Lecture 19: May 13, 2011

Imbeddings $NC(n) \hookrightarrow NC(m)$.

Let $m \leq n$, and fix $1 \leq i_1 < i_2 < \cdots < i_m = n$. Corresponding to the ("inverse") map $\mathfrak{A}^n \longrightarrow \mathfrak{A}^m$

$$(a_1, \ldots, a_n) \longmapsto (a_1 \cdots a_{i_1}, a_{i_1+1} \cdots a_{i_2}, \ldots, a_{i_{m-1}+1} \cdots a_{i_m})$$

there is an imbedding $NC(m) \hookrightarrow NC(n)$, as follows.

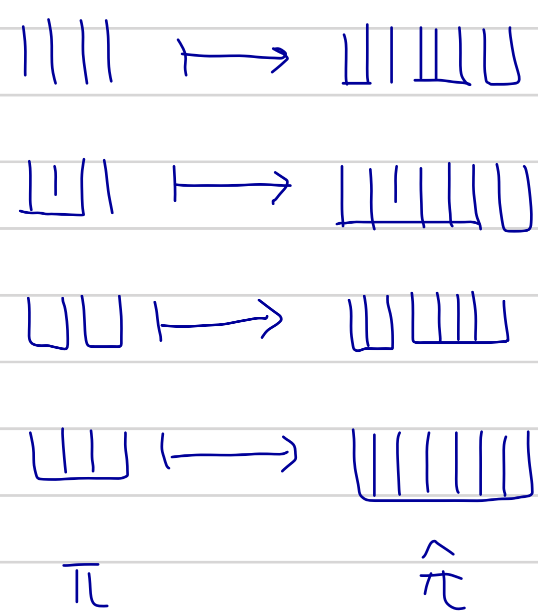
If $\pi \in NC(m)$, then define $\hat{\pi} \in NC(m)$ as follows:

$$\underbrace{\{1, \ldots, i_1\}}, \underbrace{\{i_1+1, \ldots, i_2\}}, \ldots, \underbrace{\{i_{m-1}+1, \ldots, i_m\}}$$

all in the same block of $\hat{\pi}$

and $i_k \sim_{\hat{\pi}} i_l \iff k \sim_{\pi} l$.

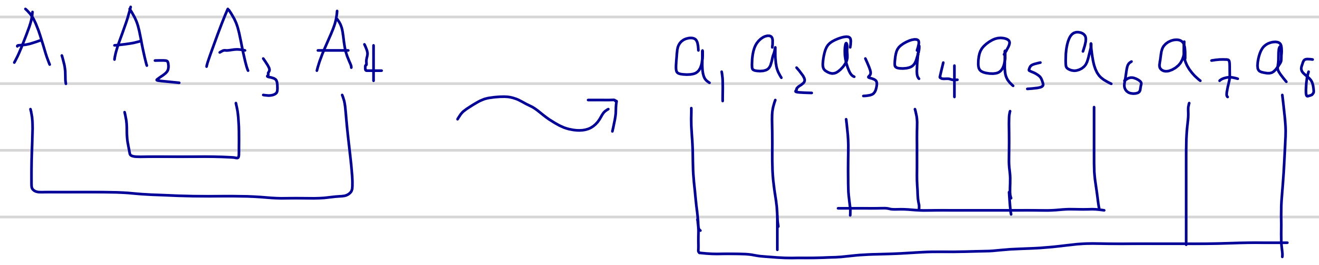
E.g. with break points $2, 3, 6, 8$, the map $NC(4) \hookrightarrow NC(8)$ takes



In general, for any break points, $1_m \mapsto 1_n$.

It is easy to check that $\hat{\pi}$ is non-crossing whenever π is. In terms of the relation to product blocks,

$$(a_1, a_2, \ldots, a_8) \mapsto (a_1 a_2, a_3, a_4 a_5 a_6, a_7 a_8) = (A_1, A_2, A_3, A_4)$$



Properties of the imbedding:

(1) It is injective. (This is clear from the definition: $i_k \sim_{\hat{\pi}} i_l \Leftrightarrow k \sim_{\pi} l$.)

(2) It is a poset homomorphism. (Again, clear from the definition.)

(3) The image of $NC(m)$ is the interval $[\hat{0}_m, 1_n]$.

Pf. Suppose $\tau \geq \hat{0}_m$. Since $\hat{0}_m$ is a refinement of τ, it is easily checked that $\tau = \hat{\sigma}$ where $\sigma \in NC(m)$ is the image of $\tau|_{\{i_1,\ldots,i_m\}}$ under the usual isomorphism between $NC(\{i_1,\ldots,i_m\})$ and $NC(m)$. ///

In other words,

<u>Theorem</u>: For fixed $1 \leq i_1 < \cdots < i_m = n$, the associated map $\pi \mapsto \hat{\pi}$ from $NC(m) \to NC(n)$ is a lattice isomorphism from $NC(m)$ onto $[\hat{0}_m, 1_n]$.

