Math 286: Fall 2022
Homework 1

Available	Friday, October 7	Due	Sunday, October 23

Turn in the homework by 11:59pm through Gradescope. Late homework will not be accepted.

1. Exercise 1.1 on p. 21 in Chung & Williams.

2. Exercise 2.4 on pp. 54-55 in Chung & Williams.

3. Exercise 2.10 on p. 55 in Chung & Williams.

4. Exercise 2.13 on p. 56 in Chung & Williams.

5. Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion. Show that $B^2 \in \Lambda^2(\mathscr{P}, B)$, and prove that

$$\int_0^t B^2\, dB = \frac{1}{3}B_t^3 - \int_0^t B_s\, ds.$$

[*Hint*: Use left-endpoint Riemann–Stieltjes sums, and show that any such sequence converges in $\mathcal{L}_B^2$ as the partition mesh tends to 0.]

6. As always, assume we have a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with filtration satisfying the usual conditions. If $p \in [1, \infty]$, then we write $\mathbb{L}^p_{\text{loc}}$ for the set of indistinguishability classes of progressively measurable stochastic processes $X\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that for all $t \geq 0$,

$$\|X\|_{p,t} := \Big\| \sup_{0\leq s\leq t} |X_s| \Big\|_p < \infty.$$

For $X, Y \in \mathbb{L}^p_{\text{loc}}$, define

$$d_p(X, Y) := \sum_{k=1}^{\infty} 2^{-k} \wedge \|X - Y\|_{p,k}.$$

Similarly, write $\mathbb{L}^0_{\text{loc}}$ for the set of indistinguishability classes of progressively measurable processes X such that

$$\sup_{0\leq s\leq t} |X_s| < \infty, \text{ for all } t \geq 0,$$

almost surely. For $X, Y \in \mathbb{L}^0_{\text{loc}}$, define

$$d_0(X, Y) := \sum_{k=1}^{\infty} 2^{-k} \wedge \mathbb{E}\left[\frac{\sup_{0\leq s\leq k} |X_s - Y_s|}{1 + \sup_{0\leq s\leq k} |X_s - Y_s|}\right].$$

You may take for granted that if $L^0(\Omega, \mathscr{F}, \mathbb{P})$ is the set of $\mathbb{P}$-almost everywhere equivalence classes of random variables $\Omega \to \mathbb{R}$, then $d(X, Y) := \mathbb{E}\big[\frac{|X-Y|}{1+|X-Y|}\big]$ is a complete metric on $L^0(\Omega, \mathscr{F}, \mathbb{P})$ that topologizes convergence in probability.

(i) Fix $p \in \{0\} \cup [1, \infty]$. Prove that $(\mathbb{L}^p_{\text{loc}}, d_p)$ is a complete metric space and that a sequence $(X^n)_{n\in\mathbb{N}}$ in $\mathbb{L}^p_{\text{loc}}$ converges to $X \in \mathbb{L}^p_{\text{loc}}$ if and only if for all $t \geq 0$, $\sup_{0\leq s\leq t} |X^n_s - X_s| \to 0$ in L^p as $n \to \infty$. (To be clear, convergence in L^0 is convergence in probability.)

(ii) Prove that if $X \in \mathbb{L}^0_{\text{loc}}$ and if τ is a stopping time, then the map $\mathbb{L}^0_{\text{loc}} \ni X \mapsto X^\tau \in \mathbb{L}^0_{\text{loc}}$ is well-defined and continuous.

(iii) Suppose that $1 < p < \infty$, and define $\mathbb{M}^{(r)c}_p$ to be the set of indistinguishability classes of (right-)continuous L^p-martingales. Prove that $\mathbb{M}^{(r)c}_p \subseteq \mathbb{L}^p_{\text{loc}}$ is closed and that

$$\tilde{d}_p(M, N) := \sum_{k=1}^{\infty} 2^{-k} \wedge E[|M_k - N_k|^p]^{\frac{1}{p}} \leq d_p(M, N) \leq \frac{p}{p-1} \tilde{d}_p(M, N),$$

for all $M, N \in \mathbb{M}^{(r)c}_p$.

(iv) Fix $p \in [1, \infty)$. Suppose $(M^n)_{n\in\mathbb{N}}$ is a sequence of (right-)continuous martingales and M is a process such that if $t \geq 0$, then $M^n_t \to M_t$ in L^p as $n \to \infty$. Without using Doob's Regularization Theorem, prove that M has a (right-)continuous modification. Assuming M is one such (right-)continuous modification, prove that if $p > 1$, then $M^n \to M$ in $\mathbb{L}^p_{\text{loc}}$ as $n \to \infty$, and that if $p = 1$, then $M^n \to M$ in $\mathbb{L}^0_{\text{loc}}$ as $n \to \infty$.