Lecture 9: April 15, 2011

Today we discuss <u>free products</u>. As a refresher, recall:

<u>Def</u>: Let $\{G_i : i \in I\}$ be a collection of groups. The free product $G = \ast_{i\in I} G_i$ is the group (unique up to isomorphism) defined by the following universal property. There are homomorphisms $\theta_i : G_i \to G$ such that, whenever H is a group and $\phi_i : G_i \to H$ are any group homomorphisms, $\exists !$ homomorphism $\Phi : G \to H$ such that

$$\begin{tikzcd} G \arrow[rr, "\Phi"] & & H \\ & G_i \arrow[ul, "\theta_i"] \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

Commutes.

By taking the case $H = G$, it is easy to see the θ_i must be injective; so we think of them just as injections of G_i as subgroups of G. The fact that <u>any</u> homomorphisms ϕ_i from the separate G_i must extend to a single homomorphism Φ required that G_i have no relations between them in G (otherwise such relations would have to be present in the image Φ in <u>any</u> group H, which is of course impossible).

Following this course, we can do the same for (unital $\ast$-) algebras.

<u>Def</u>: Let $\{\mathfrak{A}_i : i \in I\}$ be unital $\ast$-algebras over $\mathbb{C}$. Their <u>free product</u> $\mathfrak{A} \equiv \ast_{i\in I} \mathfrak{A}_i$ is the (unique up to isomorphism) unital $\ast$-algebra with the following universal property. There are homomorphisms $\theta_i : \mathfrak{A}_i \to \mathfrak{A}$ s.t., for any unital $\ast$-algebra $\mathfrak{B}$ and homomorphisms $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$, $\exists !$ homomorphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ s.t.

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "\Phi"] & & \mathfrak{B} \\ & \mathfrak{A}_i \arrow[ul, "\theta_i"] \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

Commutes.

The above comments apply here: by taking the case $\mathfrak{B} = \mathfrak{A}$, we see that the θ_i are injective, so we view $\mathfrak{A}_i$ as subalgebras of $\mathfrak{A}$. Note: here, all the $\mathfrak{A}_i$ share the same $1 \in \mathfrak{A}$, so they all intersect along $\mathbb{C}1_{\mathfrak{A}}$. Aside from this, the same "no relations" idea applies in the algebra setting.

Of course, this doesn't mean such an $\mathfrak{A}$ exists!

Construction: To see how to build $\mathfrak{A} \equiv \mathop{\ast}_{i\in I} \mathfrak{A}_i$, we look at some elements it must have.

- As $\mathfrak{A}_i \subset \mathfrak{A}$, we must have all words of the form $a_1 \cdots a_n$ $(n \in \mathbb{N})$ where (as usual) $a_j \in \mathfrak{A}_{i_j}$ and $i_1 \neq i_2 \neq \cdots \neq i_n$ (for uniqueness).
- Again for uniqueness, since $\mathbb{C}1_{\mathfrak{A}} \subseteq \mathfrak{A}_i$ for all i, we should exclude the span of $1_{\mathfrak{A}}$ from such words.

A good way to accomplish this, in line with our interests, is to specify a linear functional $\varphi_i : \mathfrak{A}_i \to \mathbb{C}$ on each algebra, with $\varphi_i(1_{\mathfrak{A}}) \in \mathbb{C}^*$ (e.g. $=1$). Then define

$$\mathring{\mathfrak{A}}_i = \operatorname{Ker} \varphi_i .$$

So, if $i_1, \dots, i_n$ are any consecutively-distinct indices in I, set

$$\mathring{W}_{i_1,\dots,i_n} = \operatorname{span}_{\mathbb{C}} \{ a_1 \cdots a_n : a_j \in \mathring{\mathfrak{A}}_{i_j} \} \subset \mathfrak{A} .$$

Then $\mathfrak{A} \supseteq \mathbb{C}1_{\mathfrak{A}} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathring{W}_{i_1,\dots,i_n}$.

