Lecture 20: May 16, 2011

Last time, we saw that freeness of subalgebras is equivalent to the vanishing of mixed cumulants between them.

But what about freeness of random variables?

Recall, $\{a_i\}_{i\in I}$ are free iff $\{\langle a_i\rangle\}_{i\in I}$ are free. Following our theorem, then, we have

$\{a_i\}_{i\in I}$ are free iff $\forall n\geq 2$, $b_1,\ldots,b_n$ s.t. $b_j\in\langle a_{i_j}\rangle$, if $(i_1,\ldots,i_n)$ is non-constant then

$$\kappa_n(b_1,\ldots,b_n)=0.$$

Now, $b_j\in\langle a_{i_j}\rangle$ so $\exists$ polynomial P_j s.t. $b_j=P_j(a_{i_j})$.
Thus, freeness of $\{a_i\}_{i\in I}$ is equivalent to:

$\forall n\geq 2$, $(i_1,\ldots,i_n)$ non-constant, and polynomials $P_1,\ldots,P_n$,

$$\kappa_n(P_1(a_{i_1}),\ldots,P_n(a_{i_n}))=0.$$

This is pretty clunky, and not much better than the definition in terms of moments. However, it can be significantly improved.

<u>Theorem</u>: Let $(\mathcal{A},\varphi)$ be a NCPS with free cumulants $\{\kappa_n\}_{n\geq 1}$. Let $\{a_i\}_{i\in I}\subseteq\mathcal{A}$. TFAE:

(1) $\{a_i\}_{i\in I}$ are free.
(2) $\forall n\geq 2$, and $(i_1,\ldots,i_n)\in I^n$ non-constant,

$$\kappa_n(a_{i_1},\ldots,a_{i_n})=0.$$

I.e. random variables are free iff their mixed cumulants vanish.

Pf. (1)$\Rightarrow$(2) is a weaker statement than the prev. theorem (it is the special case $P_1,\ldots,P_n=\mathrm{id}$).

(2) $\Rightarrow$ (1) Appealing to the previous theorem, it suffices to show that (2) implies the vanishing of mixed cumulants between the algebras generated by $\{a_i\}_{i\in I}$. So consider a mixed cumulant

$$\kappa_n(P_1(a_{i_1}), \dots, P_n(a_{i_n})).$$

By multi-linearity, this expands to a linear-comb. of terms of the form

$$\kappa_n(a_{i_1}^{k_1}, \dots, a_{i_n}^{k_n})$$

for some $k_1, \dots, k_n \in \mathbb{N}$. It $\therefore$ suffices to show all such terms are 0. We already know that if any $k_j = 0$ (so $a_{i_j}^{k_j} = 1$), $\kappa_n = 0$, so we may assume $k_j \geq 1$.

This is a cumulant with products as arguments. Let $\omega = \underbrace{|\,|\cdots|\,|}_{k_1}\;\underbrace{|\,|\cdots|\,|}_{k_2}\;\cdots\;\underbrace{|\cdots|}_{k_n}$. Then we have

$$\kappa_n(a_{i_1}^{k_1}, \dots, a_{i_n}^{k_n}) = \sum_{\substack{\pi \in NC(k) \\ \pi \vee \omega = 1_k}} \kappa_\pi(\underbrace{a_{i_1}, \dots, a_{i_1}}_{k_1}, \underbrace{a_{i_2}, \dots, a_{i_2}}_{k_2}, \dots, \underbrace{a_{i_n}, \dots, a_{i_n}}_{k_n})$$

where $k = k_1 + \cdots + k_n$. Now, by assumption (2), in order for the κ_π term to be $\neq 0$, each block of π must be contained in a block of ω. But that contradicts $\pi \vee \omega = 1_k$. So there are no π with $\kappa_\pi \neq 0$ in the sum. Thus, the sum is 0, concluding the proof. ///

Thus, under any reasonable interpretation, freeness is described as vanishing of mixed free cumulants.

Eg. Let $s_1, s_2 \in (\mathcal{A}, \varphi)$ be free, each with the semicircular distribution. Set $c = \frac{1}{\sqrt{2}}(s_1 + i s_2)$. This is the free probability version of a complex Gaussian. It is called a <u>circular</u> random variable.

<u>Note</u>: $c^* = \frac{1}{\sqrt{2}}(s_1 - i s_2)$, so c is not self-adjoint. Even worse:

$$cc^* - c^*c = \frac{1}{2}\left(s_1^2 + s_2^2 - i(s_1 s_2 - s_2 s_1)\right)$$
$$- \frac{1}{2}\left(s_1^2 + s_2^2 + i(s_1 s_2 - s_2 s_1)\right) = i(s_2 s_1 - s_1 s_2)$$

and this is non-zero: s_1, s_2 are free (and self-adjoint and non-constant), so they cannot commute. Thus c is a non-normal operator.

Hence, the interesting object for c is not its moments (which are easily checked to be all 0), but its $*$-distribution: the set of all joint-moments in c, c^*. This would be fierce to compute directly. But if we record that information as free cumulants instead, it becomes almost trivial.

