Lecture 18: May 11, 2011

Classical Cumulants: Let X be a (classical) random variable, that is pretty close to bounded: for small enough $\varepsilon > 0$,

$$\mathbb{E}(e^{\varepsilon|X|}) < \infty .$$

It follows that all the moments are finite. (The Gaussian distribution has this property.)

The (exponential) moment-generating function of X is

$$M_X(z) = \sum_{n \geq 0} \frac{\mathbb{E}(X^n)}{n!} z^n = \mathbb{E}(e^{zX})$$

which exists (and is analytic) for suff. small z.

The (classical) cumulant-generating function of X is

$$C_X(z) = \ln M_X(z)$$

well-defined & analytic near 0, as $M_X(0) = 1$.

E.g. If $\mu_X = N(0,1)$, it is a standard calculation that $M_X(z) = e^{z^2/2}$. Thus $C_X(z) = z^2/2$.

Exercise: Let $M(z) = 1 + \sum_{n \geq 1} \frac{m_n}{n!} z^n$. Calculate (or, better yet, ask Maple to calculate)

$$\ln M(z) = \underbrace{m_1}_{\kappa_1^X} z + \frac{1}{2}\underbrace{(m_2 - m_1^2)}_{\kappa_2^X} z^2 + \frac{1}{6}\underbrace{(m_3 - 3m_2 m_1 + 2m_1^3)}_{\kappa_3^X} z^3 + o(z^3)$$

If $M = M_X$, then $m_n = \mathbb{E}(X^n)$. Thus $m_2 - m_1^2 = \operatorname{Var} X$. Indeed, But further calculation will show that the next term is *not* $\sim \kappa_4^X$. So what is it?

Def: Expand $\ln M_X(z) = \sum_{n \geq 1} \frac{c_n^X}{n!} z^n$. The coefficients c_n^X are the (classical) cumulants of X.

So the classical cumulants of $N(0,1)$ are $c_n = \delta_{n2}$ – the same as the free cumulants of the semicircle law.

The relationship between free and classical probability is captured by the following exercise.

<u>(Involved) Exercise</u>. Let $(c_n^X)_{n=1}^{\infty}$ be the classical cumulants of a random variable X (which possesses an exponential moment). Form the multiplicative family over all of $\mathcal{P}(n)$ (all partitions):

$$\forall \pi \in \mathcal{P}(n),\ \pi = \{B_1, \ldots, B_r\} \quad c_\pi^X \equiv c_{|B_1|}^X \cdots c_{|B_r|}^X .$$

Show that $\mathbb{E}(X^n) = \sum_{\pi \in \mathcal{P}(n)} c_\pi^X$.

By Möbius inversion on the lattice $\mathcal{P}(n)$, it follows that c_n is given as a convolution of a moment function over the <u>full</u> lattice $\mathcal{P}(n)$. Since $NC(n) = \mathcal{P}(n)$ for $n \leq 3$, it follows that $c_n = \kappa_n$ for $n \leq 3$ — but they differ thereafter. (So free and classical probability agree "up to the third order".)

Now, classical cumulants hint at why we might be interested in the free analogue.

It is well known that if classical random variables $X_1, \ldots, X_k$ are independent, then

$$M_{X_1+\cdots+X_k}(z) = \mathbb{E}(e^{z(X_1+\cdots+X_k)}) = \mathbb{E}(e^{zX_1} \cdots e^{zX_k}) = \mathbb{E}(e^{zX_1}) \cdots \mathbb{E}(e^{zX_k})$$
$$= M_{X_1}(z) \cdots M_{X_k}(z).$$

taking logarithms, this means

$$\{X_1, \ldots, X_k\} \text{ independent} \Rightarrow C_{X_1+\cdots+X_k} = C_{X_1} + \cdots + C_{X_k}.$$

Put another way: if $\mu_1, \ldots, \mu_k$ are the distributions of $X_1, \ldots, X_k$, then $\mu_{X_1+\cdots+X_k} = \mu_1 * \cdots * \mu_k$; the above statement is that

$$C_{\mu_1 * \cdots * \mu_k} = C_{\mu_1} + \cdots + C_{\mu_k}.$$

This provides a way to compute convolutions — provided we can go back and forth between $\mu \longleftrightarrow C_\mu$. (We can; up to a log and some i's, this is just the Fourier transform.)

