Lecture 22: May 20, 2011

Random Walks on Finitely-Generated Groups

Let G be a group with generating set

$$S = \{u_1, u_1^{-1}, \dots, u_k, u_k^{-1}\}.$$

(It may be possible to generate G without all of the inverses present; we will only consider the symmetric case.)

A *random walk* in (G, S) is a sequence $(g_n)_{n \geq 0}$ in G with $g_0 = 1 \in G$, and $\forall n \geq 0, \exists s_n \in S$ s.t.

$$g_{n+1} = s_n g_n.$$

I.e. $g_{n+1} = s_n s_{n-1} \cdots s_0$, so we could record the sequence equally well by the successive "differences" $(u^n)_{n \geq 1}$.

Let us assign a (discrete) probability measure to S: i.e. $\mathbb{P}(u_k) = \mathbb{P}(u_k^{-1}) = p_k$. (Insisting u_k, u_k^{-1} are equally probable makes the random walk *symmetric*.) I.e.

$$p_1, \dots, p_k \geq 0, \quad \sum_{n=1}^{k} 2p_k = 1.$$

Then we can think of the collection of random walks on (G, S) as a Markov chain with states in G, and the transition probability $g \to h$ is $= p_n$ if $h = u_n g$ or $h = u_n^{-1} g$ for some $n \in [k]$, and 0 otherwise.

We can then ask questions like: is the walk recurrent? I.e.

$$\mathbb{P}(g_n = 1 \text{ infinitely often}) > 0? \quad (\text{If so, it} = 1)$$

E.g. Let $G = \mathbb{Z}^k$ with $S = \{\pm e_n : n \in [k]\}$, $p_n = \frac{1}{2k}\ \forall n$. The associated random walk is known as the standard symmetric walk on $\mathbb{Z}^k$. It is well known that it is recurrent iff $k \leq 2$. (Pòlya, 1920.)

In the Markov chain, the individual steps are independent — in particular, for fixed $u_{i_1}, \dots, u_{i_n}$, and $\varepsilon_1, \dots \varepsilon_n \in \{\pm 1\}$,

$$\mathbb{P}\left((g_m)_{m=1}^{\infty} : g_{\ell+1} = u_{i_\ell}^{\varepsilon_\ell} g_\ell, \ 0 \leq \ell \leq n-1\right) = p_{i_0} p_{i_1} \cdots p_{i_{n-1}}.$$

Exercise: Show that the random walk in (G, S) is recurrent iff

$$\sum_{n \geq 1} \mathbb{P}((g_m) : g_n = 1) = \infty.$$

[Hint: use the Second Borell-Cantelli lemma.]

Let's specialize (for now) to the case $p_n = \frac{1}{2k}$ $\forall n \in [k]$.
Then for any fixed initial sequence of length n,

$$\mathbb{P}((g_m) : (g_1, \dots, g_n) = \text{this fixed sequence}) = \left(\tfrac{1}{2k}\right)^n.$$

Thus for fixed $n \geq 1$,

$$\mathbb{P}(g_n = 1) = \sum_{\substack{(s_1, \dots, s_n) \in S^n \\ s_n \cdots s_1 = 1 \in G}} \underbrace{\mathbb{P}(g_1 = s_1, g_2 = s_2 s_1, \dots, g_n = s_n s_{n-1} \cdots s_1)}_{\text{always } = \left(\frac{1}{2k}\right)^n}$$

$$= \left(\tfrac{1}{2k}\right)^n \#\{(s_1, \dots, s_n) \in S^n : s_n \cdots s_1 = 1 \in G\}.$$

Hence, by the exercise, random walks in (G, S) are recurrent iff

$$\sum_{n \geq 1} \left(\tfrac{1}{2k}\right)^n \#\{(s_1, \dots, s_n) \in S^n : s_n \cdots s_1 = 1 \in G\} = \infty.$$

So we are left with an enumeration problem.

E.g. in $\mathbb{Z}$, we are counting the set of lattice paths in $(\pm 1)^n$ whose sum is 0. We already counted this Lecture 7: $\# = \binom{n}{n/2}$ if n is even (0 if n odd).

So, with $k=1$, we're looking at

$$\sum_{n\geq 0} \underbrace{\frac{1}{2^{2n}}\binom{2n}{n}}_{\text{the terms are } \sim \frac{1}{\sqrt{\pi n}}} \sim \sum_{n\geq 0} \frac{1}{\sqrt{\pi n}} = \infty.$$

The calculations for $\mathbb{Z}^2, \mathbb{Z}^3, \dots$ are quite a bit more involved.

