Math 286: Fall 2022
Homework 2

Available	Wednesday, October 26	Due	Sunday, November 13

Turn in the homework by 11:59pm through Gradescope. Late homework will not be accepted.

1. Exercise 4.1 on pp. 90-91 in Chung & Williams.

2. Exercise 4.2 on p. 91 in Chung & Williams.

3. Exercise 5.2 on p. 112 in Chung & Williams.

4. Exercise 5.6 on p. 113 in Chung & Williams.

5. Let M be a local martingale. Suppose that, for each $t \in \mathbb{R}_+$, the family of random variables

$$\{M_t^\tau : \tau \text{ is a stopping time}\}$$

is uniformly integrable. Prove that M is actually a martingale.

6. Let M be a RCLL L^2-martingale.

(i) Prove that if $X, Y \in \mathcal{EP}$ and $\mu_M(\{(t,\omega) \in \mathbb{R}_+ \times \Omega : X_t(\omega) \neq Y_t(\omega)\}) = 0$, then the processes $\int_0^\cdot X\,dM$ and $\int_0^\cdot Y\,dM$ (both defined as in Observation 2 from the Lecture 6 notes on Canvas) are indistinguishable.

(ii) Let N be another RCLL L^2-martingale. Prove that if M is indistinguishable from N, then $\mu_M = \mu_N$, $\Lambda^2(\mathcal{P}, M) = \Lambda^2(\mathcal{P}, N)$, and $\int_0^\cdot X\,dM = \int_0^\cdot X\,dN$ as elements of $\mathbb{M}_2^{\mathrm{rc}}$ whenever $X \in \Lambda^2(\mathcal{P}, M) = \Lambda^2(\mathcal{P}, N)$. Conclude that if M and N are, more generally, RCLL local L^2-martingales that are indistinguishable, then $\Lambda(\mathcal{P}, M) = \Lambda(\mathcal{P}, N)$ and $\int_0^\cdot X\,dM = \int_0^\cdot X\,dN$ as members of $\mathbb{L}^0_{\mathrm{loc}}$ whenever $X \in \Lambda(\mathcal{P}, M) = \Lambda(\mathcal{P}, N)$.

7. Let $(\mathscr{F}_t)_{t\geq 0}$ be a filtrations satisfying the usual assumptions, and let $M = (M_t)_{t\geq 0}$ be an adapted, L^2, right continuous process with $M_0 = 0$, $\mathbb{E}(M_t) = 0$ for all $t \geq 0$, and with *independent increments*: $M_t - M_s$ is independent from $\mathscr{F}_s$ for $0 \leq s < t < \infty$. Prove that M is an L^2-martingale, that $A_t = \mathbb{E}(M_t^2)$ is a right-continuous increasing (deterministic) process, and that $N_t = M_t^2 - A_t$ is a martingale.

8. Let B be a Brownian motion on $\mathbb{R}$, and suppose that $f\colon \mathbb{R}_+ \to \mathbb{R}$ is a measurable function such that $\int_0^t f(s)^2\,ds < \infty$, for all $t \geq 0$.

(i) Prove that the (deterministic) process f belongs to $\Lambda^2(\mathcal{P}, B)$.

(ii) Prove that if $f \in L^2(\mathbb{R}_+)$, then $\lim_{t\to\infty} \int_0^t f(s)\,dB_s$ exists in L^2. Does it exist a.s.?

(iii) Prove that

$$X_t := \exp\left\{\int_0^t f(s)\,dB_s - \frac{1}{2}\int_0^t f(s)^2\,ds\right\}$$

is an L^2-martingale, and compute $\mathbb{E}[X_t^2]$. Prove also that if $f \in L^2(\mathbb{R}_+)$, then X is an L^2-bounded martingale.