It will turn out that the $\mathcal{R}$-transform will play the same role for <u>free</u> convolution. To get there, we need a kind of analogue of the multi-variate moment-gen. function (and free cumulants).

Moment and Free Cumulant Functionals

Let $a_1, \ldots, a_n$ be elements of $\mathcal{A}$. Their joint moment

$(a_1, \ldots a_n) \longmapsto \varphi(a_1 a_2 \cdots a_n)$ is a multi-linear functional.

Denote it by $\varphi_n : \mathcal{A}^n \to \mathbb{C}$. So we have a family not of complex numbers but of multi-linear functionals.

Note:

$$\varphi_n(a, a, \ldots, a) = \varphi(a^n) = \varphi_n^a.$$

Now, we turned φ_n^a into a multiplicative family. We can do the same with the functionals φ_n.

<u>Def</u>: Let $B \subseteq [n]$, $B = \{i_1 < i_2 < \cdots < i_k\}$. Define function

$$\varphi_n(B) : \mathcal{A}^n \to \mathbb{C} \text{ by}$$

$$\varphi_n(B)(a_1, \ldots, a_n) = \varphi_k(a_{i_1}, \ldots, a_{i_k}) = \varphi(a_{i_1} \cdots a_{i_k}).$$

Then for $\pi \in NC(n)$, $\pi = \{B_1, \ldots, B_r\}$, define

$$\varphi_\pi : \mathcal{A}^n \to \mathbb{C} \quad : \varphi_\pi = \varphi_n(B_1) \cdots \varphi_n(B_r).$$

Eg. $\varphi_{\sqcup\!\sqcup\!\sqcup\,\sqcup\,|}(a_1, \ldots, a_7) = \varphi(a_1 a_3 a_4)\varphi(a_2)\varphi(a_5 a_6)\varphi(a_7)$.

<u>Remark</u>: The functions $\varphi_n(B)$ are not multi-linear; only in the variables in B. Since any partition π includes all the variables in disjoint blocks, φ_π <u>is</u> multi-linear.

<u>Remark</u>: $\varphi_\pi(a, a, \ldots, a) = \varphi(a^{|B_1|}) \cdots \varphi(a^{|B_r|}) = \varphi_\pi^a$.

So we have "polarized" the one-variable notion of a multiplicative family.

Def: The <u>free cumulant functionals</u> $\kappa_n: \mathfrak{A}^n \to \mathbb{C}$ are defined by

$$\kappa_n = \sum_{\sigma \in NC(n)} \mu_n(\sigma, 1_n) \varphi_\sigma .$$

Remark: By fixing a particular n-tuple in $\mathfrak{A}^n$, we can apply the preceeding machinery of multiplicative families to verify that

$$\kappa_\pi = \sum_{\sigma \le \pi} \mu_n(\sigma, \pi) \varphi_\sigma$$

is a multiplicative family, with $\kappa_{1_n} = \kappa_n$. In particular, this means that

$$\boxed{\varphi_n = \varphi_{1_n} = \sum_{\pi \in NC(n)} \kappa_\pi}$$

Again we have the moment-cumulant formula - this time on the level of functionals.

E.g. $n=1$: $\kappa_1 = \varphi$.

$n=2$: $\kappa_2(a,b) = \mu_2(11, \sqcup)\varphi_{11}(a,b) + \mu_2(\sqcup,\sqcup)\varphi_{\sqcup}(a,b)$

$= -\varphi(a)\varphi(b) + \varphi(ab) = \mathrm{Cov}(a,b).$

$n=3$: $\kappa_3(a_1,a_2,a_3) = \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) - \varphi(a_2)\varphi(a_1a_3)$

$- \varphi(a_3)\varphi(a_1a_2) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3)$

Naturally, $\kappa_n(a,a,\dots,a) = \kappa_n^a$ from our earlier notation.

So, the functionals κ_n are a different way to organize the information contained in the joint moments. But what good are they? Here is the first clue.

Proposition: Let $\{\kappa_n\}_{n=1}^\infty$ be the free cumulant functionals associated to a $\mathbb{C}$-algebra $\mathfrak{A}$ with state φ. Then $\varphi: \mathfrak{A} \to \mathbb{C}$ is a homomorphism iff $\kappa_n = 0$ $\forall n \ge 2$.

