Lecture 21: May 18, 2011

We now have a simple description of freeness of random variables in terms of their free cumulants. Here is a trivial but important corollary.

Cor: Let $x, y \in (\mathcal{A}, \varphi)$ be free. Then $\forall n \geq 1$,

$$\kappa_n^{x+y} = \kappa_n^x + \kappa_n^y.$$

Pf. Recall that $\kappa_n^a = \kappa_n(a, \ldots, a)$. The case $n=1$ is just

$$\kappa_1^{x+y} = \varphi(x+y) = \varphi(x) + \varphi(y) = \kappa_1^x + \kappa_1^y$$

For $n \geq 2$, using the multi-linearity of κ_n,

$$\kappa_n^{x+y} = \kappa_n(x+y, x+y, \ldots, x+y)$$

$$= \sum_{\eta \in [2]^n} \kappa_n(x_{\eta_1}, x_{\eta_2}, \ldots, x_{\eta_n})$$

where $x_1 = x$
$x_2 = y$

all of these are mixed cumulants, except $\eta = (1, \ldots, 1)$ and $\eta = (2, \ldots, 2)$. So all the middle terms are 0 by freeness, and thus

$$= \kappa_n(x, x, \ldots, x) + \kappa_n(y, y, \ldots, y) = \kappa_n^x + \kappa_n^y.$$

///

The key fact here is that

Cor: Let $x, y \in (\mathcal{A}, \varphi)$ be free. Then $\forall$ suff. small z,

$$\mathcal{R}_{x+y}(z) = \mathcal{R}_x(z) + \mathcal{R}_y(z)$$

Pf. Recall that $\mathcal{R}_a(z) = \sum_{n \geq 0} \kappa_{n+1}^a z^n$. The result is trivial. ///

Now, the $\mathcal{R}$-transform bears an important relation to the Stieltjes transform. Recall:

For $x \in (\mathcal{A}, \varphi)$, $G_x = G_{\mu_x}$ is analytic in a nbhd of ∞, and satisfies $z \cdot G_x(z) \to 1$ as $|z| \to \infty$. It is defined by

$$G_x(z) = \int_{\mathbb{R}} \frac{\mu_x(dt)}{z-t} = \sum_{n \geq 0} \frac{\varphi(x^n)}{z^{n+1}}$$

actually analytic on $\mathbb{C} - \mathrm{supp}\, \mu_x$ — holds on a nbhd of ∞.

We can recover μ_x from G_x by the Stieltjes inversion formula

$$-\frac{1}{\pi} \text{w-}\lim_{\varepsilon \downarrow 0} \operatorname{Im} G_x(t+i\varepsilon)\, dt = \mu_x(dt).$$

AND, most importantly, for all suff. small z,

$$(*) \qquad G_x\left(\mathcal{R}_x(z) + \frac{1}{z}\right) = z. \qquad (*)$$

So, we now have an "algorithm" for computing $\mu \boxplus \nu$ ($= \mu_{x+y}$ when $\mu_x = \mu$ and $\mu_y = \nu$):

(1) Compute G_μ, G_ν.

(2) Compute $\mathcal{R}_\mu, \mathcal{R}_\nu$ from (1) using $(*)$.

(3) By definition, $\mathcal{R}_{\mu \boxplus \nu} = \mathcal{R}_{x+y} = \mathcal{R}_\mu + \mathcal{R}_\nu$.

(4) Compute $G_{\mu \boxplus \nu}$ from $\mathcal{R}_{\mu \boxplus \nu}$ using $(*)$.

(5) Compute $\mu \boxplus \nu$ from (4) using the Stieltjes inversion formula.

The only non-routine steps are (2),(4) which require inverting an analytic function. Let's look at some examples.

Eg. Suppose $\mu = \delta_\alpha, \nu = \delta_\beta$, for $\alpha, \beta \in \mathbb{R}$. Here $\mu = \mu_a$ and $\nu = \mu_b$ where $a = \alpha, b = \beta$ are constants. Since constants are always free, we know $\mu \boxplus \nu = \mu_{a+b} = \delta_{\alpha+\beta}$ already – here is a sanity check:

(1) $G_\alpha(z) = \frac{1}{z-\alpha}$, $G_\beta(z) = \frac{1}{z-\beta}$.

(2) So $w = \mathcal{R}_\alpha(z)$ satisfies $\frac{1}{w+\frac{1}{z}-\alpha} = z$, i.e.

$$zw + 1 - \alpha z = 1, \text{ so } z(w-\alpha) = 0 \ \forall \text{small } z$$

i.e. $\mathcal{R}_\alpha(z) = w = \alpha$. Similarly $\mathcal{R}_\beta(z) = \beta$.

