Measure Extension Theorem

\((\mathcal{F}, \mathcal{A}, \mu)\) premeasure space

\(\mu^* : 2^\mathcal{F} \to [0, \infty]\) Carathéodory outer measure

\[\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in \mathcal{A}, \ E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}\]

Theorem: (Fréchet, Carathéodory, Hopf, Kolmogorov... 1920s) There is a \(\sigma\)-field \(\mathcal{M} \supseteq \mathcal{A}\) s.t. \(\mu^*|_{\mathcal{M}}\) is a measure.

Standard approach: \(\mathcal{M} = \{E \subseteq \mathcal{F} : \mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c), \forall T \subseteq \mathcal{F}\}\)

- Show it is a \(\sigma\)-field, containing \(\mathcal{A}\)
- Show \(\mu^*\) is countably additive on it

Requires new tool: Monotone Class Theorem or Dynkin's \(\pi\)-\(\lambda\) Theorem
Advantage: works for all premeasures.
Disadvantage: finicky, technical, unmotivated: too clever by half.

Driver's Approach: restrict to finite premeasures.

$$(\Omega, A, \mu) \quad \mu : A \rightarrow [0, \infty)$$

Want extension: $\overline{\mu} : \overline{A}$

1. Make $2^\Omega$ into a topological space.
2. Define $\overline{A}$ to be the closure of $A$.
3. Prove $\mu : A \rightarrow [0, \infty)$ is sufficiently continuous, extends to closure.
4. Use topological tools to show $\overline{A}$ is a $\sigma$-field, and $\overline{\mu}$ is a measure.

It will turn out that $\overline{\mu} = \mu^*$ on $\overline{A}$.
Metric Spaces

\[ d : X \times X \to [0, \infty) \]

1. \( d(x, y) = 0 \iff x = y \)
2. \( d(x, y) = d(y, x) \)
3. \( d(x, z) \leq d(x, y) + d(y, z) \)

A sequence \((x_n)_{n=1}^{\infty}\) in \(X\) has a limit \(x\) if

\[ \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \quad d(x_n, x) < \varepsilon. \]

Given \(V \subseteq X\), the closure \(\overline{V}\) is the set of limits of sequences in \(V\).

A set \(V\) is closed if \(\overline{V} = V\).

A function \(f : V \to \mathbb{R}\) is Lipschitz if \(\exists K \in (0, \infty)\) s.t. \(|f(x) - f(y)| \leq K d(x, y)\).

**Prop:** If \(f\) is Lipschitz on a nonempty \(V \subseteq X\), then there is a unique Lipschitz extension \(\overline{f} : \overline{V} \to \mathbb{R}\) (with the same Lipschitz constant \(K\)).

\[ \overline{f} |V = f. \]
The Outer Pseudo-Metric

\((\Sigma, \mathcal{A}, \mu)\) finite premeasure space

\(\mu^*: 2^\Omega \rightarrow [0, \mu(\Omega)]\) Carathéodory outer measure

\[\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in \mathcal{A}, E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}\]

**Def:** \(d_\mu : 2^\Omega \times 2^\Omega \rightarrow [0, \mu(\Omega)]\)

\[d_\mu(E, F) = \mu^*(E \Delta F)\]

**Prop:** \(d_\mu\) is a pseudo-metric on \(2^\Omega\).

**Pf:**
Key Properties of the Outer Pseudo-Metric

1. \( \forall A, B \in 2^{\mathcal{D}} \quad d_\mu(A, B) = d_\mu(A^c, B^c) \).

2. \( \forall \{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \in 2^{\mathcal{D}} \):

   \[
   (a) \quad d_\mu(\bigcup_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} d_\mu(A_n, B_n) \\
   (b) \quad d_\mu(\bigcap_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} d_\mu(A_n, B_n).
   \]

\( \text{Def.} \)
Lemma: If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, and $A_n \in \mathcal{A}$ with $A_n \uparrow A$, then $d_\mu(A_n, A) = \mu^*(A) - \mu(A_n) \to 0$.

Pf. Let $D_n = A_n \setminus A_{n-1}$. Then $A = \bigcup_{n=1}^{\infty} D_n$, and by definition of $\mu^*$

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu(D_n)$$

Cor: If $A_n \in \mathcal{A}$ and $A_n \uparrow A$, then $A \in \bar{\mathcal{A}}$. 
Theorem: If \((\Omega, \mathcal{A}, \mu)\) is a finite premeasure space, then the closure \(\bar{A}\) of the field \(A\) in the pseudo-metric space \((2^\Omega, d\mu)\) is a \(\sigma\)-field.

Pf.