Lecture 23: May 23, 2011

The Free Poisson Distribution

In classical probability, the Poisson distribution is a probability measure on $\mathbb{Z}_+$. We can be a little more generous and let the jumps be in $\alpha\mathbb{Z}_+$ for some $\alpha \in \mathbb{R}$.

<u>Def</u>: The classical Poisson law with jump-size $\alpha \in \mathbb{R}$ and "rate" $\lambda \geq 0$ is

$$\nu_c = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_{k\alpha}.$$

Except in the trivial cases $\alpha = 0$ or $\lambda = 0$ this law is not compactly-supported. It is also discrete, making it quite different from the laws we've seen in free probability. Nonetheless, there is a free version of this law. To get at it, we need to think more about infinite-divisibility.

In fact, it turns out that ν_c is "infinitesimally" infinitely-divisible.

<u>Exercise</u>: For fixed $\lambda \geq 0$, $\alpha \in \mathbb{R}$, note that

$$\left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_\alpha$$

is a probability measure when $N \geq \lambda$. Prove that

$$w\text{-}\lim_{N \to \infty} \left(\left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_\alpha \right) = \nu_c \text{ (with rate } \lambda\text{, jump-size } \alpha\text{)}.$$

This suggests a way to define the <u>free-Poisson</u> distribution with rate $\lambda \geq 0$ and jump-size $\alpha \in \mathbb{R}$.

Theorem: Let $\lambda \geq 0$, $\alpha \in \mathbb{R}$. For $N \geq \lambda$, define

$$\nu_N = \left[\left(1-\frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_\alpha\right]^{\boxplus N}$$

Then the limit in distribution of ν_N exists. This law, ν_f, is the free-Poisson distribution with "rate" λ and "jump-size" α. It has $\mathcal{R}$-transform

$$\mathcal{R}_{\nu_f}(z) = \frac{\lambda\alpha}{1-\alpha z}.$$

Pf. We will show the nominally weaker claim that for each $n \geq 1$, the n^{th} moment of ν_N converges to the n^{th} moment of ν_f. (To show the stronger claim of convergence in distribution, it suffices to show ν_N and ν_f are all supported in a fixed compact interval, which can be done - we will address it next time.)

The following Lemma is very useful in general.

Lemma: Let μ_N, μ be compactly-supported probability measures. Then

$$\int t^n \mu_N(dt) \to \int t^n \mu(dt) \quad \forall n$$

iff $\quad K_n(\mu_N) \to K_n(\mu) \quad \forall n.$

(Since the n^{th} cumulant is a degree n polynomial in the first n moments, and vice versa, this is a triviality.) ///

So, it suffices to show that $K_n(\nu_N) \to K_n(\nu_f)$ $\forall n$.

Note,
$$\mathcal{R}_{\nu_f}(z) = \frac{\lambda\alpha}{1-\alpha z} = \sum_{n\geq 0} \underbrace{\lambda\alpha^{n+1}}_{K_{n+1}} z^n$$

So $K_n(\nu_f) = \lambda\alpha^n$.

Now, let $\theta_N = (1-\frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\delta_\alpha$ (for $N \geq \lambda$), so that $\nu_N = \theta_N^{\boxplus N}$.

$$\therefore \int t^n \, \theta_N(dt) = \frac{\lambda}{N}\alpha^n.$$

From here, we could work out $\kappa_n(\theta_N)$ explicitly. But it suffices to note that

$$\kappa_n(\theta_N) = \sum_{\pi \in NC(n)} m_\pi(\theta_N)\,\mu_n(\pi, 1_n)$$

$$= \frac{\lambda}{N}\alpha^n + \sum_{\pi < 1_n} m_\pi(\theta_N)\,\mu_n(\pi, 1_n)$$

where m_π is the multiplicative function determined by the sequence of moments of θ_N. Well, for $\pi < 1_n$, $m_\pi(\theta_N)$ is a product of at least two moments, each of which scales with $1/N$. Since there is (of course) an upper bound on the first n moments, this means

$$\kappa_n(\theta_N) = \frac{\lambda}{N}\alpha^n + O\left(\frac{1}{N^2}\right).$$

But then

$$\kappa_n(\nu_N) = \kappa_n(\theta_N^{\boxplus N}) = N \cdot \kappa_n(\theta_N) = \lambda\alpha^n + O\left(\tfrac{1}{N}\right).$$

The result follows. ///

Remark: Actually, we have to be a little careful: we have shown that all cumulants converge — but we take on faith that they converge to the free cumulants of a probability measure. (This is required to apply the lemma.) So we should check that the $\mathcal{R}$-transform in question is the $\mathcal{R}$-transform of a probability measure.

