Math 286: Fall 2018
Homework 2

Available | Friday, November 16 | Due | Friday, November 30

Turn in the homework by 11:00am in lecture on November 30. Late homework will not be accepted.

1. Exercise 5.11 on p. 115 in Chung & Williams.
2. Exercise 6.5 on p. 138 in Chung & Williams.
4. The **Lévy area process** \( L \) is defined by

\[
L_t = \int_0^t (X \, dY - Y \, dX)
\]

where \( X, Y \) are independent Brownian motions. (If \( X_t, Y_t \) were smooth functions, by Green’s theorem \( L_t \) would equal twice the signed area of the region enclosed by the path \( s \mapsto (X_s, Y_s) \) for \( 0 \leq s \leq t \) followed by the radial path from \( (X_t, Y_t) \) back to \( (0, 0) \).)

Fix \( u \in \mathbb{R} \), and let \( \alpha, \beta : \mathbb{R}_+ \to \mathbb{R} \) be \( C^1 \) functions. Define

\[
V_t = iuL_t - \frac{1}{2} \alpha(t)(X_t^2 + Y_t^2) + \beta(t).
\]

(a) Prove that \( e^{V_t} \) is a local martingale if \( \alpha \) and \( \beta \) solve the following system of ODEs:

\[
\dot{\beta} = \alpha \quad \text{and} \quad \dot{\alpha} = \alpha^2 - u^2.
\]

(b) The unique solution \((\alpha, \beta)\) to these ODEs satisfying the *final* condition \( \alpha(T) = \beta(T) = 0 \) is

\[
\alpha(t) = u \tanh(u(T - t)) \quad \text{and} \quad \beta(t) = -\log \cosh(u(T - t)).
\]

(You need not prove this.) With these \( \alpha \) and \( \beta \), show that \( \{\exp V_t\}_{t \leq T} \) is actually a martingale.

(c) Use the fact that \( \mathbb{E}(\exp V_t) \) is constant on \([0, T]\) to prove that the characteristic function of the Lévy area process is given by

\[
\mathbb{E}
\left[
\exp\left(\tfrac{iuL_T}\right)
\right] = \frac{1}{\cosh(uT)},
\quad
T \geq 0, \; u \in \mathbb{R}.
\]
5. Let $X$ be a semi-martingale, and let $\mathbb{R} \times \mathbb{R}_+ \ni (u, t) \mapsto Y_t(u)$ be a continuous function such that $t \mapsto Y_t(u)$ is adapted for each $u \in \mathbb{R}$.

(a) Let $g \in C_c^{\infty}(\mathbb{R})$, and let $Z_t = \int_{\mathbb{R}} g(u) Y_t(u) \, du$. Prove that

$$\int_0^t Z_s \, dX_s = \int_{\mathbb{R}} g(u) \left( \int_0^t Y_s(u) \, dX_u \right) \, du \quad \text{a.s.}$$

(b) Let $\psi \in C_c^{\infty}(\mathbb{R})$ with $\psi \geq 0$ and $\int_{\mathbb{R}} \psi(u) \, du = 1$. Set $\psi_k(u) = k\psi(ku)$, so that $\psi_k$ forms an approximate identity sequence. Prove that, as $k \to \infty$,

$$\int_0^t (\psi_k \ast Y_s)(u) \, dX_s \to \int_0^t Y_s(u) \, dX_s \quad \text{for all } u \in \mathbb{R}, \quad \text{a.s.}$$

6. In this exercise, we prove the time-dependent version of Itô’s formula (namely allowing the function of the semi-martingale to also explicitly depend on time). Let $X$ be a semi-martingale.

(a) Suppose $f \in C^2(\mathbb{R}_+ \times \mathbb{R}^n)$ (meaning that there is a $C^2$ function on $\mathbb{R} \times \mathbb{R}^n$ whose restriction to $\mathbb{R}_+ \times \mathbb{R}^n$ is $f$). Prove the Itô formula

$$f(t, X_t) - f(0, X_0) = \int_0^t \frac{\partial f}{\partial s}(s, X_s) \, ds + \sum_{j=1}^n \int_0^t \frac{\partial f}{\partial x_j}(s, X_s) \, dX_s + \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) \, d[X^i, X^j]_s.$$

[Hint: apply the Itô formula to the process $Y_t = (t, X_t)$.]

(b) Prove this formula holds more generally if $f \in C^{1,2}$ (i.e. it is $C^1$ in time and $C^2$ in space). [Hint: use an approximate identity to smooth out $f$ to make it $C^{\infty}$ in both space and time, then use Problem 6(b) to remove the mollification.]