In fact, this will be an equality: if $a_1, \dots, a_n$ are any elements of $\mathfrak{A}_{i_1}, \dots, \mathfrak{A}_{i_n}$ (with $i_1, \dots, i_n$ arbitrary), in the product $a_1 \cdots a_n$ we first combine adjacent elements in the same algebra, thus we assume wlog that $i_1 \neq i_2 \neq \cdots \neq i_n$. Then we center $a_j = \mathring{a}_j + \varphi(a_j)1$ with $\mathring{a}_j \in \mathring{\mathfrak{A}}_{i_j}$, and so

$$a_1 \cdots a_n = (\mathring{a}_1 + \varphi(a_1)1) \cdots (\mathring{a}_n + \varphi(a_n)) = \mathring{a}_1 \cdots \mathring{a}_n + \sum_{j=1}^{n} \varphi(a_j) \mathring{a}_1 \cdots \widehat{\mathring{a}_j} \cdots \mathring{a}_n + \cdots$$

$$\cdots + \varphi(a_1) \cdots \varphi(a_n) 1$$

$$\in \mathring{W}_{i_1,\dots,i_n} \oplus \bigoplus_{j=1}^{n} \mathring{W}_{i_1,\dots,\hat{i}_j,\dots,i_n} \oplus \cdots$$

$$\cdots \oplus \mathbb{C}1_{\mathfrak{A}} .$$

This is basically how to construct $\mathfrak{A}$. But if we really want to show the universal property holds, it is convenient to define the product between different algebras using another universal construction: the tensor-product.

Def / Construction: Given unital $\mathbb{C}$-algebras $\{\mathfrak{A}_i\}_{i\in I}$ and linear functionals $\varphi_i : \mathfrak{A}_i \to \mathbb{C}$ with $\varphi(1_{\mathfrak{A}_i}) = 1$, define the free product

$$\mathfrak{A} = \underset{i\in I}{*} \mathfrak{A}_i = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{i_1\neq\cdots\neq i_n} \mathring{\mathfrak{A}}_{i_1} \otimes \cdots \otimes \mathring{\mathfrak{A}}_{i_n}$$

where $\mathring{\mathfrak{A}}_i = \ker \varphi_i$.

This defines $\mathfrak{A}$ as a vector-space. The product can then be specified using $\otimes$ and the product within each algebra. For example, if $a_1 \otimes \cdots \otimes a_n$ and $b_1 \otimes \cdots \otimes b_m$ are words with $a_n \in \mathring{\mathfrak{A}}_{i_n}$, $b_1 \in \mathring{\mathfrak{A}}_{j_1}$ and $i_n \neq j_1$, we simply have

$$a_1 \otimes \cdots \otimes a_n \cdot b_1 \otimes \cdots \otimes b_m = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m .$$

Linearity of $\otimes$ makes this compatible with the decomposition $a_i = \mathring{a}_i + \varphi(a_i)$, as can be easily worked out.

But if the middle letters are from the same algebra, greater care must be taken. For the lowest case, take

$a_1, b_1 \in \mathring{\mathfrak{A}}_1$ and $a_2, b_2 \in \mathring{\mathfrak{A}}_2$. What is $a_1 \otimes a_2 \cdot b_2 \otimes b_1$?

If we can do this, the general case $a_1, b_1 \in \mathfrak{A}_1$, $a_2, b_2 \in \mathfrak{A}_2$ follows by decomposing $a_1 = \mathring{a}_1 + \varphi_1(a_1) 1$ etc. The naïve attempt would be

$$a_1 \otimes a_2 \cdot b_2 \otimes b_1 \overset{?}{=} a_1 \otimes (a_2 b_2) \otimes b_1$$

but this is not in $\mathring{\mathfrak{A}}_2$.