As such, we can use partial Möbius inversion in this lattice to calculate free cumulants with products as arguments.

<u>Theorem</u>: Let $1 \leq i_1 < \cdots < i_m = n$, and define $\pi \mapsto \hat{\pi}$ as above. Then

$$\kappa_m(a_1 \cdots a_{i_1}, a_{i_1+1} \cdots a_{i_2}, \ldots, a_{i_{m-1}+1} \cdots a_{i_m}) = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \hat{0}_m = 1_n}} \kappa_\pi(a_1, a_2, \ldots, a_n).$$

Pf. Let $A_j = a_{i_{j-1}+1} \cdots a_{i_j}$ for $1 \leq j \leq m$ (where $i_0 \equiv 1$). From the moment cumulant formula, for $\tau \in NC(m)$,

$$\kappa_\tau(A_1, \ldots, A_m) = \sum_{\pi \leq \tau} \varphi_\pi(A_1, \ldots, A_m)\, \mu_m(\pi, \tau).$$

The simple interaction between φ_m and the product in $\mathcal{A}$ can be summarized as

$$\varphi_\pi(A_1,\dots,A_m) = \varphi_{\hat\pi}(a_1,\dots,a_n).$$

Now, $\pi\mapsto\hat\pi$ is a lattice isomorphism $NC(m)\to[\hat 0_m, 1_n]$ in $NC(n)$, and so by the properties of Möbius functions,

$$\mu_m(\pi,\tau) = \mu_n(\hat\pi,\hat\tau).$$

Thus

$$\kappa_\tau(A_1,\dots,A_m) = \sum_{\substack{\pi\in NC(m)\\ \pi\le\tau}} \varphi_{\hat\pi}(a_1,\dots,a_n)\,\mu_n(\hat\pi,\hat\tau)$$

$$= \sum_{\underset{=\hat\pi}{\sigma}\in[\hat 0_m,1_n]} \varphi_\sigma(a_1,\dots,a_n)\,\mu_n(\sigma,\hat\tau).$$

By partial Möbius inversion in the lattice $NC(n)$, this gives

$$\boxed{\kappa_\tau(A_1,\dots,A_m) = \sum_{\substack{\pi\in NC(n)\\ \pi\vee\hat 0_m=\hat\tau}} \kappa_\pi(a_1,\dots,a_n).}$$

The special case $\tau=1_m$ (so $\hat\tau=1_n$) proves the theorem. ///

Eg. $\kappa_2(\underbrace{a_1a_2}_{\{1,2\}},\underbrace{a_3}_{\{3\}}) = \sum_{\substack{\pi\in NC(3)\\ \pi\vee\{\{1,2\},\{3\}\}=\{\{1,2,3\}\}}} \kappa_\pi(a_1,a_2,a_3).$

Can easily check by exhaustion here that the candidates are $\pi\in\{\{\{1,3\},\{2\}\},\ \{\{1\},\{2,3\}\},\ \{\{1,2,3\}\}\}$. This yields

$$\kappa_2(a_1a_2,a_3) = \kappa_{\{\{1,3\},\{2\}\}}(a_1,a_2,a_3) + \kappa_{\{\{1\},\{2,3\}\}}(a_1,a_2,a_3) + \kappa_{\{\{1,2,3\}\}}(a_1,a_2,a_3)$$

$$= \kappa_1(a_2)\kappa_2(a_1,a_3) + \kappa_1(a_1)\kappa_2(a_2,a_3) + \kappa_3(a_1,a_2,a_3).$$

Easily checked; not easily guessable.

The join in $NC(n)$ is a tricky thing. Fortunately, not so much when joining with $\hat{0}_m$.

<u>Exercise</u>: Let $\omega \in NC(n)$ be an <u>interval partition</u>: all blocks in ω are of the form $\{i, i+1, \ldots, i+k\}$. Show that, for any $\pi \in NC(n)$,

$$\pi \vee_{NC(n)} \omega = \pi \vee_{P(n)} \omega$$

and that this common join is described thus: $i \sim_{\pi\vee\omega} j$ iff $\exists\, i_2, i_3, \ldots, i_r$ s.t. $i \sim_\pi i_2 \sim_\omega i_3 \sim_\pi i_4 \sim_\omega \cdots \sim_\omega i_{r-1} \sim_\pi j$.

(Note: can always start and end with π, since we don't rule out the possibility that $i_2 = i$ or $i_r = j$.)

