Lecture 5: April 6, 2011

Let's take a direct look at realizing $\mathbb{C}G$ as an operator algebra, for discrete G.

$\|h\|_2^2 = \sum_x |h(x)|^2$

Def: $\ell^2(G) = \{h : G \to \mathbb{C} : \sum_{x \in G} |h(x)|^2 < \infty\} \equiv H_G$.

this sum makes sense since G is discrete (ergo countable)

H_G is a Hilbert space, and $\mathbb{C}G$ acts on it basically the same way it acts on itself: through convolution:

for $f \in \mathbb{C}G$, $h \in H_G$, define $f * h \in H_G$

$$f * h\,(x) = \sum_{y \in G} f(xy^{-1}) h(y).$$

because $f \in \mathbb{C}G$ is finitely-supported, this is just a finite sum, so at least $|f(h)(x)| < \infty$ for all x. We need to check that $f * h \in H_G = \ell^2(G)$.

First, it's convenient to reverse the order of multiplication in the convolution product:

$$f * h\,(x) = \sum_y f(xy^{-1}) h(y) = \sum_z f(z) h(z^{-1}x) \quad (\neq h * f(x) \text{ if } G \text{ is not abelian})$$

$z = xy^{-1}$ runs through all of G when y does

Now, for fixed $z \in G$, define $g_z(x) = f(z) h(z^{-1}x)$. Thus $f * h = \sum_z g_z$. So we use the triangle inequality for $\| \ \|_2$:

$$\|f*h\|_2 = \|\sum_z g_z\|_2 \leq \sum_z \|g_z\|_2.$$

Well, $\|g_z\|_2^2 = \sum_x |g_z(x)|^2 = \sum_x |f(z)|^2 |h(z^{-1}x)|^2 = |f(z)|^2 \sum_x |h(z^{-1}x)|^2 = |f(z)|^2 \|h\|_2^2$

b/c $\sum_x = \sum_{z^{-1}x}$

Hence $\|g_z\|_2 = |f(z)|\,\|h\|_2$, and so

$$\|f*h\|_2 \leq \sum_{z \in G} |f(z)| \cdot \|h\|_2$$

finite, as f is finitely-supported.

Thus, $f*(\cdot) \in \mathcal{B}(H_G)$.

Note that $f*\mathbb{1}_1 = f$ and $\mathbb{1}_1 \in \ell^2(G)$, so the injection $f \mapsto f*$ is one-to-one. Also, it is easy to see it is an algebra homomorphism – the action on ℓ^2 is defined by $*$, exactly as in $\mathbb{C}G$. So we've id'ed $\mathbb{C}G$ as a subalgebra of $\mathcal{B}(H_G)$. Next we look at the adjoint.

<u>Exercise</u>: Let f^* denote the usual involution on $\mathbb{C}G$: $f^*(x) = \overline{f(x^{-1})}$. Show that, for all $h,k \in \ell^2(G)$, $f \in \mathbb{C}G$,

$$\langle f^* * h, k \rangle_{\ell^2(G)} = \langle h, f*k \rangle_{\ell^2(G)}.$$

Thus, the $*$ on $\mathbb{C}G$ corresponds to the $*$ on $\mathcal{B}(H_G)$ under the identification of $\mathbb{C}G \hookrightarrow \mathcal{B}(H_G)$ via $f \mapsto f*$. I.e. this inclusion is a $*$-algebra homomorphism.

So, $\mathbb{C}G$ is an operator algebra, acting on H_G. To complete the GNS picture, we only need to see that φ_G is a vector state. Well, indeed:

$$f*\mathbb{1}_1(x) = \sum_y f(xy^{-1})\,\mathbb{1}_1(y) = f(x)$$

$$\text{so } \langle f*\mathbb{1}_1, \mathbb{1}_1 \rangle_{\ell^2(G)} = \sum_x f*\mathbb{1}_1(x)\cdot\overline{\mathbb{1}_1(x)} = f*\mathbb{1}_1(1) = f(1) = \varphi_G(f).$$

Since $\mathbb{1}_1 \in \ell^2(G)$ is a unit vector, this shows $\varphi_G = \varphi_{\mathbb{1}_1}$ is a vector state on $\mathbb{C}G \subset \ell^2(G)$. So we have fully realized $\mathbb{C}G$ in the GNS framework.