So we center it: $a_2 b_2 = (a_2 b_2)^\circ + \varphi_2(a_2 b_2)$. Using this, the above answer would be

$$= a_1 \otimes (a_2 b_2)^\circ \otimes b_1 + \varphi_2(a_2 b_2)\, a_1 \otimes b_1$$

but this combination $\mathring{\mathfrak{A}}_1 \otimes \mathring{\mathfrak{A}}_1$ never appears in $\mathfrak{A}$ (adjacent 1s).

The trick is: in this process, whenever two factors from the same algebra appear next to each other separated by a $\otimes$, remove the $\otimes$. So the next "approximation" is

$$a_1 \otimes a_2 \cdot b_2 \otimes b_1 \overset{?}{=} a_1 \otimes (a_2 b_2)^\circ \otimes b_1 + \varphi_2(a_2 b_2)\, a_1 b_1 .$$

This turns out to be correct, but we need to recenter to put it into the right decomposed form:

$$a_1 \otimes a_2 \cdot b_2 \otimes b_1 = a_1 \otimes (a_2 b_2)^\circ \otimes b_1 + \varphi_2(a_2 b_2)\,(a_1 b_1)^\circ + \varphi_2(a_2 b_2)\varphi_1(a_1 b_1).$$
$$\in \mathring{\mathcal{O}}_1 \otimes \mathring{\mathcal{O}}_2 \otimes \mathring{\mathcal{O}}_1 \oplus \mathring{\mathcal{O}}_1 \oplus \mathbb{C} \subset \mathcal{O}.$$

Through this process, the product on $\mathcal{O}$ is defined inductively. It is now a simple matter to check that it is well-defined and makes $\mathcal{O}$ into a $\mathbb{C}$-algebra, whose unit is $1 \in \mathbb{C}$.

As for the involution (to make it a $*$-algebra), it suffices to define $*$ on the "word spaces" $\mathring{\mathcal{O}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{O}}_{i_n}$ for $i_1 \neq \cdots \neq i_n$. There is, of course, only one way to define it so that it is an involution on $\mathcal{O}$ and reduces to the $*$ on each $\mathcal{O}_i$:

$$\mathring{\mathcal{O}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{O}}_{i_n} \ni a_1 \otimes \cdots \otimes a_n \mapsto (a_1 \otimes \cdots \otimes a_n)^* = a_n^* \otimes \cdots \otimes a_1^*.$$

It is now just an easy matter to check that $(a \cdot b)^* = b^* \cdot a^*$ for $a, b \in \mathcal{O}$. Thus, $\mathcal{O}$ is a unital $*$-algebra that "contains" all the $\mathcal{O}_i$: the injections are

$$\theta_i : \mathcal{O}_i \longrightarrow \mathbb{C} \oplus \mathring{\mathcal{O}}_i \subset \mathcal{O}$$
$$a \longmapsto \varphi_i(a) + \mathring{a}$$

Remark: This construction seems to depend on the particular choices of functionals φ_i. Indeed, the specific decomposition above depends on them, but any other choices of φ_i (with $\varphi_i(1) \neq 0$) would yield an isomorphic object, because:

Prop: this construction of $\mathcal{O} = \underset{i \in I}{\ast} \mathcal{O}_i$ indeed has the universal property described on page 1.

Pf. Exercise. [Hint: lookup and use the universal property for tensor products of rings.]

Remark: Since the universal object $\ast \mathcal{O}_i$ is uniquely-defined up to isomorphisms, this shows any choices of φ_i will give the same object. Indeed, φ_i just determines the specific form of θ_i.

Now, in our world (of NCPS's), having specific choices of linear functionals $\varphi_i : \mathfrak{A}_i \to \mathbb{C}$ around is quite natural anyhow. This leads us to

Free Products of NCPS's.

