Turn in the homework by 11:00am in lecture on Wednesday. Late homework will not be accepted.

1. Exercise 1.1 on p. 21 in Chung & Williams.
2. Exercise 1.4 on p. 22 in Chung & Williams.
3. Exercise 2.4 on pp. 54-55 in Chung & Williams.
4. Exercise 2.8 on p. 55 in Chung & Williams.
5. Exercise 2.9 on p. 55 in Chung & Williams.
8. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(\{\mathcal{F}_t\}_{t \geq 0}\) be a right-continuous filtration for which \(\mathcal{F}_t\) contains all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\) for each \(t \geq 0\). Fix \(T > 0\). Define

\[
\mathcal{M}_T^{(r)c} := \{(\text{right)-continuous } L^2\text{-martingales } M \text{ on } [0, T]\}
\]

equipped with the inner product

\[
\langle M, N \rangle_T := \mathbb{E}[M_T N_T].
\]

(More precisely: for \(\langle \cdot, \cdot \rangle_T\) to be a non-degenerate inner product, \(\mathcal{M}_T^{(r)c}\) must be defined as the space of equivalence classes of (right-)continuous \(L^2\)-martingales \(M\) on \([0, T]\), where two such martingales \(M\) and \(N\) are equivalent if they are indistinguishable, i.e. if \(\mathbb{P}(M_t = N_t \forall t \in [0, T]) = 1\).)

Prove that \(\mathcal{M}_T^{(r)c}\) is a Hilbert space, and that the endpoint evaluation map

\[
e_T : \mathcal{M}_T^{(r)c} \to L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \quad \text{defined by } \quad e_T(M) = M_T
\]

is an isometry. In the right-continuous case \(\mathcal{M}_T^{(r)c}\), show that \(e_T\) is unitary (i.e. surjective isometry); in the continuous case \(\mathcal{M}_T^{c}\), show that \(e_T\) is not surjective.

9. Let \(B = (B_t)_{t \geq 0}\) be a standard Brownian motion. Show that \(B^2 \in \Lambda^2(\mathcal{P}, B)\), and prove that

\[
\int_0^t B^2 dB = \frac{1}{3}B^3_t - \int_0^t B_t dt.
\]

[Hint: Use left-endpoint Riemann–Stieltjes sums, and show that any such sequence converges in \(L^2_B\) as the partition mesh tends to 0.]