Relation to Group Algebras

Let G be a finitely-generated group. Consider the NCPS

$$(\mathbb{C}G, \varphi_G)$$

(recall: $\varphi_G(f) = f(1)$ — i.e. $\varphi_G(\sum_g \alpha_g \cdot g = \alpha_1)$)

Let $S = \{u_1^{\pm 1}, \dots, u_k^{\pm 1}\}$ be a symmetric generating set. Define

$$\Delta_S = \Delta = u_1 + u_1^{-1} + u_2 + u_2^{-1} + \dots + u_k + u_k^{-1} \in \mathbb{C}G.$$

<u>Prop</u>: $\varphi_G(\Delta^n) = \#\{(s_1, \dots, s_n) \in S^n : s_n \cdots s_1 = 1 \in G\}$.

Pf. $\Delta^n = \sum$ all words $s_1 \cdots s_n$ with $s_i \in S$, $1 \leq i \leq n$.

The state φ_G applied to any such word yields either 0 (if the word does not reduce to $1 \in G$) or 1 (if it does reduce to $1 \in G$). ///

So, the numbers we need to calculate are moments: random walks in G are recurrent iff

$$\sum_{n\geq 1} \left(\frac{1}{2k}\right)^n \varphi_G(\Delta^n) = \infty.$$

<u>Exercise</u> (for operator algebra buffs): Show r.w. in (G,S) transient iff $\Delta_S - |S|$ is invertible in C^*G.

Exercise: The <u>Cayley graph</u> of (G,S) has vertices G and edges (g,h) when $gh^{-1} \in S$. Elements of $\mathbb{C}G$ can be viewed as functions on the Cayley graph. The <u>Laplacian</u> L_S on the Cayley graph of (G,S) is
$$L_S f(x) = \sum_{y \sim x} [f(y) - f(x)]$$
Show that $L_S f = (\Delta_S - |S|) * f$.

Random Walks in Free Groups

Let $\mathbb{F}_k$ be a free group with generators $\{u_n\}_{n=1}^{k}$.

In $\mathbb{C}\mathbb{F}_k$, the random variables $x_n = u_n + u_n^{-1} = u_n + u_n^*$ have distribution μ_{x_n} = arcsine law, as we showed in Lecture 6.

Combining with the result from the last lecture, if $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$ is the symmetric Bernoulli law, then $\mu_{x_n} = \mu \boxplus \mu$, $n \in [k]$.

Moreover, in $\mathbb{C}\mathbb{F}_k$, $u_1, \ldots, u_k$ are $*$-free. Thus $x_1, \ldots, x_k$ are free. And so
$$\Delta_k = x_1 + \cdots + x_k \text{ is a free sum.}$$
I.e. $\mu_{\Delta_k} = \mu_{x_1} \boxplus \cdots \boxplus \mu_{x_k} = (\mu \boxplus \mu)^{\boxplus k} = \mu^{\boxplus 2k}$.

We now have the tools to compute this, which will allow us to calculate the desired moments. As calculated last day,
$$\mathcal{R}_\mu(z) = \frac{-1 + \sqrt{1+4z^2}}{2z}; \quad \mathcal{R}_{\mu^{\boxplus 2k}}(z) = k \cdot \frac{-1 + \sqrt{1+4z^2}}{z}.$$
We can thus compute the Stieltjes transform G_{Δ_k}.

Let $w = \mathcal{R}_{\mu^{\boxplus 2k}}(z) = k\dfrac{\sqrt{1+4z^2}-1}{z}$. Then

$$G_{\Delta_k}\left(w+\tfrac{1}{z}\right) = G_{\mu^{\boxplus 2k}}\left(w+\tfrac{1}{z}\right) = z.$$

So $u = w + \frac{1}{z} = \dfrac{k\sqrt{1+4z^2}}{z} - \dfrac{(k-1)}{z}$, i.e. $(uz+k-1)^2 = k^2(1+4z^2)$

$$u^2z^2 + 2(k-1)uz + (k-1)^2 = k^2 + 4k^2z^2$$

$$\therefore\ (u^2-4k^2)z^2 + 2(k-1)uz - 2k+1 = 0.$$

I.e.
$$z = \frac{-2(k-1)u \pm \sqrt{4(k-1)^2u^2 - 4(u^2-4k^2)(1-2k)}}{2(u^2-4k^2)}$$

$$= \frac{-(k-1)u \pm k\sqrt{u^2-4(2k-1)}}{u^2-4k^2}.$$

I.e.
$$G_{\Delta_k}(z) = \frac{-(k-1)z \pm k\sqrt{z^2-4(2k-1)}}{z^2-4k^2}$$

Sanity check: @ $k=1$, this reduces to

$$G_{\Delta_1}(z) = G_{u+u^*}(z) = \frac{\pm\sqrt{z^2-4}}{z^2-4} = \frac{\pm 1}{\sqrt{z^2-4}}$$

as we saw last day, with the correct sign being $+$.
In general, the condition

$$z\cdot G_{\Delta_k}(z) \to 1 \text{ as } |z|\to\infty$$

shows that the $+$ sign is correct for all $k \geq 1$.