Now, suppose we have a collection of non-commutative prob. spaces

$$\{(\mathfrak{A}_i, \varphi_i)\}_{i \in I}.$$

The φ_i give us a concrete representation of the $*$-algebra $\mathop{*}_{i \in I} \mathfrak{A}_i \equiv \mathfrak{A}$. The question is, is there a faithful positive state φ on $\mathfrak{A}$ such that $\varphi|_{\mathfrak{A}_i} = \varphi_i$? This would be the free version of "product measure".

In fact, there is an obvious choice for how such φ must be defined.

Def / Thm: Let $\varphi \equiv \mathop{*}_{i \in I} \varphi_i$ be defined on $\mathfrak{A} = \mathop{*}_{i \in I} \mathfrak{A}_i$ by:

- $\varphi(1) = 1$
- $\varphi|_{\mathfrak{A} \ominus \mathbb{C}} = 0$.

Then φ is a state st. $\varphi|_{\mathfrak{A}_i} = \varphi_i$. Moreover, the algebras $\mathfrak{A}_i$ are ($*$-)free wrt φ.

Pf. φ is defined to be a linear functional on $\mathfrak{A}$ with $\varphi(1) = 1$, so φ is a state. Moreover, for $a \in \mathfrak{A}_i$, the decomposition of $a \in \mathfrak{A}$ is $a = \varphi_i(a) + a - \varphi_i(a) \in \mathbb{C} \oplus \mathfrak{A}_i$, so $\varphi(a) = \varphi(\varphi_i(a)) + \varphi(\text{something in } \mathfrak{A} \ominus \mathbb{C})$

$$= \varphi_i(a).$$

Now, if $a_1, \ldots, a_n$ are in $\mathfrak{A}$ with $a_j \in \mathfrak{A}_{i_j}$ where $i_1 \neq i_2 \neq \cdots \neq i_n$, and such that $\varphi(a_j) = 0$, then $\varphi_{i_j}(a_j) = 0$ so $a_j \in \mathring{\mathfrak{A}}_{i_j}$, which means

$$a_1 \cdots a_n \in W_{i_1, \ldots, i_n} = \mathring{\mathfrak{A}}_{i_1} \otimes \cdots \otimes \mathring{\mathfrak{A}}_{i_n} \subset \mathfrak{A} \ominus \mathbb{C}$$

Thus, by definition $\varphi(a_1 \cdots a_n) = 0$. Hence, the $\mathfrak{A}_i$ are free in $(\mathfrak{A}, \varphi)$. Since they are $*$-algebras, they are $*$-free. ///

Exercise: Let G_i be groups. Show that $(\ast \mathbb{C}G_i, \ast \varphi_{G_i})$ is isomorphic to $(\mathbb{C}(\ast G_i), \varphi_{\ast G_i})$.

Remark: It is easy to see that φ is the only state on $\mathfrak{A}$ that reduces to φ_i on $\mathfrak{A}_i$ such that the $\mathfrak{A}_i$ are free in $(\mathfrak{A}, \varphi)$.

HOWEVER: we have yet to show that φ is positive & faithful.

To proceed with these properties, we actually need an important result from (classical) matrix theory.

Def: Let $A, B \in M_n(\mathbb{C})$. The Schur product $A \circ B \in M_n(\mathbb{C})$ is the pointwise product

$$[A \circ B]_{ij} = A_{ij} B_{ij} \quad 1 \leq i, j \leq n.$$

Lemma: If $A, B \in M_n(\mathbb{C})$ are positive, then so is $A \circ B$.