(3) So $\mathcal{R}_{\mu\boxplus\nu}(z) = \alpha+\beta$.

(4) Thus $G_{\mu\boxplus\nu}(\alpha+\beta+\frac{1}{z}) = z$. Let $u = \alpha+\beta+\frac{1}{z}$.

Then $z = \frac{1}{u-(\alpha+\beta)}$, and so

$$G_{\mu\boxplus\nu}(u) = \frac{1}{u-(\alpha+\beta)}$$

(5) We don't need the Stieltjes inversion formula here – it is immediate that

$$\frac{1}{z-(\alpha+\beta)} = G_{\delta_{\alpha+\beta}}(z).$$

✓

Eg. Consider the semicircular law of variance $t > 0$:

$$\gamma_t(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2}\,\mathbb{1}_{|x| \le 2\sqrt{t}}\,dx.$$

It is a simple calculus exercise to check that $\mathrm{Var}\,\gamma_t = t$. As is true with a classical Gaussian, γ_t is a dilation of γ_1: if x has the standard semicircular distribution $\mu_x = \gamma_1$, then $\mu_{\sqrt{t}x} = \gamma_t$.

So, what is $\gamma_s \boxplus \gamma_t$?

(1) First we compute $G_{\gamma_t} \equiv G_t$. From the definition, this is fairly complicated:

$$G_t(z) = \frac{1}{2\pi}\int_{-2\sqrt{t}}^{2\sqrt{t}} \frac{\sqrt{4t-u^2}}{z-u}\,du$$

can be done as a contour integral; requires cleverness.

But, actually, we can skip right to

(2) $\mathcal{R}_{\gamma_t}$. First, we already know $\mathcal{R}_{\gamma_1}$: if $\mu_x = \gamma_1$, then $K_x^n = \delta_{n2}$. Thus

$$\mathcal{R}_{\gamma_1}(z) = \mathcal{R}_x(z) = \sum_{n\geq 0} K_{n+1}^x z^n = z.$$

Now, as we noted above, $\gamma_t = \mu_{\sqrt{t}x}$, and so

$$\mathcal{R}_{\gamma_t}(z) = \mathcal{R}_{\sqrt{t}x}(z) = \sum_{n\geq 0} K_{n+1}^{\sqrt{t}x} z^n$$

where $K_n^{\sqrt{t}x} = K_n(\sqrt{t}x, \ldots, \sqrt{t}x) = t^{n/2} K_n^x$.

$$\therefore \mathcal{R}_{\gamma_t}(z) = \sum_{n\geq 0} t^{(n+1)/2} K_{n+1}^x z^n = \sqrt{t}\,\mathcal{R}_{\gamma_1}(\sqrt{t}z).$$

(I.e., in general, $\mathcal{R}_{\alpha x}(z) = \alpha \mathcal{R}_x(\alpha z)$.)

In particular, this means $\mathcal{R}_{\gamma_t}(z) = tz$.

From here, we can go back and calculate G_{γ_t} if we wish:

$$G_{\gamma_t}\left(tz + \frac{1}{z}\right) = z.$$

Let $u = tz + \frac{1}{z}$. So $tz^2 - uz + 1 = 0$, so

$$z = \frac{1}{2t}\left(u \pm \sqrt{u^2-4t}\right).$$

i.e.

$$G_{\gamma_t}(u) = \frac{u \pm \sqrt{u^2-4t}}{2t}$$

as $\lim_{|u|\to\infty} G_{\gamma_t}(u) = 0$, we must have the minus sign.

(3) Thus $\mathcal{R}_{\gamma_t \boxplus \gamma_s}(z) = \mathcal{R}_{\gamma_t}(z) + \mathcal{R}_{\gamma_s}(z) = tz + sz$.

Note, this is the same as $(t+s)z = \mathcal{R}_{\gamma_{t+s}}(z)$.

Thus, without going through (4)–(5), we see immediately that

$$\gamma_t \boxplus \gamma_s = \gamma_{t+s}. \Leftarrow \{\gamma_t\}_{t\geq 0} \text{ forms a free convolution semigroup.}$$

This is precisely analogous to the classical case for Gaussians, under (usual) convolution.

Remark: In particular, this means that for any $n \geq 1$,

$$\underbrace{\gamma_{1/n} \boxplus \cdots \boxplus \gamma_{1/n}}_{n \text{ times}} = \gamma_1$$

This says that the semicircle law is (freely) <u>infinitely divisible</u>.

