
2. Let $B$ be a Brownian motion on $\mathbb{R}$, and let $X = [X^1, X^2]^T$ be the diffusion process satisfying the SDE

$$
\begin{align*}
    dX^1_t &= -\frac{1}{2}X^1_t \, dt - X^2_t \, dB_t \\
    dX^2_t &= -\frac{1}{2}X^2_t \, dt + X^1_t \, dB_t
\end{align*}
$$

with $X^1(0) = 0, X^2(0) = 1$.

(a) What is the generator of the process $X$? Is it elliptic?

(b) Prove that, with probability 1, $X_t$ is in the unit circle for all $t > 0$.

3. Let $T < 1$, and consider the SDE

$$
\begin{align*}
    dX_t &= -\frac{X_t}{1-t} \, dt + dB_t \\
    X_0 &= x
\end{align*}
$$

(where $X$ and $B$ take values in $\mathbb{R}^d$). Find an explicit (strong up to time $T$) solution. Show that it is a Gaussian process, and compute the mean and covariance. [When $x = 0$, this process is called the Brownian Bridge on $[0, 1]$.]

4. Consider the 1-dimensional SDE

$$
\begin{align*}
    dX_t &= X^2_t \, dB_t + X^3_t \, dt \\
    X_0 &= x
\end{align*}
$$

Let $\tau_x = \inf\{t \geq 0: B_t = \frac{1}{x}\}$ for $x \neq 0$ and $\tau_0 = +\infty$. Prove that, for $t \in [0, \tau_x)$, a solution is given by

$$
X_t = \frac{x}{1-xB_t}.
$$

To be precise: for each fixed $x > 0$, and $0 < \epsilon < \frac{1}{x}$, let

$$
\tau_{x,\epsilon} = \inf\{t \geq 0: B_t = \frac{1}{x} - \epsilon\}.
$$

Prove that $\tau_{x,\epsilon} \uparrow \tau_x$ as $\epsilon \downarrow 0$, and that for each such $\epsilon > 0$, the process $X$ defined above satisfies

$$
X_{t \wedge \tau_{x,\epsilon}} = x + \int_0^{t \wedge \tau_{x,\epsilon}} X^2_s \, dB_s + \int_0^{t \wedge \tau_{x,\epsilon}} X^3_s \, ds.
$$