Pf. As usual, positive means $A = XX^*$ and $B = YY^*$ for some matrices $X, Y \in M_n(\mathbb{C})$. But here we utilize standard linear alg.: we can alternatively characterize positivity of a matrix as

$$A \geq 0 \iff A = A^* \text{ and has all } \geq 0 \text{ eigenvalues}$$
$$\iff \langle Av, v \rangle \geq 0 \text{ for all } v \in \mathbb{C}^n.$$

So we will check the last condition for $A \circ B$. Fix $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$. Then

$$\langle A \circ B v, v \rangle = \sum_{j=1}^n [A \circ B v]_j \bar{v}_j = \sum_{i,j=1}^n [A \circ B]_{ij} v_i \bar{v}_j$$
$$= \sum_{i,j=1}^n A_{ij} B_{ij} v_i \bar{v}_j.$$

Now, as $A \geq 0$, $A = XX^*$ for some $X \in M_n(\mathbb{C})$. That is:

$$[A]_{ij} = [XX^*]_{ij} = \sum_{k=1}^n [X]_{ik} [X^*]_{kj} = \sum_{k=1}^n X_{ik} \bar{X}_{jk}.$$

Thus: $$\langle A \circ B v, v \rangle = \sum_{i,j,k=1}^n X_{ik} \bar{X}_{jk} B_{ij} v_i \bar{v}_j = \sum_{k=1}^n \left(\sum_{i,j=1}^n B_{ij} (v_i X_{ik}) \overline{(v_j X_{jk})} \right)$$

$$w_k \equiv (v_1 X_{1k}, \ldots, v_n X_{nk}) \longrightarrow \quad = \sum_{k=1}^n \langle B w_k, w_k \rangle \geq 0. \quad ///$$

We need one more little lemma, which is a polarization of the definition of positive in a NCPS.

<u>Lemma</u>: Let $(\mathfrak{A},\varphi)$ be a NCPS, and let $a_1,\dots,a_n \in \mathfrak{A}$. Then the matrix

$$[A]_{ij} = \varphi(a_i a_j^*)$$

is positive.

Pf. For any $v \in \mathbb{C}^n$, we have

$$\langle Av,v\rangle = \sum_{i,j=1}^{n} A_{ij} v_i \bar{v}_j = \sum_{i,j=1}^{n} \varphi(a_i a_j^*) v_i \bar{v}_j$$
$$= \varphi\Big(\sum_{i,j=1} v_i a_i \cdot (v_j a_j)^*\Big) = \varphi\Big(\Big(\sum_{i=1}^{n} v_i a_i\Big)\Big(\sum_{i=1}^{n} v_i a_i\Big)^*\Big)$$

and this is ≥ 0 b/c $\varphi \geq 0$. ///

<u>Remark</u>: The definition of $\varphi \geq 0$ is the $n=1$ case, so in fact the statement of the Lemma is equivalent to $\varphi \geq 0$.

<u>Theorem</u>: If $(\mathfrak{A}_i,\varphi_i)$ are NCPS's, then $(\ast\mathfrak{A}_i, \ast\varphi_i)$ is a NCPS.

Pf. This is a fairly involved proof, that uses a lot of the tools we've developed. In particular, we will use the orthogonality relations of freeness we developed in Lecture.

Let $a \in \mathfrak{A} = \ast\mathfrak{A}_i$. Then $a = \sum_{n=0}^{N} \sum_{i_1\neq\cdots\neq i_n}^{\text{finite}} a_{i_1\cdots i_n}$, where $a_{i_1\cdots i_n} \in \mathring{\mathfrak{A}}_{i_1}\otimes\cdots\otimes\mathring{\mathfrak{A}}_{i_n}$.

Then

$$aa^* = \sum_{n,m=0}^{N} \sum_{i_1\neq\cdots\neq i_n}^{\text{finite}} \sum_{j_1\neq\cdots\neq j_m}^{\text{finite}} a_{i_1\cdots i_n} a_{j_1\cdots j_m}^*.$$

Taking φ, we look at inner-products $\varphi(a_{i_1\cdots i_n} a_{j_1\cdots j_m}^*)$.

By the orthogonality relations, $\varphi(\;\cdot\;)=0$ unless $(i_1,\dots,i_n)=(j_1,\dots,j_m)$.