As with classical probability, the freely infinitely-divisible laws form an important class of measures. They have a complete description in terms of a Lévy–Khintchine-type representation. This is one of the advanced directions we might choose to follow next.

Eg. Let's look again at the symmetric Bernoulli law: $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$. We can compute $\mu \boxplus \mu$.

(1) $G_\mu(z) = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right) = \frac{z}{z^2-1}$.

(2) As already computed in Lecture 17,

$$\mathcal{R}_\mu(z) = \frac{-1 + \sqrt{1+4z^2}}{2z}.$$

Thus, $\mathcal{R}_{\mu\boxplus\mu}(z) = \frac{\sqrt{1+4z^2}-1}{z}$. Indeed, in general, the free convolution powers are

$$\mathcal{R}_{\mu^{\boxplus n}}(z) = \frac{n}{2}\,\frac{\sqrt{1+4z^2}-1}{z}.$$

(4) Thus, setting $u_n = \frac{n}{2}\frac{\sqrt{1+4z^2}-1}{z} + \frac{1}{z}$, we have

$$G_{\mu^{\boxplus n}}(u) = z.$$

For the time being, let us stick to the case $n=2$.

$$u = u_2 = \frac{\sqrt{1+4z^2}-1}{z} + \frac{1}{z} = \frac{\sqrt{1+4z^2}}{z}$$

So $u^2z^2 = 1+4z^2$, and so $(u^2-4)z^2 = 1$ i.e.

$$G_{\mu\boxplus\mu}(u) = z = \frac{\pm 1}{\sqrt{u^2-4}}.$$

(5) Now, applying the Stieltjes inversion formula:

$$-\frac{1}{\pi}\operatorname{Im} G_\mu(t+i\varepsilon) = \mp\frac{1}{\pi}\operatorname{Im}\frac{1}{\sqrt{(x+i\varepsilon)^2-4}}$$

Letting $\varepsilon \downarrow 0$, we get $\mp\frac{1}{\pi}\cdot\begin{cases} -1/\sqrt{4-x^2}, & |x|\leq 2 \\ 0, & |x|>2 \end{cases}$

And since this density must be ≥ 0, the correct sign for $G_{\mu\boxplus\mu}$ is $\frac{+1}{\sqrt{z^2-4}}$.

Thus:

$$\mu\boxplus\mu(dt) = \frac{1}{\pi\sqrt{4-t^2}}\mathbb{1}_{|t|\leq 2}\,dt.$$

This is the arcsine law. (In Lecture 6, we saw that it is the law of $a+a^*$ if a is an ∞-order element in a group G, in $(\mathbb{C}G, \varphi_G)$; more on this next day.)

<u>Remarks</u> (1) This shows free convolution can smooth out singularities in a way that classical convolution cannot: here μ is discrete, but $\mu \boxplus \mu$ is continuous, with a smooth density! Compare:

$$\mu * \mu = \frac{1}{4}\delta_{-2} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_2$$ ← records prob dist. of random walk of length 2; next time, we'll see that $\mu \boxplus \mu$ also has to do with a random walk.

(2) Also shows that $\boxplus$ does not distribute over convex combinations the way $*$ does, ie

$$\left(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2\right) * \nu = \frac{1}{2}\mu_1 * \nu + \frac{1}{2}\mu_2 * \nu$$

BUT: $\left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\right) \boxplus \mu = \mu \boxplus \mu = $ arcsine law

while $\frac{1}{2}\delta_{-1} \boxplus \mu + \frac{1}{2}\delta_1 \boxplus \mu = ?$

⌈ If $\alpha \in \mathbb{R}$, $\delta_\alpha \boxplus \mu_x = \mu_{a+x}$ if $\mu_a = \delta_\alpha$ and a, x are free. But if $a = \alpha$ is const, it is automatically free from any x, so

$$\delta_\alpha \boxplus \mu_x = \mu_{x+\alpha} = T_\alpha(\mu_x)$$

$$= \delta_\alpha * \mu_x$$

↖ translation by α: $T_\alpha(\mu)(B) = \mu(B - \alpha)$. ⌋

In particular, tracing through means that

$$\frac{1}{2}\delta_1 \boxplus \mu + \frac{1}{2}\delta_{-1} \boxplus \mu = \frac{1}{2}\delta_1 * \mu + \frac{1}{2}\delta_{-1} * \mu = \underset{\text{discrete}}{\underset{\uparrow}{\mu * \mu}}$$

Free convolution is a <u>non-linear</u> convolution. It has wonderful properties — at the expense of losing its "easy-to-calculate" linear character.