Note: <u>any</u> $*$-subalgebra of $\mathcal{B}(H)$ has the positive dominating property that if $x = x^*$ $\exists\, \lambda_x > 0$ with $\lambda_x 1 - x \geq 0$ (e.g. take $\lambda_x = \|x\|$).

this means ≥ 0 in $\mathcal{B}(H)$ – i.e. $\exists\, b \in \mathcal{B}(H)$ s.t. $\lambda_x 1 - x = bb^*$. But there is <u>no</u> guarantee $b \in$ the original algebra. It is a (possibly hard) question whether $b \in \mathbb{C}G$ in this example.

<u>Remark</u>: Consider the restriction of this inclusion to $G < \mathbb{C}G$: $x \in G \hookrightarrow \mathbb{1}_x \in \mathbb{C}G$.

$$\mathbb{1}_x * h(u) = \sum_y \mathbb{1}_x(y)\,h(y^{-1}u)$$
$$= h(x^{-1}u).$$

This action $G \curvearrowright \ell^2(G)$ by $x\cdot h(u) = h(x^{-1}u)$ is called the Left-Regular Representation $\Lambda(G)$. Note: $\|x\cdot h\|_2^2 = \sum_y |h(x^{-1}y)|^2 = \sum_z |h(z)|^2 = \|h\|_2^2$, so Λ is a <u>unitary</u> representation. We will come back to it later.

Distributions of Random Variables

In classical probability (Ex 1), the information contained in a single random variable can be summarized by its distribution or law.

Def: Given $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, μ_X is the probability measure on $\mathbb{C}$ (or $\mathbb{R}$ in the $\mathbb{R}$-valued case) given by

$$\mu_X(\underset{\substack{\uparrow \\ \text{Borel subset of } \mathbb{C}}}{B}) = \mathbb{P}(X \in B) = \mathbb{P}(\underbrace{X^{-1}(B)}_{\substack{\text{an event in } \mathcal{F}, \text{ since} \\ X \text{ is measurable.}}})$$

We are primarily concerned with bounded random variables, meaning there is some compact set $K \subset \mathbb{C}$ s.t. $\text{range}(X) \subseteq K$ with probability 1. This means the distribution μ_X is supported in K. I.e. we will be thinking mostly about compactly-supported prob. measures.

μ_X is a probability measure on $\mathbb{C}$. This means the set $\mathcal{F}$ of events is the set of Borel subsets of $\mathbb{C}$. In this case, we can use calculus to describe "most" such measures.

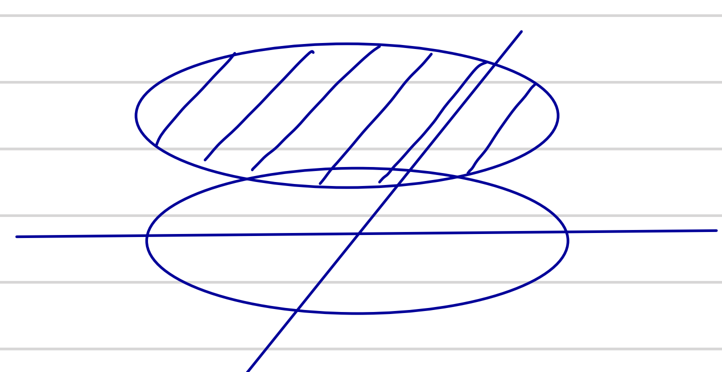
A probability density ρ is a function $\rho : \mathbb{C} \to [0, \infty)$ that is integrable and such that

