Measurement

To keep things simple, let's talk about measuring angle segments in the unit circle.

\[ S = \{ z \in \mathbb{C} : |z| = 1 \} \]

For a given subset \( E \subseteq S \) we'd like to assign a number

\[ P(E) = \]
Properties of Measurements

1. \( P(S) = P \colon 2^S \rightarrow \) ______

2. If \( E_1, E_2 \subseteq S \) are ______
   then \( P(E_1 \cup E_2) = \) ______

2'. If \( \{E_j\}_{j=1}^{\infty} \) are ______
   then \( P\left( \bigcup_{j=1}^{\infty} E_j \right) = \) ______

3. If \( E_1, E_2 \subseteq S^4 \) are ______
   then \( P(E_1) \) \( P(E_2) \)

Theorem:
Proof. If $E \leq S$ and $u \in S$ then $E$ and $uE = \{u \circ e : e \in E\}$ are congruent. \(\therefore\) by (3), $P(E) = P(uE)$ $\forall u \in S$.

Now, consider $S \triangleright T := \{e^{2\pi i t} : t \in \mathbb{Q}\}$ (countable).

$S / T = \{\text{equivalence classes in S} \}$

where $z \sim w \iff z = uw$ for some $u \in T$.

Choose exactly 1 representative element $q$ from each equivalence class, and let $\Phi = \{q_i\} \subset S$ be the collection of all these representatives.

Claim: $S = \bigsqcup_{u \in \Phi} u \Phi$. 
Contradicting Calculus?

The measurement function \( P \), satisfying (1), (2'), (3) is used daily in Calculus!

\[
P(E) =
\]

So how can it fail to exist?

The answer lies in an important subtlety: the definition of the Riemann integral only works over "nice" sets. The set \( \Phi \) is not nice!

Much of this quarter will be spent extending the Riemann integral. BUT there's only so far it can be extended.
The Moral of the Story

$p : 2^\mathbb{S} \to [0,1]$

This might seem like a bad sign... but it is actually a foundational truth for Kolmogorov's probability theory (that we now embark on developing).

In short: we don't always have complete information about the world, which means there may be some events we simply cannot assign probabilities to.

As to the unmeasurable sets...
**Banach-Tarski Paradox** (1942) or $\mathbb{R}^d$ for $d \geq 3$

Given any two subsets $E, F \subseteq \mathbb{R}^3$ with nonempty interior, there are finite partitions

\[
E = E_1 \cup E_2 \cup \ldots \cup E_n
\]
\[
F = F_1 \cup F_2 \cup \ldots \cup F_n
\]

such that $E_j$ is congruent to $F_j$ for $1 \leq j \leq n$.

**Robinson’s Doubling Theorem** (1947)

If $E$ is a solid ball in $\mathbb{R}^3$, and $F$ is two disjoint balls of the same radius, then Banach-Tarski works explicitly with $n = 5$. 