\pi_1\sqcup\cdots\sqcup\pi_s\\ \pi_j\in NC(i_j)}}\beta_s\, g_{i_1}(\pi_1)\cdots g_{i_s}(\pi_s).$$

$B=\{1,i_1+2,i_1+i_2+3,\dots,i_1+\cdots+i_s+s\}$

The internal summation factors as

$$\beta_s\sum_{\pi_1\in NC(i_1)}g_1(\pi_{i_1})\cdots\sum_{\pi_s\in NC(i_s)}g_s(\pi_{i_s})=\beta_s\alpha_{i_1}\cdots\alpha_{i_s}.$$

Thus

$$u(z)=1+\sum_{n=1}^{\infty}\alpha_n z^n=1+\sum_{n=1}^{\infty}\sum_{s=1}^{n}\sum_{\substack{i_1,\dots,i_s\geq 0\\ i_1+\cdots+i_s=n-s}}\beta_s\alpha_{i_1}\cdots\alpha_{i_s}z^n$$

$$=1+\sum_{n=1}^{\infty}\sum_{s=1}^{n}\sum_{\substack{i_1,\dots,i_s\geq 0\\ i_1+\cdots+i_s=n-s}}(\beta_s z^s)(\alpha_{i_1}z^{i_1})\cdots(\alpha_{i_s}z^{i_s})$$

$$=1+\sum_{s=1}^{\infty}\beta_s z^s\Big(\sum_{i=0}^{\infty}\alpha_i z^i\Big)^s=V(z\,u(z)).$$

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The $\mathcal{R}$-transform

Remember the Stieltjes transform of a distribution

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\,\mu(dt), \quad z \in \mathbb{C}_+$$

$$= \sum_{n \geq 0} \int_{\mathbb{R}} t^n \mu(dt)\; z^{-n-1} \quad \text{for suff. large } z.$$

Thus, if $a \in (\mathcal{A}, \varphi)$ has associated multiplicative moment family φ_π^a, if $u(z) = 1 + \sum_{n\geq 1} \varphi_{1_n}^a z^n$ (for small z) then

$$z \cdot u(z) = G_{\mu_a}\left(\tfrac{1}{z}\right) \quad \text{i.e.} \quad u(z) = \tfrac{1}{z} G_a\left(\tfrac{1}{z}\right).$$

Def: the R-transform of a (or μ_a), $\mathcal{R}_a = \mathcal{R}_{\mu_a}$, is

$$\mathcal{R}_a(z) = \sum_{n=0}^{\infty} \kappa_{n+1}^a z^n.$$

So, E.g., if s is semicircular, $\mathcal{R}_s(z) = z$. (This is the reason for the shift; otherwise $\mathcal{R}_s$ would be z^2.)

If κ_π^a is the multiplicative family of free cumulants of a, and $v(z) = 1 + \sum_{n \geq 1} \kappa_{1_n}^a z^n$, then we have

$$\mathcal{R}_a(z) = \tfrac{1}{z}(v(z) - 1) \quad \text{i.e.} \quad v(z) = z\mathcal{R}_a(z) + 1.$$

Now, $\varphi^a = \kappa^a * \zeta$, so by the theorem, $u\left(\frac{z}{v(z)}\right) = v(z)$.
In other words,

$$\frac{v(z)}{z} G_a\left(\frac{v(z)}{z}\right) = u\left(\frac{z}{v(z)}\right) = v(z)$$

for suff. small z. Since $v(z) \neq 0$ for small z, we can cancel:

$$z = G_a\left(\frac{v(z)}{z}\right) = G_a\left(\frac{z\mathcal{R}_a(z)+1}{z}\right).$$

I.e. Theorem: For sufficiently small z,

$$\boxed{G_a\left(\mathcal{R}_a(z) + \tfrac{1}{z}\right) = z.}$$

Eg. If $\mu_a = \frac{1}{2}(\delta_1 + \delta_{-1})$, then

$$G_a(z) = \int \frac{1}{z-t}\,\mu_a(dt) = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right) = \frac{z}{z^2-1}.$$

So, letting $w = \mathcal{R}_a(z)$,

$$z = G_a\left(w + \tfrac{1}{z}\right) = \frac{w+\frac{1}{z}}{(w+\frac{1}{z})^2 - 1} = \frac{z^2 w + z}{(zw+1)^2 - z^2}$$

i.e. $(zw+1)^2 - z^2 = zw+1$, so $zw+1 = \frac{1}{2}(1 \pm \sqrt{1+4z^2})$.

i.e. $\mathcal{R}_a(z) = \dfrac{-1 \pm \sqrt{1+4z^2}}{2z}$. must be + b/c $\mathcal{R}_a$ is analytic @ 0.

Now we can use the binomial theorem to calculate the free cumulants; after all, $z\mathcal{R}_a(z) = \sum_{n\geq 1} \kappa_n^a z^n$.

$$\sqrt{1+4z^2} = \sum_{k=0}^{\infty} \frac{1}{k!}\,\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-k+1\right) z^{2k}$$

from which we can easily check that $\kappa_n^a = 0$ if n is odd, while for $n = 2m$,

$$\kappa_{2m} = (-1)^{m-1} C_{m-1}.$$

<u>Remark</u>: We will generally not calculate cumulants explicitly, but rather work with the $\mathcal{R}$-transform directly.

So, now we have tools to compute free cumulants. The question remains: why would we want to? We will begin to answer that next day. In fact, cumulants and the $\mathcal{R}$-transform have <u>everything</u> to do with freeness!

<u>Exercise</u>: Recall that, in a lattice L, $\zeta_L^{*(k+1)}(0_L, 1_L) = \# L^{(k)}$ (multichains of length $k-1$).

Let $C_n^{(k)} = \# NC(n)^{(k)}$ and set $u_k(z) = 1 + \sum_{n=1}^{\infty} C_n^{(k)} z^n$.

Use the functional eq'n's for multiplicative systems to show

$$u_k(z\,u_{k+1}(z)) = u_{k+1}(z).$$

By induction, conclude that $u_k(z) = 1 + z\,u_k(z)^{k+1}$, $k \geq 1$.

From here, conclude that the coefficients $C_n^{(k)}$ satisfy

$$C_n^{(k)} = \sum_{\substack{i_1, \dots, i_{k+1} \geq 0 \\ i_1 + \cdots + i_{k+1} = n-1}} C_{i_1}^{(k)} \cdots C_{i_{k+1}}^{(k)}$$

and so they are the Fuss–Catalan numbers.