So, since $\hat{0}_m$ is an interval partition, the condition $\pi \wedge \hat{0}_m = 1_n$ says (since 1_n has only one block) that for any two blocks $V, W \in \hat{0}_m$, $\exists\, i_1, i_2, \ldots, i_r$ with

$$V \ni i_1 \sim_\pi i_2 \sim_{\hat{0}_m} i_3 \sim_\pi \cdots \sim_{\hat{0}_m} i_{r-1} \sim_\pi i_r \in W.$$

I.e. <u>π couples the blocks of $\hat{0}_m$</u>.

We now have all the tools we need to characterize freeness in terms of free cumulants. The first step is the following lemma.

<u>Lemma</u>: Let $(\mathcal{A}, \varphi)$ be a NCPS with associated free cumulants $\{\kappa_n\}_{n\geq 1}$. Let $a_1, \ldots, a_n \in \mathcal{A}$. If $\exists\, i \in [n]$ s.t. $a_i = 1$, then

$$\kappa_n(a_1, \ldots, a_n) = 0, \quad \forall n \geq 2.$$

Pf. To simplify notation, assume $a_n = 1$; the general case is analogous. So, we wish to show $\kappa_n(a_1, \ldots, a_{n-1}, 1) = 0$. We proceed by induction. The base case is

$$\kappa_2(a, 1) = \varphi(a \cdot 1) - \varphi(a)\varphi(1) = 0.$$

Now, presume we know $\forall k<n$ $\forall a_1,\dots,a_k \in \mathcal{A}$ that

$$\kappa_k(a_1,\dots,a_{k-1},1) = 0.$$

Using the moment-cumulant formula,

$$\varphi(a_1\cdots a_{n-1}1) = \sum_{\pi\in NC(n)} \kappa_\pi(a_1,\dots,a_{n-1},1)$$

$$= \sum_{\pi<1_n} \kappa_\pi(a_1,\dots,a_{n-1},1) + \kappa_n(a_1,\dots,a_{n-1},1).$$

In this sum, the terms are products of κ_k's. The final entry 1 must appear as a factor in each term; by the inductive hypothesis, the term is 0 unless $\{n\}\in\pi$, in which case $\pi=\sigma\cup\{n\}$ and

$$\kappa_\pi(a_1,\dots,a_{n-1},1) = \kappa_\sigma(a_1,\dots,a_{n-1})\kappa_1(1)$$

σ an arbitrary el't of $NC(n-1)$.

Thus

$$\varphi(a_1\cdots a_{n-1}) = \sum_{\sigma\in NC(n-1)} \kappa_\sigma(a_1,\dots,a_{n-1}) + \underbrace{\kappa_n(a_1,\dots,a_{n-1},1)}_{=0}.$$

but by the moment-cumulant formula, these are equal.

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This can be viewed as a special case of the following.

<u>Theorem</u>: Let $(\mathcal{A},\varphi)$ be a NCPS with free cumulant functionals $\{\kappa_n\}_{n\geq1}$. Let $\{\mathcal{A}_i\}_{i\in I}$ be unital subalgebras of $\mathcal{A}$. TFAE:

(1) $\{\mathcal{A}_i\}_{i\in I}$ are free.

(2) $\forall n\geq 2$, $\forall a_j\in\mathcal{A}_{i_j}$ for $j\in[n]$, $i_j\in I$, $\kappa_n(a_1,\dots,a_n)=0$ whenever $(i_1,\dots,i_n)$ is not a constant sequence.

Note: there is no assumption that the a_i's come from consecutively distinct subalgebras. The statement is that freeness is equivalent to the vanishing of all <u>mixed</u> free cumulants.

Note also that the a_i's need not be centred.

<u>Pf</u>. (2) $\Rightarrow$ (1) Suppose mixed free cumulants vanish. Let $a_j \in \mathcal{A}_{i_j}$ where $i_1 \neq i_2 \neq \cdots \neq i_n$, and where $\varphi(a_j) = 0$. Then

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \underbrace{K_\pi(a_1, \ldots, a_n)}_{}$$

$\therefore$ each term contains a $K_{p+1}(a_\ell, \ldots, a_{\ell+p})$ $\Leftarrow$ $\left\{\begin{array}{l} \text{a product over the blocks of } \pi. \\ \text{As } \pi \text{ is non-crossing, at least one} \\ \text{block is an interval } \{\ell, \ell+1, \ldots, \ell+p\}. \end{array}\right.$

if $p=0$, this is $K_1(a_\ell) = \varphi(a_\ell) = 0$.
if $p \neq 0$, this $= 0$ by the assumption of (2), as $i_\ell \neq i_{\ell+1}$.

