Outer Pseudo-Metric Closure (§6.2 in Driver)

\[
(\Omega, A, \mu) \text{ finite premeasure space.}
\]

\[
\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A_j) : A_j \in A, E \subseteq \bigcup_{j=1}^{\infty} A_j \right\} \forall E \in 2^\Omega
\]

\[
d_\mu(E, F) = \mu^*(E \Delta F)
\]

**Theorem:** The closure \(\bar{A}\) of \(A\) in the pseudo-metric space \(\left(2^\Omega, d_\mu\right)\) is a \(\sigma\)-field.

Now, we've proved that \(\mu^*_A = \mu\). So, for \(A, B \in A\),

\[
d_\mu(A, B)
\]

**Prop:** \(\mu\) extends to a unique \(\text{Lip-1}\) function \(\tilde{\mu} : \bar{A} \to [0, \mu(\Omega)]\).
Def: Given $\mathcal{E} \subseteq 2^\Omega$, $\mathcal{E}_c = \{\text{countable unions of elements of } \mathcal{E}\}$

Note: $\mathcal{E}_c$ is automatically closed under countable unions.

If $\mathcal{E}$ is closed under finite intersections, so is $\mathcal{E}_c$.

Restatement of Lemma (from last time):
If $(\Omega, A, \mu)$ is a finite premeasure space, then $\overline{A}_c = \overline{A}$, and $\overline{\mu} = \mu^*$ on $A_c$.

Pf. We showed that if $A \in A_n \uparrow A$ then $d_\mu(A_n, A) = \mu^*(A) - \mu(A_n) \to 0$. 
Prop: Let $(\Omega, \mathcal{A}, \mu)$ be a finite premeasure space. For $B \in 2^\Omega$, TFAE:

1. $B \in \overline{\mathcal{A}}$
2. $\forall \varepsilon > 0$, $\exists C \in \mathcal{A}_\varepsilon$ s.t. $B \subseteq C$ and $\mu^*(C \setminus B) = d_\mu(B, C) < \varepsilon$.

Pf. (2) $\Rightarrow$ (1): Select a sequence $C_n \in \mathcal{A}_\varepsilon$ s.t. $d_\mu(B, C_n) < \frac{1}{n}$; then $C_n \rightarrow B$ and so $B \in \overline{\mathcal{A}_\varepsilon}$.

(1) $\Rightarrow$ (2):
Cor: Let $(\Omega, \mathcal{A}, \mu)$ be a finite premeasure space. Then $\mu^* = \bar{\mu}$ on $\overline{\mathcal{A}}$.

Pf. Let $B \in \overline{\mathcal{A}}$. $\bar{\mu}(B) =$
Theorem: If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, then $\tilde{\mu} : \mathcal{A} \to [0, \infty]$ is a measure.

Pf. We will show that $\tilde{\mu}$ is finitely-additive on $\bar{\mathcal{A}}$. Once we've done that, we've shown $\tilde{\mu}$ is a finitely-additive measure on the $\sigma$-field $\bar{\mathcal{A}}$, and it is countably super-additive. But by the prev. Corollary, $\tilde{\mu} = \mu^*$ on $\bar{\mathcal{A}}$, and $\mu^*$ is countably subadditive.