Math 251C: Spring 2019
Homework

Available	Sunday, May 19	Due	Wednesday, June 12

Turn in the homework by 1:00pm to Kemp's mailbox.

1. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$, and suppose that $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ decompose as the direct sum of two Lie subalgebras. Show that there are closed, simply connected subgroups $H_1, H_2 \subseteq G$ such that $\mathrm{Lie}(H_j) = \mathfrak{h}_j$ for $j = 1, 2$, and G is isomorphic to $H_1 \times H_2$ as a Lie group.

2. Let $Q \in \mathrm{SO}(3)$. Show that 1 is an eigenvalue of Q. Let v by an eigenvector with eigenvalue 1; show that Q is a rotation in the plane $v^\perp$.

3. Use the spectral theorem to show that the exponential map $\exp\colon \mathfrak{u}(n) \to \mathrm{U}(n)$ is surjective, but not injective.

4. Let M be a connected smooth manifold, and suppose (E_1, π_1) and (E_2, π_2) are two simply connected smooth covering spaces for M. Show that there is a unique diffeomorphism $\Phi\colon E_1 \to E_2$ such that $\pi_2 \circ \Phi = \pi_1$.

5. Let X, Y, Z be the usual basis for the Heisenberg Lie algebra $\mathfrak{h}(3, \mathbb{R})$, satisfying $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$. Let V be the subspace of $\mathfrak{h}(3, \mathbb{R})$ spanned by $\{X, Y\}$, which is *not* a Lie subalgebra. Let $G = \langle \exp V \rangle$ be the subgroup generated by the exponential image of V. Show that $G = \mathrm{H}(3, \mathbb{R})$.

6. Let G be a Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$, let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra, and let $H \subseteq G$ be the unique connected Lie subgroup with $\mathrm{Lie}(H) = \mathfrak{h}$. Suppose there exists a simply connected *compact* Lie group K with $\mathrm{Lie}(K) \cong \mathfrak{h}$. Show that H is then closed. Is it necessarily true that $H \cong K$?

7. Recall that $\mathrm{SU}(2)$ is the diffeomorphic image of the 3-sphere $\mathbb{S}^3 = \{(\alpha, \beta) \in \mathbb{C}^2 \colon |\alpha|^2 + |\beta|^2 = 1\}$ under the map

$$F(\alpha, \beta) = \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}.$$

For each $U \in \mathbb{SU}(2)$, let $\mathbf{v}_U = F^{-1}(U)$ denote the corresponding unit vector in $\mathbb{C}^2$. Let θ_U be the "polar angle": the angle in $[0, \pi]$ that $\mathbf{v}_U$ makes with the north pole $(1, 0)$.

(a) Suppose $U \in \mathrm{SU}(2)$ has eigenvalues $e^{i\theta}$ and $e^{-i\theta}$, with $\theta \in [0, \pi]$. Show that $\theta_U = \theta$. [*Hint:* take the trace.]

(b) Conclude that two matrices are conjugate in $\mathrm{SU}(2)$ if and only if they have the same polar angle.

8. Show that every Lie group homomorphism $\mathbb{T}^k \to \mathbb{S}^1$ has the form

$$(u_1, \ldots, u_k) \mapsto u_1^{m_1} \cdots u_k^{m_k}$$

for some integers $m_1, \ldots, m_k \in \mathbb{Z}$. [Hint: consider the induced Lie algebra homomorphism.]

9. Consider the compact connected Lie group $\mathrm{SO}(n)$ with $n \geq 3$. Let H be the abelian subgroup consisting of diagonal matrices in $\mathrm{SO}(n)$. Show that H is a maximal abelian subgroup, but that H is not contained in a maximal torus. [*Hint*: If A and B are commuting matrices and λ is an eigenvalue of A with eigenspace E_λ, then $B(E_\lambda) \subseteq E_\lambda$.]

10. In contrast to the previous exercise: show that any maximal abelian subgroup of $\mathrm{SU}(n)$ ($n \geq 2$) is conjugate to a subgroup of the diagonal subgroup; hence, every maximal abelian subgroup of $\mathrm{SU}(n)$ is contained in a maximal torus.

11. Let $\mathfrak{h}(3, \mathbb{C}) = \mathfrak{h}(3, \mathbb{R})_{\mathbb{C}}$ (complex strictly upper-triangular 3×3 matrices).

(a) Show that the maximal abelian subalgebras of $\mathfrak{h}(3, \mathbb{C})$ are precisely the 2-dimensional subalgebras that contain the center.

(b) Show that $\mathfrak{h}(3, \mathbb{C})$ does not have any Cartan subalgebras (according to our definition: i.e. no maximal abelian subalgebra $\mathfrak{h}$ such that ad_H is diagonalizable for all $H \in \mathfrak{h}$).

12. Give an example of a maximal commutative subalgebra of $\mathfrak{sl}(2; \mathbb{C})$ that is not a Cartan subalgebra.

13. Consider the root system D_n ($n \geq 2$) realized as the set of vectors $\{\pm e_j \pm e_k \colon j < k\}$ in $\mathbb{R}^n$. Show that the Weyl group of D_n is the group of transformations of $\mathbb{R}^n$ expressible as a composition of a permutation of the entries and an even number of sign changes.

14. Show that the Weyl group of the root system A_n does not contain $-I$ unless $n = 1$.

15. Let K be a compact Lie group, and let (V, Π) be a representation. For each matrix $A \in \mathrm{End}(V)$, define $f_A \colon K \to \mathbb{C}$ by $f_A(x) = \mathrm{Tr}(\Pi(x)A)$.

(a) Show that $f_{\Pi,A}$ is a representative function for K, and that the character of Π is of the form f_A for some A.

(b) Show that if Π is irreducible then $f_A \equiv 0$ iff $A = 0$.

(c) Show that f_A is $C^\infty(K)$. If $\mathfrak{k}$ is the Lie algebra of K and $X \in \mathfrak{k}$, with associated left-invariant vector field $\widetilde{X}$, then

$$\widetilde{X} f_A = f_{\Pi_*(X)A}.$$