$$\int_{\mathbb{C}} \rho(z) \, |dz| = 1.$$

Eg. Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. The uniform probability measure on $\mathbb{D}$ is

$$\mu(\underset{\substack{\uparrow \\ \text{Borel set}}}{B}) = \frac{1}{\pi} \underset{\substack{\uparrow \\ \text{Lebesgue measure (Area)}}}{\lambda}(B \cap \mathbb{D}) = \int_B \mathbb{1}_{\mathbb{D}}(z) \, |dz|$$

So μ has density $\rho(z) = \frac{1}{\pi} \mathbb{1}_{\mathbb{D}}(z)$.



(We will come across this distribution plenty in the material to come, in free probability and random matrices.)

If ρ is a probability density, then $\mu(B) = \int_B \rho(z)\,|dz|$ is a probability measure. Not every probability measure on $\mathbb{C}$ has this form, but all can be <u>approximated</u> by densities. So it is fine (for most of our purposes) to think just in terms of densities.

A notable exception is: if X is a real-valued r.v., then for $B \subseteq \mathbb{C}$,

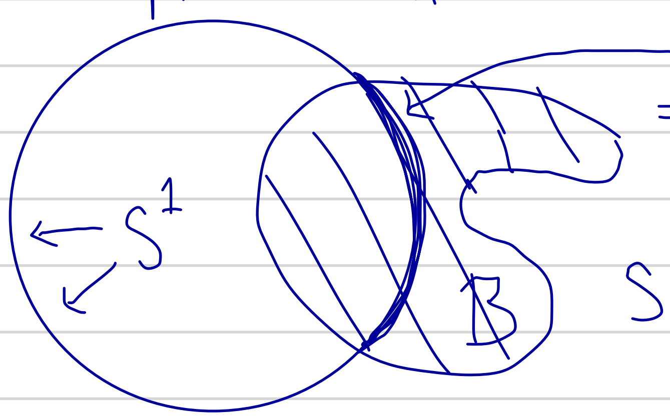
$$\mu_X(B) = \mathbb{P}(X \in B) = \mathbb{P}(X \in B \cap \mathbb{R}) \leftarrow \text{So } \mu_X \text{ is supported in } \mathbb{R}, \text{ and cannot have a density on } \mathbb{C}.$$

In this case, μ_X may have a density on $\mathbb{R}$: $\rho : \mathbb{R} \to [0,\infty)$ s.t.

$$\mu_X(B) = \int_B \rho(x)\,dx, \quad B \subseteq \mathbb{R}.$$

Similarly, a probability measure on $\mathbb{C}$ might be supported on a different lower-dimensional subset, where it has a density. Here is an important example we will see:

This is sometimes called the <u>Haar</u> measure on S^1.



$\mu(B) = \frac{1}{2\pi}$ (length of the arc $B \cap S^1$).

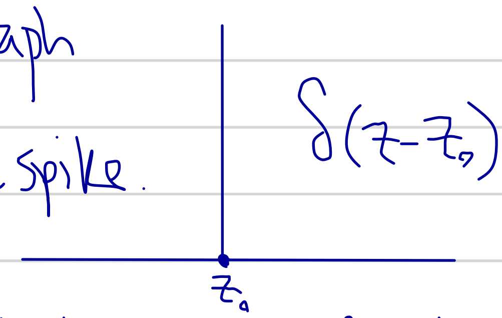
So "$d\mu = \frac{1}{2\pi} d\theta$".

A measure might be supported on an even smaller subset: for example point masses

$$\delta_{z_0}(B) = \begin{cases} 1, & z_0 \in B \\ 0, & z_0 \notin B \end{cases}$$

This measure has no density... but if it did, it would be what Physicists call a Dirac Delta Function.

the graph is an infinite spike.