$\therefore$ Each term in the sum $= 0$, so $\varphi(a_1 \cdots a_n) = 0$.
Thus $\{\mathcal{A}_i\}_{i \in I}$ are free.

(1) $\Rightarrow$ (2) First note that if we are in the setup of freeness, everything works easily. I.e. Suppose that $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\varphi(a_j) = 0 \ \forall j \in [n]$, then

$$K_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \underbrace{\varphi_\pi(a_1, \ldots, a_n)}_{} \mu_n(\pi, 1_n)$$

again, this product contains an interval partition $\varphi_{p+1}(a_\ell, \ldots, a_{\ell+p}) = \varphi(a_\ell \cdots a_{\ell+p}) = 0$ by freeness.

The thrust of this direction of the proof is that we can relax the $\varphi(a_j) = 0$ and $i_1 \neq i_2 \neq \cdots \neq i_n$ conditions.

First: the "centred" condition is of no consequence. By the Lemma, if any argument is 1, then $K_n = 0$. By multi-linearity, this means

$$\begin{aligned} K_n(a_1,\dots,a_n) &= K_n(a_1-\varphi(a_1)1,\dots,a_n-\varphi(a_n)1) \\ &= K_n(\mathring{a}_1,\dots,\mathring{a}_n). \end{aligned}$$

So, the assumption that a_j are centred can be removed.

Now, the tricky part is showing that "consecutively-distinct" can be replaced with "mixed". For this, we proceed by induction. For $n=2$: if a_1, a_2 are free, we know $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$; thus $K_2(a_1,a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2) = 0$.

Now, let $n \geq 3$, and suppose we have shown (2) to hold for all K_k with $k<n$. We now consider $K_n(a_1,\dots,a_n)$, where $a_j \in \mathcal{A}_{i_j}$ and $(i_1,\dots,i_n)$ is non-const. First, if it happens to be true that $i_1 \neq i_2 \neq \cdots \neq i_n$, then as we showed above $K_n(a_1,\dots,a_n)=0$. If this is not the case, then we can multiply adjacent entries until it is: that is $\exists\, j_1 < j_2 < \cdots < j_m = n$ s.t.

$$i_1 = i_2 = \cdots = i_{j_1},\ i_{j_1+1} = \cdots = i_{j_2},\ \dots,\ i_{j_{m-1}} = \cdots = i_{j_m}$$

are all distinct. I.e. $a_1,\dots,a_{j_1}$ are all in the same $\mathcal{A}_{i_{j_1}}$
$a_{j_1+1},\dots,a_{j_2}$ " " $\mathcal{A}_{i_{j_2}}$
$\dots$ $a_{j_{m-1}+1},\dots,a_{j_m}$ " " $\mathcal{A}_{i_{j_m}}$

So set $A_1 = a_1 \cdots a_{j_1}$, $A_2 = a_{j_1+1}\cdots a_{j_2}, \dots, A_m = a_{j_{m-1}+1}\cdots a_{j_m}$. Note, since $(i_1,\dots,i_n)$ is not constant, $m \geq 2$. Our result on cumulants with products as arguments says

$$\begin{aligned} K_m(A_1,\dots,A_m) &= \sum_{\substack{\pi \in NC(n) \\ \pi \vee \omega = 1_n}} K_\pi(a_1,\dots,a_n) \\ &= K_n(a_1,\dots,a_n) + \sum_{\substack{\pi < 1_n \\ \pi \vee \omega = 1_n}} K_\pi(a_1,\dots,a_n) \end{aligned}$$

where $\omega = \{\{1,\dots,j_1\},\dots,\{j_{m-1}+1,\dots,j_m\}\}$.

Now, by construction, $A_1, \ldots, A_m$ come from consecutively distinct subalgebras, and so as shown above $K_m(A_1, \ldots, A_m) = 0$. Now, for $\pi < 1_n$, $K_\pi(a_1, \ldots, a_n)$ is a product of K_k's with $k < n$. By the inductive hypothesis, such $K_k = 0$ unless all its arguments come from the same subalgebra. In other words

$$K_n(a_1, \ldots, a_n) = -\sum_{\pi \in D_w} K_\pi(a_1, \ldots, a_n)$$

where $D_w = \{\pi < 1_n : \pi \vee w = 1_n$ & the function $j \mapsto i_j$ is const. on the blocks of $\pi.\}$

But the condition $\pi \vee w = 1_n$ means that π couples the blocks of w. In particular, as $i_{j_1} \not\sim_w i_{j_1+1}$, $i_{j_1} \sim_\pi i_{j_1+1}$. But this is impossible since $i_j \neq i_{j+1}$. So $D_w = \emptyset$. Thus

$K_n(a_1, \ldots, a_n) = 0$, concluding the proof. ///