We start by computing the putative Stieltjes transform.

$$\mathcal{R}(z) = \frac{\lambda\alpha}{1-\alpha z}.$$ So we solve

$$G\Big(\underbrace{\frac{\lambda\alpha}{1-\alpha z}+\frac{1}{z}}_{u}\Big)=z$$

So $\lambda\alpha z + 1 - \alpha z = uz(1-\alpha z)$

i.e. $\alpha u z^2 + [(\lambda-1)\alpha - u]z + 1 = 0.$

i.e. $$G(u) = z = \frac{u-(\lambda-1)\alpha \pm \sqrt{(u-(\lambda-1)\alpha)^2 - 4\alpha u}}{2\alpha u}.$$

It is convenient to complete the square in the u variable inside the discriminant, yielding

$$(u-\alpha(\lambda+1))^2 - 4\lambda\alpha^2.$$

Now, if G is to be the Stieltjes transform of a probability measure, we must have

$$\lim_{|u|\to\infty} u\cdot G(u) = 1.$$

So the two terms (both $O(|u|)$) must cancel in the numerator, meaning we must have the minus sign. Thus

$$G(z) = \frac{z-(\lambda-1)\alpha - \sqrt{(z-\alpha(\lambda+1))^2 - 4\lambda\alpha^2}}{2\alpha z}.$$

So, is this the Stieltjes transform of a probability measure? The easiest way to tell is to apply the Stieltjes inversion formula and find the putative measure.

The form of the answer actually depends on the sign of $\lambda-1$.

$$G(z) = \frac{1}{2\alpha} + \frac{1-\lambda}{2z} - \frac{\sqrt{(z-\alpha(1+\lambda))^2 - 4\lambda\alpha^2}}{2\alpha z}.$$

The discriminant is purely real when $z \in \mathbb{R}$ and

$$0 \leq (z-\alpha(1+\lambda))^2 - 4\lambda\alpha^2$$

i.e. $z \in \mathbb{R}$, $|z - \alpha(1+\lambda)| \geq 2\sqrt{\lambda}\,\alpha$

i.e. $z \notin [\alpha(1-\sqrt{\lambda})^2, \alpha(1+\sqrt{\lambda})^2]$.

So this term contributes a non-zero density only on this interval. That is, set

$$\tilde{\nu}_f(dt) = \frac{1}{2\pi\alpha t}\sqrt{4\lambda\alpha^2 - (t-\alpha(1+\lambda))^2}\; \mathbb{1}_{[\alpha(1-\sqrt{\lambda}),\, \alpha(1+\sqrt{\lambda})^2]}(t)\, dt.$$

When $\lambda > 1$, one can check that $\int \tilde{\nu}_f(dt) = 1$, and as the other terms in G are real on $\mathbb{R}$, we have $\nu_f = \tilde{\nu}_f$ in this case.

But when $0 \leq \lambda \leq 1$, $\int \tilde{\nu}_f(dt) = 1 - \frac{1}{\lambda}$. In this case, note that the first terms in G include

$$\frac{1-\lambda}{z} = \underset{\geq 0}{(1-\lambda)}\, G_{\delta_0}(z)$$

Indeed, in this case, we get a point mass @ 0.

The final result is:

<u>Thm</u>: The free Poisson law ν_f is given by

$$\nu_f = \begin{cases} \tilde{\nu}_f, & \lambda > 1 \\ (1-\lambda)\delta_0 + \lambda\tilde{\nu}_f, & 0 \leq \lambda \leq 1 \end{cases}.$$

It is a probability measure on $\mathbb{R}$, with $\mathcal{R}$-transform

$$\mathcal{R}_{\nu_f}(z) = \frac{\alpha\lambda}{1-\alpha z}.$$

Remark: Except (sometimes) at 0, ν_f is a continuous measure, much unlike its classical counterpart. In particular, the terms "rate" and "jump-size" aren't very meaningful here; they are two parameters in this family of laws.

Remark: One can check that the classical cumulants of the classical Poisson law ν_c are $c_n(\nu_c) = \lambda\alpha^n$. This highlights what is known as the Bercovici-Pata bijection: If α_n is a sequence, it is the sequence of classical cumulants of a measure μ iff it is also the sequence of free cumulants of a measure $P\mu$.

Eg. $P(\text{Gaussian}) = \text{Semicircle}$
$P(\nu_c) = \nu_f$

This bijection has some surprising properties, however. Consider the case $\lambda=1$ in ν_f:

$$\nu_\alpha(dt) = \frac{1}{2\pi\alpha t}\sqrt{\underbrace{4\alpha^2-(t-2\alpha)^2}_{=4\alpha t-t^2}}\;\mathbb{1}_{[0,4\alpha]}(t)\,dt$$

$$= \frac{1}{2\pi\alpha\sqrt{t}}\sqrt{4\alpha-t}\;\mathbb{1}_{[0,4\alpha]}(t)\,dt$$

Does this look familiar? Suppose X is semicircular with variance α

$$\mu_X(dt) = \frac{1}{2\pi\alpha}\sqrt{4\alpha-t^2}\;\mathbb{1}_{[-2\sqrt{\alpha},2\sqrt{\alpha}]}(t)\,dt.$$

It is a standard exercise from undergraduate probability (and first-year calculus) that if X has (even) density f_X, then

$$f_{X^2}(t) = \frac{1}{\sqrt{t}}f_X(\sqrt{t}).$$

I.e. Thm: If $X \sim S(0,\alpha)$, the $S^2 \sim \nu_f(\lambda=1,\alpha)$.

This relation does not hold in the classical case - the square of a normal r.v. is not Poissonian.

Remark: The combinatorial explanation is that

$$\# NC_2(2n) = \# NC(n)$$

BUT

$$\# \mathcal{P}_2(2n) \neq \# \mathcal{P}(n)$$

I.e. relations like this happen in free probability but not in classical probability because the lattice $NC(n)$ has more structure / symmetry than the lattice $\mathcal{P}(n)$.