Math 286: Fall 2022
Homework 3

Available	Friday, November 18	Due	Friday, December 9

Turn in the homework by 11:59pm through Gradescope. Late homework will not be accepted.

1. The **Lévy area process** L is defined by

$$L_t = \int_0^t (X\,dY - Y\,dX)$$

where X, Y are independent Brownian motions. (If X_t, Y_t were smooth functions, by Green's theorem L_t would equal twice the signed area of the region enclosed by the path $s \mapsto (X_s, Y_s)$ for $0 \le s \le t$ followed by the radial path from (X_t, Y_t) back to $(0,0)$.)

Fix $u \in \mathbb{R}$, and let $\alpha, \beta\colon \mathbb{R}_+ \to \mathbb{R}$ be C^1 functions. Define

$$V_t = iuL_t - \frac{1}{2}\alpha(t)(X_t^2 + Y_t^2) + \beta(t).$$

(a) Prove that e^{V_t} is a local martingale if α and β solve the following system of ODEs:

$$\dot{\beta} = \alpha \quad \text{and} \quad \dot{\alpha} = \alpha^2 - u^2.$$

(b) The unique solution (α, β) to these ODEs satisfying the *final* condition $\alpha(T) = \beta(T) = 0$ is

$$\alpha(t) = u \tanh(u(T-t)) \quad \text{and} \quad \beta(t) = -\log\cosh(u(T-t)).$$

(You need not prove this.) With these α and β, show that $\{\exp V_t\}_{t \le T}$ is actually a martingale.

(c) Use the fact that $\mathbb{E}(\exp V_t)$ is constant on $[0,T]$ to prove that the characteristic function of the Lévy area process is given by

$$\mathbb{E}\left[e^{iuL_T}\right] = \frac{1}{\cosh(uT)}, \quad T \ge 0, u \in \mathbb{R}.$$

2. Let B be a standard Brownian motion on $\mathbb{R}$. Suppose that $f \in C^1(\mathbb{R})$ and there is a finite set $F \subseteq \mathbb{R}$ such that $f \in C^2(\mathbb{R} \setminus F)$. Prove that if $\sup_{x \in \mathbb{R}\setminus F} |f''(x)| < \infty$, then

$$f(B) = f(B_0) + \int_0^{\cdot} f'(B_t)\,dB_t + \frac{1}{2}\int_0^{\cdot} f''(B_t)\,dt$$

no matter what value we assign to $f''(x)$ when $x \in F$. (Hint: Choose a sequence $(f_k)_{k\in\mathbb{N}}$ in $C^2(\mathbb{R})$ such that $f_k \to f$ uniformly, $f_k' \to f'$ uniformly, $\sup_{m\in\mathbb{N}} \|f_m''\|_\infty < \infty$, and $f_k''(x) \to f''(x)$ as $k \to \infty$ whenever $x \in \mathbb{R} \setminus F$.)

3. Let B be a standard Brownian motion on $\mathbb{R}$. For $x \in \mathbb{R} \setminus \{0\}$, define

$$\tau_x := \inf\{t \geq 0\colon B_t = \tfrac{1}{x}\}.$$

Define

$$X_t = \begin{cases} \dfrac{x}{1 - xB_t} & t \in [0, \tau_x) \\ 0 & t \notin [0, \tau_x) \end{cases}$$

Prove that X "solves" the SDE $dX_t = X_t^3\,dt + X_t^2\,dB_t$ with $X_0 = x$ in the following sense. For $\epsilon > 0$, define

$$\tau_{x,\epsilon} := \inf\{t \geq 0\colon |B_t - \tfrac{1}{x}| \leq \epsilon\}.$$

Then

$$dX_t^{\tau_{x,\epsilon}} = X_t^3 \mathbb{1}_{[0,\tau_{x,\epsilon}]}(t,\cdot)\,dt + X_t^2 \mathbb{1}_{[0,\tau_{x,\epsilon}]}(t,\cdot)\,dB_t$$

and $\tau_{x,\epsilon} \nearrow \tau_x$ a.s. as $\epsilon \searrow 0$.

4. Let B be a Brownian motion on $\mathbb{R}$, and let $X = (X^1, X^2)$ be the diffusion process satisfying the SDE

$$\begin{aligned} dX_t^1 &= -\frac{1}{2} X_t^1\,dt - X_t^2\,dB_t \\ dX_t^2 &= -\frac{1}{2} X_t^2\,dt + X_t^1\,dB_t \end{aligned}$$

with $X^1(0) = 1, X^2(0) = 0$.

(a) What is the generator of the process X? Is it elliptic?

(b) Prove that, with probability 1, X_t is in the unit circle for all $t > 0$.

5. Let $T < 1$, and consider the SDE

$$\begin{aligned} dX_t &= -\frac{X_t}{1-t}\,dt + dB_t \\ X_0 &= x \end{aligned}$$

(where X and B take values in $\mathbb{R}^d$). Find an explicit (strong up to time T) solution. Show that it is a Gaussian process, and compute the mean and covariance. [When $x = 0$, this process is called the *Brownian Bridge* on $[0, 1]$.]