Semi-Algebras of Sets

A collection $\mathcal{S} \subseteq 2^{\Omega}$ is a semi-algebra if

1. $\emptyset \in \mathcal{S}$
2. If $A, B \in \mathcal{S}$ then $A \cap B \in \mathcal{S}$
3. If $A \in \mathcal{S}$ then $A^c$ is a finite disjoint union of elements from $\mathcal{S}$.

The canonical example is

$$\mathcal{S} = \{ (a, b) : -\infty \leq a < b \leq \infty \}$$

Prop: If $\mathcal{S}$ is a semi-algebra over $\Omega$, then $A(\mathcal{S}) = \{ \text{finite disjoint unions of sets from } \mathcal{S} \}$

Prop: If $\chi : \mathcal{S} \to [0, \infty]$ is additive over disjoint unions, then $\chi \left( \bigcup_{j=1}^{n} E_j \right) := \sum_{j=1}^{n} \chi(E_j)$ is a well-defined finitely-additive measure on $A(\mathcal{S})$. 

Prof:
Stieltjes Premeasures?

For \( F : \mathbb{R} \to \mathbb{R} \) non-decreasing, \( X_F(a,b] = F(b) - F(a) \)
is additive, and so extends to a finitely-additive measure on \( \mathcal{B}_{\mathbb{R}}(\mathbb{R}) \).

It is not a premeasure if \( F \) fails to be right-continuous, as we saw.

Fortunately, the converse is true.

**Theorem:** The finitely-additive measure \( X_F \) is a premeasure (i.e., is countably additive) on \( \mathcal{B}_{\mathbb{R}}(\mathbb{R}) \) iff \( F \) is right-continuous on \( \mathbb{R} : \]
\[
\lim_{s \to 0^+} F(a+s) = F(a) .
\]
Prop: Let \( S \subseteq 2^\omega \) be a semi-algebra.
A finitely-additive measure \( \chi : A(S) \to [0,\infty] \) is a premeasure iff it is countably subadditive on \( S \):
\[
E = \bigsqcup_{j=1}^\omega E_j \text{ in } S \Rightarrow \chi(E) \leq \sum_{j=1}^\omega \chi(E_j)
\]

Pf.
\((\Rightarrow)\) Premeasures are countably additive.

\((\Leftarrow)\) Finitely-additive measures are always countably superadditive, so it suffices to prove that \( \chi \) is countably subadditive on \( A = A(S) \)

\[
A = \bigsqcup_{n=1}^\omega A_n
\]
we now show that $\chi_{\mathcal{F}} : A(\mathcal{C}_F) \to \mathcal{B}_F(1\mathcal{I}) \to [0, \infty)$ is a premeasure by showing it is countably subadditive on the semi-algebra $\mathcal{C}_F = \{a_j b_j\}$.

$$(a_j b_j = \lim_{j \to \infty} (a_j b_j) = (a_j b_j) = \frac{1}{a_j b_j} = \frac{1}{b}.$$
\[ X_f \text{ is a premeasure on } \mathcal{B}_{\text{c}}(\mathbb{R}). \]

Notably:  
\[ F(x) = x \quad \chi_{(a,b]} = b-a \]

\[ \chi(E + a) \]

Lebesgue premeasure.