Any free cumulant $\kappa_n(c^{\varepsilon_1}, c^{\varepsilon_2}, \ldots, c^{\varepsilon_n})$ $(\varepsilon_j \in \{1, *\})$ expands as a linear combination of 2^n terms

$$\kappa_n(s_{i_1}, s_{i_2}, \ldots, s_{i_n}) \qquad (\text{b/c } c, c^* = \tfrac{1}{\sqrt{2}}(s_1 \pm i s_2))$$

Since s_1, s_2 are free, the only terms here that are non-zero must have $(i_1, \ldots, i_n)$ constant – i.e. only

$$\kappa_n(s_1, \ldots, s_1) = \kappa_n^{s_1} = \delta_{n2} = \kappa_n^{s_2} = \kappa_n(s_2, \ldots, s_2).$$

Thus, all terms $\kappa_n(c^{\varepsilon_1}, \ldots, c^{\varepsilon_n}) = 0$ if $n=2$.
Moreover, among the 4 $n=2$ terms:

$$\kappa_2(c,c) = \frac{1}{2}\kappa_2(s_1 + i s_2, s_1 + i s_2) = \frac{1}{2}\Big(\kappa_2(s_1, s_1) + \frac{1}{2} i \cancel{\kappa_2(s_1, s_2)}$$
$$+ \frac{1}{2} i \cancel{\kappa_2(s_2, s_1)} - \frac{1}{2}\kappa_2(s_2, s_2)\Big)$$
$$= \frac{1}{2} - \frac{1}{2} = 0.$$

Similar calculations show $\kappa_2(c^*, c^*) = 0$, while $\kappa_2(c, c^*) = \kappa_2(c^*, c) = 1$. I.e., among all $*$-cumulants of c, only $\kappa_2(c, c^*) = \kappa_2(c^*, c) = 1$ are $\neq 0$.

This provides a complete description of μ_{c,c^*}. We can use it, if we like, to compute the joint moments of c, c^* via the moment-cumulant formula:

$$\varphi(c^{\varepsilon_1}c^{\varepsilon_2}\cdots c^{\varepsilon_n}) = \sum_{\pi\in NC(n)} \kappa_\pi(c^{\varepsilon_1}, c^{\varepsilon_2}, \ldots, c^{\varepsilon_n})$$

Since $\kappa_n = 0$ if $n \neq 2$, the only non-zero terms come from $\pi \in NC_2(n)$. Because of the $\kappa_2(c,c) = \kappa_2(c^*, c^*) = 0$ condition, what this amounts to is

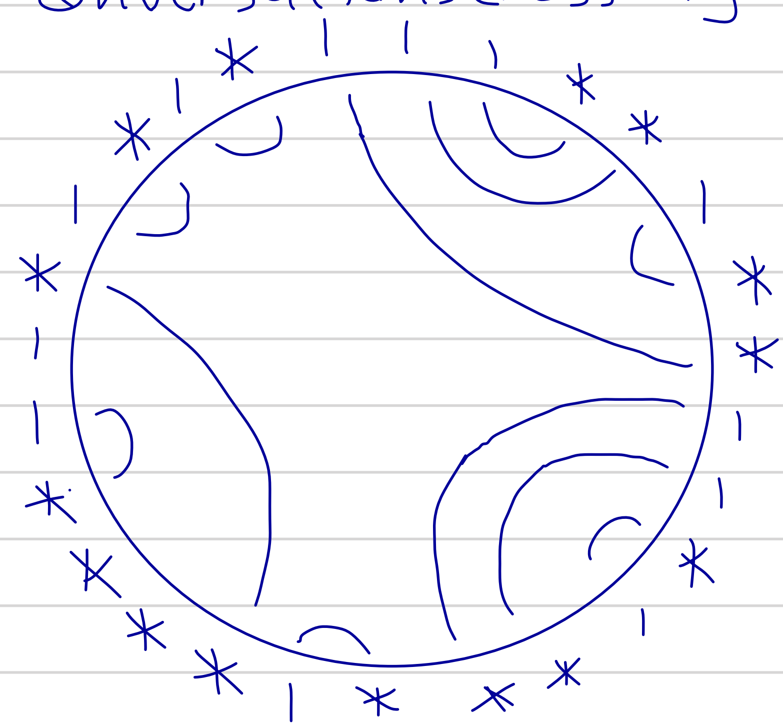
$$= \#\{\pi \in NC_2(n) : \{i,j\}\in\pi \Rightarrow \varepsilon_i \neq \varepsilon_j\}.$$

I.e. the moments count the number of non-crossing pairings that match c's with c^*'s. An easy consequence is that $\#\{j : \varepsilon_j = 1\} = \#\{j : \varepsilon_j = *\}$ if the moment $\neq 0$.

I like to describe this enumeration problem in the following Medieval terms.

The Knights & Ladies of the Round Table

King Arthur has m Knights. He throws a soirée to which he invites m Ladies. All $2m$ attendees sit randomly around the Round Table. How many ways can they pair off (Knights with Ladies) so that without any conversations crossing?



Exercises

1. Show that $\varphi((cc^*)^m) = C_m$.

2. Show that $\varphi((c^p c^{*p})^m) = C_m^{(p)}$.

   [Hint: Find a bijection between the NC-pairings of the string $(1^p *^p)^m$ and the set of multi-chains $NC(m)^{(p)}$.]

3. Attend my combinatorics seminar talk next Tuesday May 24, where I will talk about harder enumeration problems, and associated algebraic combinatorics.