And... there are even weirder measures that are not finite combinations of these. But we won't come across any in this course.

So, with all these different kinds of measures, how are we supposed to recognize the distribution of a random variable <u>algebraically</u>, in an NCPS?

Moments

The definition of μ_X relies heavily on the realization of X as a function $\Omega \to \mathbb{C}$. But, fortunately, there is an algebraic way to recognize the distribution. For now, we stick to the real-valued case.

Prop: Let X be a bounded $\mathbb{R}$-valued random variable. Let μ_X be its distribution. Then (the change-of-variable formula asserts that) if $f: \mathbb{R} \to \mathbb{R}$ is a continuous function,

$$\mathbb{E}(f(X)) = \int_{\mathbb{R}} f(x)\, \mu_X(dx)$$

What's more, μ_X is determined by its integrals againts polynomials. In particular, there is a unique probability measure μ on $\mathbb{R}$ such that, for $n \geq 1$,

$$\int_{\mathbb{R}} x^n \mu(dx) = \mathbb{E}(X^n).$$

This uniqueness means the following: given some sequence $m_1, m_2, m_3, \ldots$ in $\mathbb{R}$, there is at most one compactly-supported probability measure μ on $\mathbb{R}$ with the property that

$$m_n = \int_{\mathbb{R}} x^n \mu(dx), \quad n = 1, 2, 3, \ldots$$

The general abstract question of which sequences exactly are moment sequences (for some probability measure) is a difficult one. (There is a general "positive-definite" condition that is known, but it is all-but-impossible to verify.) We will not concern ourselves with this question here. Rather, we simply define:

Def: Let $(\mathcal{A}, \varphi)$ be a NCPS, and let $x = x^* \in \mathcal{A}$. The distribution μ_x of x (if it exists) is the unique compactly-supported prob. measure on $\mathbb{R}$ with moments given by $\varphi(x^n)$:

$$\int_{\mathbb{R}} t^n \mu_x(dt) = \varphi(x^n), \quad n = 1, 2, 3, \ldots$$

We can dispense with the question of <u>whether</u> such a measure exists provided we are in the operator-algebra context.

<u>Exercise</u>: (Only for those who've taken functional analysis.)

Let $\mathfrak{A} \subseteq \mathcal{B}(H)$ be a $*$-subalgebra, and let φ be a faithful positive state on $\mathfrak{A}$ that is continuous wrt the weak*-topology on $\mathcal{B}(H)$.

(a) Show that φ extends uniquely to a faithful, positive state $\overline{\varphi}$ on $W^*(\mathfrak{A})$ (the weak*-closure of $\mathfrak{A}$ in $\mathcal{B}(H)$), and that this extension is tracial iff φ is.

(b) Let $x \in \mathfrak{A}$ be self-adjoint, with spectral resolution (i.e. projection-valued measure) E^x. Show that E^x is supported in $W^*(\mathfrak{A})$. Show that $\overline{\varphi} \circ E^x$ is a probability measure on $\mathbb{R}$, with support $= \mathrm{spec}(x)$.

(c) Show that
$$\int_{\mathbb{R}} t^n \, \overline{\varphi} \circ E^x(dt) = \varphi(x^n), \quad n = 1, 2, 3, \ldots$$

The point of this exercise is that in the case that our NCPS is an operator algebra (and the state is continuous in the appropriate sense), then every self-adjoint element indeed possesses a distribution determined by its moments.

This is a technical point. The interesting part is:

(1) How do we determine μ_x from the moments of x, constructively?

(2) How do we calculate the moments of x?

It is in answering these questions that we'll get our first glimpse at how combinatorics is used in this subject.

<u>Exercise</u>: Let $A \in M_n(\mathbb{C})$ be self-adjoint, with eigenvalues $\lambda_1, \ldots, \lambda_n$. In the NCPS $(M_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr})$, show that
$$\mu_A = \frac{1}{n}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_n}).$$