Pf. If $\kappa_2 = 0$, this means $\forall a,b \in \mathfrak{A}$ $0 = \kappa_2(a,b) = \varphi(ab) - \varphi(a)\varphi(b)$ So φ is a homomorphism. Conversely, if φ is a homom., b/c $\pi \in NC(n)$ includes each index in $[n]$ exactly once we have

$$\varphi_\pi(a_1, \dots, a_n) = \cdots \varphi(a_{i_1} \cdots a_{i_k}) \cdots = \varphi(a_1) \cdots \varphi(a_n).$$

That is, $\varphi_\pi = \varphi_{0_n}$ $\forall \pi \in NC(n)$. Thus for $n \geq 2$

$$K_n = \sum_{\pi \in NC(n)} \mu_n(\pi, 1_n)\,\varphi_\pi = \qquad_{0_n \leq \pi \leq 1_n} \; n(\pi, 1_n)\Big)\varphi_{0_n} = 0 \cdot \varphi_{0_n}$$

b/c $0_n \neq 1_n$ when $n \geq 2$ (here we use the recurrence relation for the Möbius function). ///

<u>Remark</u>: A corollary is that $K_2 = 0 \Rightarrow K_n = 0$ $\forall n \geq 0$. This is as <u>functionals</u>. On the level of random variables, recall (from lecture 3) that if $(\mathcal{A}, \varphi)$ is a NCPS and $a \in \mathcal{A}$ is self-adjoint, then

$$0 = K_2^a = K_2(a,a) = \varphi(a^2) - \varphi(a)^2 \Rightarrow \varphi((a - \varphi(a))^2) = 0$$
$$\Rightarrow a = \varphi(a) \text{ is constant.}$$

But then $\mu_a = \delta_a$, and (as we calculated last time) it follows that $K_n^a = 0$ $\forall n \geq 2$. A similar argument will show that, even if a is not self-adjoint, if $K_2(a, a^*) = 0$ then a is constant, meaning $K_n(a^{\varepsilon_1}, \ldots, a^{\varepsilon_n}) = 0$ for $\varepsilon_i \in \{1, *\}$. We will talk about $*$-cumulants soon.

Free cumulants of Products

Let $a_1, \ldots, a_n \in \mathcal{A}$. The multilinear functional $\varphi_n : \mathcal{A}^n \to \mathbb{C}$ is defined by the (associative) product in $\mathcal{A}$. So, e.g.

$$\varphi_2(a_1 a_2, a_3) = \varphi((a_1 a_2) a_3) = \varphi(a_1(a_2 a_3)) = \varphi_2(a_1, a_2 a_3)$$

and both are equal to $\varphi_3(a_1, a_2, a_3)$. In general, if $1 \leq i_1 < i_2 < \cdots < i_m = n$, then

$$\varphi_m(a_1 \cdots a_{i_1}, a_{i_1+1} \cdots a_{i_2}, \ldots, a_{i_{m-1}+1} \cdots a_{i_m})$$
$$= \varphi_n(a_1, a_2, \ldots, a_{n-1}, a_n).$$

Predictably, the interaction between φ_n and the product in $\mathcal{A}$ is very simple. The interaction with K_n, on the other hand, is not as easy.

E.g. $\kappa_2(a_1 a_2, a_3) = \varphi(a_1 a_2 a_3) - \varphi(a_1 a_2)\varphi(a_3)$

$\kappa_2(a_1, a_2 a_3) = \varphi(a_1 a_2 a_3) - \varphi(a_1)\varphi(a_2 a_3)$

Not equal, and in fact share no obvious relationship.

In fact, there is a general relationship between

$\kappa_m(a_1 \cdots a_{i_1}, a_{i_1+1} \cdots a_{i_2}, \ldots, a_{i_{m-1}+1} \cdots a_{i_m})$

and the cumulants $\{\kappa_\pi(a_1, \ldots, a_n) : \pi \in NC(n)\}$.

After all, this can be expanded as a sum of mixed moments, and then the nice properties of the φ_n wrt the product in $\mathcal{A}$ can be used, followed by the moment-cumulant formula, to get such an expansion. The wonderful thing is: the expansion actually achieved is simple and tractable. We'll explore it in the next lecture.