1.2 Algebraic Structures of Subsets (II.4 in Driver)

Start with a sample space \( \Omega \) (any set).

**Definition:** A collection, \( F \subseteq 2^\Omega \), is a \______ if

1. \( \Omega \in F \), \( \emptyset \in F \)
2. If \( E \in F \), then \( E^c \in F \).
3. If \( E_1, \ldots, E_n \in F \), then \( \bigcup_{j=1}^n E_j \in F \).
4. If \( E_1, \ldots, E_n \in F \), then \( \bigcap_{j=1}^n E_j \in F \).

If, instead of 3, we have the stronger

3'. If \( \{ E_j \} \) is a countable set of events in \( F \), then \( \bigcup_{j=1}^\infty E_j \in F \).

then we call \( F \) a \______.
Examples

Eg. \( \mathcal{F} = 2^\Omega \)

Eg. \( \mathcal{F} = \{ \emptyset, \Omega \} \)

Eg. \( \Omega = \{ 1, 2, 3 \} \quad \mathcal{F} = \{ \emptyset, \{1\}, \{2,3\}, \Omega \} \)

Eg. \( \mathcal{F} = \{ B \subseteq \Omega : B \text{ is countable or } B^c \text{ is countable} \} \)

Lemmas: If \( I \) is any index set and \( \{ \mathcal{F}_i : i \in I \} \) are \( \sigma \)-fields over \( \Omega \), then \( \bigcap_{i \in I} \mathcal{F}_i \) is a \( \sigma \)-field.

Pf. 1. \( \Omega \)

2. \( E \in \bigcap_{i \in I} \mathcal{F}_i \)

3. \( \{ E_n \}_{n=1}^\infty \in \bigcap_{i \in I} \mathcal{F}_i \)
Prop: Let $\mathcal{E} \subseteq 2^\Omega$ be any collection of subsets of $\Omega$.
There is a unique smallest $\sigma$-field $\sigma(\mathcal{E})$ that contains $\mathcal{E}$. It is called the $\sigma$-field generated by $\mathcal{E}$.

Pf.

Ex. $\Omega = \{1, 2, 3\}$, $\mathcal{E} = \{\emptyset, \{1\}, 2\} \Omega^3$

Exercise Let $\mathcal{E}_1, \mathcal{E}_2 \subseteq 2^\Omega$. Show that $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$ iff:

$\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$
The Borel $\sigma$-Field

Let $X$ be a topological space (e.g. Euclidean space $\mathbb{R}^d$). The Borel $\sigma$-field $\mathcal{B}(X)$ is the $\sigma$-field generated by the open subsets in $X$.

$$\mathcal{B}(X) = \sigma \{\text{open subsets of } X\}$$

Events in $\mathcal{B}(X)$ are called Borel sets.

**Fun Fact!** For $X = \mathbb{R}^d$, every open set $U$ is a countable union of open balls

$$U = \bigcup_{i=1}^{\infty} B(x_i; r_i)$$

$$\therefore \mathcal{B}(\mathbb{R}^d) = \sigma \{\text{open balls in } \mathbb{R}^d\}$$

$d = 1$: $\mathcal{B}(\mathbb{R}) = \sigma \{(a, b): a < b\}$ =