$\Big\{$ is a lin. comb. of products $a_1\cdots a_n$ with $a_\ell \in \mathring{\mathfrak{A}}_{i_\ell}$

is a lin. comb. of products $a'_m\cdots a'_1$ with $a'_\ell \in \mathring{\mathfrak{A}}_{j_\ell}$

$$\Rightarrow \varphi(aa^*) = \sum_{n=0}^{N} \sum_{i_1\neq\cdots\neq i_n}^{\text{finite}} \varphi(a_{i_1\cdots i_n} a_{i_1\cdots i_n}^*)$$

$\Big\{$ Thus, only have to prove that $\varphi|_{\mathring{\mathfrak{A}}_{i_1}\otimes\cdots\otimes\mathring{\mathfrak{A}}_{i_n}}$ is ≥ 0 and faithful.

So, fix a specific $(i_1,\dots,i_n)$ with $i_1\neq i_2\neq\cdots\neq i_n$. Let $b\in\mathring{\mathfrak{A}}_{i_1}\otimes\cdots\otimes\mathring{\mathfrak{A}}_{i_n}$.

<u>Positivity</u>: Decompose $b=\sum_{k=1}^{L} a_1^k a_2^k\cdots a_n^k \quad a_j^k\in\mathring{\mathfrak{A}}_{i_j}$. So

$$\begin{aligned}\varphi(bb^*) &= \sum_{k,l=1}^{L}\varphi(a_1^k\cdots a_n(a_n^l)^*\cdots(a_1^l)^*) && \text{again by the orthogonality relations}\\ &= \sum_{k,l=1}^{L}\varphi(a_1^k a_1^{l*})\cdots\varphi(a_n^k a_n^{l*}) && \text{since } \varphi|_{\mathfrak{A}_i}=\varphi_i\\ &= \sum_{k,l=1}^{L}\varphi_{i_1}(a_1^k a_1^{l*})\cdots\varphi_{i_n}(a_n^k a_n^{l*}).\end{aligned}$$

So, for $1\le m\le n$, let $B_m\in M_L(\mathbb{C})$ be the matrix $[B_m]_{kl}=\varphi_{i_m}(a_m^k a_m^{l*})$. By the second lemma, $B_m\geq 0$ since $\varphi_{i_m}\geq 0$. Hence, by induction on the first lemma, the Schur product $B_1\odot\cdots\odot B_n\geq 0$. Thus, taking the vector $\mathbb{1}=(1,\dots,1)\in\mathbb{C}^n$,

$$0\le\langle B_1\odot\cdots\odot B_n\mathbb{1},\mathbb{1}\rangle=\sum_{k,l=1}^{L}[B_1\odot\cdots\odot B_n]_{kl}\mathbb{1}_k\bar{\mathbb{1}}_l=\sum_{k,l=1}^{L}[B_1]_{kl}\cdots[B_n]_{kl}$$
$$=\varphi(bb^*).$$

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<u>Faithful-ity</u>: Proceed by induction on n. The case $n=0$ is just the statement $|z|^2=0\Rightarrow z=0$ for $z\in\mathbb{C}$; for $n=1$ it is just the (assumed) statement that $\varphi_i>0$. So, assume we know φ is faithful on words of length $n-1$. We can decompose

$$\mathring{\mathfrak{A}}_{i_1}\otimes\cdots\otimes\mathring{\mathfrak{A}}_{i_n}\ni b=x_1y_1+\cdots+x_my_m \text{ for some } m,\ x_j\in\mathring{\mathfrak{A}}_{i_1},\ y_j\in\mathring{\mathfrak{A}}_{i_2}\otimes\cdots\otimes\mathring{\mathfrak{A}}_{i_n}.$$

Note: If $x_1,\dots,x_m$ are linearly-dependent, we can regroup the decomposition shrinking m; so wlog they are linearly-independent. So, if $\varphi(bb^*)=0$,

$$0=\varphi(bb^*)=\sum_{k,l=1}^{m}\varphi(x_ky_ky_l^*x_l^*)=\sum_{k,l=1}^{m}\varphi(x_kx_l^*)\varphi(y_ky_l^*)\quad\text{(by the orthogonality relations)}$$

