1. Exercise 5.11 on p. 115 in Chung & Williams.

2. Exercise 6.5 on p. 138 in Chung & Williams.


4. The Lévy area process $L$ is defined by

$$L_t = \int_0^t (X \, dY - Y \, dX)$$

where $X, Y$ are independent Brownian motions. (If $X_t, Y_t$ were smooth functions, by Green's theorem $L_t$ would equal twice the signed area of the region enclosed by the path $s \mapsto (X_s, Y_s)$ for $0 \leq s \leq t$ followed by the radial path from $(X_t, Y_t)$ back to $(0, 0)$.)

Fix $u \in \mathbb{R}$, and let $\alpha, \beta: \mathbb{R}_+ \to \mathbb{R}$ be $C^1$ functions. Define

$$V_t = iuL_t - \frac{1}{2} \alpha(t)(X_t^2 + Y_t^2) + \beta(t).$$

(a) Prove that $e^{V_t}$ is a local martingale if $\alpha$ and $\beta$ solve the following system of ODEs:

$$\dot{\beta} = \alpha \quad \text{and} \quad \dot{\alpha} = \alpha^2 - u^2.$$  

(b) The unique solution $(\alpha, \beta)$ to these ODEs satisfying the final condition $\alpha(T) = \beta(T) = 0$ is

$$\alpha(t) = u \tanh(u(T - t)) \quad \text{and} \quad \beta(t) = -\log \cosh(u(T - t)).$$

(You need not prove this.) With these $\alpha$ and $\beta$, show that $\{\exp V_t\}_{t \leq T}$ is actually a martingale.

(c) Use the fact that $\mathbb{E}(\exp V_t)$ is constant on $[0, T]$ to prove that the characteristic function of the Lévy area process is given by

$$\mathbb{E} \left[ e^{iut} \right] = \frac{1}{\cosh(uT)}, \quad T \geq 0, u \in \mathbb{R}.$$
5. Let $X$ be a semi-martingale, and let $\mathbb{R} \times \mathbb{R}_+ \ni (u, t) \mapsto Y_t(u)$ be a continuous, uniformly bounded function such that $t \mapsto Y_t(u)$ is adapted for each $u \in \mathbb{R}$.

(a) Let $g \in C_c^\infty(\mathbb{R})$, and let $Z_t = \int_{\mathbb{R}} g(u) Y_t(u) \, du$. Prove that
\[
\int_0^t Z_s \, dX_s = \int_{\mathbb{R}} g(u) \left( \int_0^t Y_s(u) \, dX_s \right) \, du \quad \text{a.s.}
\]

(b) Let $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \geq 0$ and $\int_{\mathbb{R}} \psi(u) \, du = 1$. Set $\psi_k(u) = k \psi(ku)$, so that $\psi_k$ forms an approximate identity sequence. Prove that, as $k \to \infty$,
\[
\int_0^t (\psi_k \ast Y_s)(u) \, dX_s \to \int_0^t Y_s(u) \, dX_s \quad \text{for all } u \in \mathbb{R}, \quad \text{a.s.}
\]

6. In this exercise, we prove the time-dependent version of Itô’s formula (namely allowing the function of the semi-martingale to also explicitly depend on time). Let $X$ be a semi-martingale.

(a) Suppose $f \in C^2(\mathbb{R}_+ \times \mathbb{R}^n)$ (meaning that there is a $C^2$ function on $\mathbb{R} \times \mathbb{R}^n$ whose restriction to $\mathbb{R}_+ \times \mathbb{R}^n$ is $f$). Prove the Itô formula
\[
f(t, X_t) - f(0, X_0) = \int_0^t \frac{\partial f}{\partial s}(s, X_s) \, ds + \sum_{j=1}^n \int_0^t \frac{\partial f}{\partial x_j}(s, X_s) \, dX_s + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) \, d[X^i, X^j]_s.
\]

[Hint: apply the Itô formula to the process $Y_t = (t, X_t)$.] (b) Prove this formula holds more generally if $f \in C^{1,2}_b$ (i.e. it is $C^1$ in time and $C^2$ in space, and all derivatives of $f$ are uniformly bounded). [Hint: use an approximate identity to smooth out $f$ to make it $C^2_b$ or better in both space and time, then use Problem 6(b) to remove the mollification.]

Note: the theorem in part 6(b) actually holds for $f \in C^{1,2}$, with no uniform boundedness assumption. One can prove this the same way by proving the results of problem 5 without the uniform boundedness assumption. But this is quite a bit trickier. Alternatively, it is probably easier to prove the general result of 6(b) from scratch, the way we proved Itô’s formula for time-independent test functions. To see a detailed proof of this, see Seppalainen.