Now, we know that

$$G_{\Delta_k}(z) = \sum_{n\geq 0} \frac{\varphi_G(\Delta_k^n)}{z^{n+1}}$$

These are the #s we want to compute!

So, we should compute the moment series.

$$\sum_{n\geq 0} \varphi(\Delta_k^n) z^n = M_{\Delta_k}(z) = \frac{1}{z} G_{\Delta_k}\left(\frac{1}{z}\right)$$

$$= \frac{1}{z} \cdot \frac{-(k-1)\frac{1}{z} + k\sqrt{1/z^2 - 4(2k-1)}}{\frac{1}{z^2} - 4k^2}$$

$$= \frac{-(k-1) + k\sqrt{1-4(2k-1)z^2}}{1-4k^2z^2}$$

← the Taylor coefficients of this function count # walks returning to 1 in $\mathbb{F}_k$.

Note: this is an even function, reflecting the fact that any loop must have even length (b/c there are no relations in $\mathbb{F}_k$).

Exercise: Show that

$$M_{\Delta_k}(z) = 1 + \sum_{n=0}^{\infty} \left[(2k)^{2(n+1)} - 2k(2k-1) B_k^n \right] z^{2(n+1)}$$

where $B_k^n = \sum_{\ell=0}^{n} (2k)^{2(n-\ell)} (2k-1)^\ell C_\ell$

and C_ℓ is the Catalan number.

This is about as good a "closed" formula there is. So, Eg.

#loops of length $2n$ in $\mathbb{F}_k$

$$\left.\begin{aligned} 2n &= 2 : && 2k \\ 2n &= 4 : && 2k(4k-1) \\ 2n &= 6 : && 4k(10k^2-6k+1) \\ 2n &= 8 : && 2k(112k^3 - 112k^2 + 40k - 5) \end{aligned}\right\} \text{ all irreducible over } \mathbb{Z}$$

This is not the best approach to answering the question of recurrence/transience of random walks in $\mathbb{F}_k$. Actually, to answer that question, it is better to calculate $\mu^{\boxplus 2k}$ from the Steiltjes transform.

$$G_{\Delta_k}(z) = \frac{-(k-1)z}{z^2-4k^2} + \frac{k\sqrt{z^2-4(2k-1)}}{z^2-4k^2}$$

this is purely real for $z \in \mathbb{R}$, with problem points $z = \pm 2k$

for $z \in \mathbb{R}$, this is purely real when $|z| \geq 2\sqrt{2k-1}$

since $|\pm 2k| \geq 2\sqrt{2k-1}$ $\forall k$, no problem pts

$$\therefore \lim_{\varepsilon \downarrow 0} \frac{-1}{\pi} \operatorname{Im} G_{\Delta_k}(t+i\varepsilon) = \frac{k\sqrt{4(2k-1)-t^2}}{\pi(4k^2-t^2)} \mathbb{1}_{|t| \leq 2\sqrt{2k-1}}.$$

Thus, we have computed the density of $\mu^{\boxplus 2k}$. Call it ρ_k. The exact form is unimportant — what matters is the support.

$$\#\{\text{loops of length } n \text{ in } \mathbb{F}_k\} = \varphi_{\mathbb{F}_k}(\Delta_k^n) = \int_{\mathbb{R}} t^n \mu^{\boxplus 2k}(dt)$$

$$= \int_{-2\sqrt{2k-1}}^{2\sqrt{2k-1}} t^n \rho_k(t)\,dt \leq \int_{-2\sqrt{2k-1}}^{2\sqrt{2k-1}} (2\sqrt{2k-1})^n \rho_k(t)\,dt$$

$$= (2\sqrt{2k-1})^n$$

This upper bound shows that

<u>Theorem</u>: For $k \geq 2$, random walks in $\mathbb{F}_k$ are transient.

Pf. $\sum_{n\geq 1} \left(\frac{1}{2k}\right)^n \varphi_{\mathbb{F}_k}(\Delta_k^n) \leq \sum_{n \geq 1} \frac{1}{(2k)^n}(2\sqrt{2k-1})^n < \infty$

because $\frac{2\sqrt{2k-1}}{2k} < 1$ when $k \geq 2$. As we saw, this means the random walk is not recurrent. ///

<u>History</u>: This was first proved in 1958 in Harry Kesten's PhD thesis; it <u>started</u> the field of random walks on groups. Our proof is approx. 1/100 the size of the original.