Now, as proved above, $\varphi\geq0$, so by the second lemma, the matrix $[\varphi(y_ky_l^*)]_{kl}$ is positive. This means we can write it in the form BB^*. If $[B]_{kl}=\beta_{kl}$, this means that we can write

$$\varphi(y_ky_l^*)=\sum_{p=1}^{m}\beta_{kp}\bar{\beta}_{lp},\quad 1\le k,l\le m.$$

$$\therefore 0 = \varphi(bb^*) = \sum_{k,\ell=1}^{m} \varphi(x_k x_\ell^*)\varphi(y_k y_\ell^*) = \sum_{k,\ell,p=1}^{m} \varphi(x_k x_\ell^*)\beta_{kp}\overline{\beta_{\ell p}}$$

$$= \sum_{p=1}^{m} \varphi\Big(\sum_{k,\ell=1}^{m} \beta_{kp} x_k \cdot (\beta_{\ell p} x_\ell)^*\Big)$$

$$= \sum_{p=1}^{m} \varphi\Big(\big(\sum_{k=1}^{m} \beta_{kp} x_k\big)\cdot\big(\sum_{k=1}^{m} \beta_{kp} x_k\big)^*\Big).$$

Thus, since $\varphi \geq 0$, this sum of ≥ 0 terms vanishing implies that each of the m terms vanishes. But the terms are

$$\forall p \in \{1,\dots,m\} \quad 0 = \varphi\Big(\big(\sum_{k=1}^{m} \beta_{kp} x_k\big)\big(\sum_{k=1}^{m} \beta_{kp} x_k\big)^*\Big) = \varphi_{i_1}(\text{'' ''}) \quad \text{b/c the } x_k \in \mathring{\mathcal{A}}_{i_1}$$

$$\Rightarrow \sum_{k=1}^{m} \beta_{kp} x_k = 0 \quad \text{b/c } \varphi_{i_1} \text{ is faithful}$$

$$\Rightarrow \forall k,p \;\; \beta_{kp} = 0 \quad \text{b/c the } x_k \text{ are linearly-independent.}$$

But remember how β_{kp} was defined: $\varphi(y_k y_\ell^*) = \sum_{p=1}^{m} \beta_{kp}\overline{\beta_{\ell p}}$. In particular,

$$0 = \sum_{p=1}^{m} \beta_{kp}\overline{\beta_{kp}} = \varphi(y_k y_k^*) \Rightarrow y_k = 0 \;(\forall k) \quad \text{b/c } \varphi \text{ is faithful on } \mathring{\mathcal{A}}_{i_2} \otimes \cdots \otimes \mathring{\mathcal{A}}_{i_n}$$

(by inductive hypothesis).

Thus $b = x_1 y_1 + \cdots + x_m y_m = 0$. This concludes the proof. ///

So we have (finally!) shown that $(\ast \mathcal{A}_i, \ast \varphi_i)$ is a NCPS.

Thus, we can prove that:

<u>Theorem</u>: Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of compactly-supported pr probability measures on $\mathbb{R}$. There exists a NCPS $(\mathcal{A}, \varphi)$ containing a sequence $(x_n)_{n=1}^{\infty}$ of self-adjoint elements with $\mu_{x_n} = \mu_n$ such that the x_n are all <u>free</u>.

<u>Pf</u>. Note, in $L^\infty(\mathbb{R}, \mu_n)$, the distribution of the coordinate r.v. $X(t) = t$ is μ_n. Thus, take

$$(\mathcal{A}, \varphi) = \Big(\mathop{\ast}_{n=1}^{\infty} L^\infty(\mathbb{R}, \mu_n), \; \mathop{\ast}_{n=1}^{\infty} \mathbb{E}_{\mu_n}\Big).$$

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