RANDOM MATRICES WITH LOG-RANGE CORRELATIONS, AND LOG-SOBOLEV INEQUALITIES

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ABSTRACT. Let $X_N$ be a symmetric $N \times N$ random matrix whose $\sqrt{N}$-scaled entries are uniformly square integrable. We prove that if the entries of $X_N$ can be partitioned into independent subsets each of size $o(\log N)$, then the empirical eigenvalue distribution of $X_N$ converges weakly to its mean in probability. If the entries are bounded, the convergence is almost sure; if the entries are Gaussian, we prove almost sure convergence with larger blocks of size $o(N^2/\log N)$. This significantly extends the best previously known results on convergence of eigenvalues for matrices with correlated entries, where the partition subsets are blocks of size $O(1)$. We also prove the strongest known convergence results for eigenvalues of band matrices.

We prove these results developing a new log-Sobolev inequality, generalizing the first author’s introduction of mollified log-Sobolev inequalities: we show that if $Y$ is a bounded random vector and $Z$ is a standard normal random vector independent from $Y$, then the law of $Y + tZ$ satisfies a log-Sobolev inequality for all $t > 0$, and we give bounds on the optimal log-Sobolev constant.

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1. INTRODUCTION

Random matrix theory is primarily interested the convergence of statistics associated to the eigenvalues (or singular values) of $N \times N$ matrices whose entries are random variables with a prescribed joint distribution. The field was begun by Wigner in [37, 38], in which he studied the mean bulk behavior of the eigenvalues of what is now called a Gaussian Orthogonal Ensemble GOE$_N$. This is the Gaussian case of a more general class of random matrices now called Wigner ensembles: symmetric random matrices $X_N$ such that the entries of $\sqrt{N}X_N$ are i.i.d. random variables (modulo the symmetry constraint) with sufficiently many finite moments. There are also corresponding complex Hermitian ensembles, non-symmetric / non-Hermitian ensembles, as well as a parallel world of matrices generalizing the GOE$_N$, defined not via the distribution of entries but rather by invariance properties of the joint distribution. In this paper, we take Wigner ensembles as the starting point.

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Given a symmetric matrix $X_N$, enumerate its eigenvalues $\lambda_1^N \leq \cdots \leq \lambda_N^N$ in nondecreasing order. The empirical spectral distribution (ESD) of $X_N$ is the random point measure

$$
\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j^N}.
$$

Integrating $\mu_N$ against the indicator function $1_B$ yields the random variable counting the number of eigenvalues in $B$ (building up the histogram of the eigenvalues of $X_N$). In general, the random variables $\int f \, d\mu_N$ for test functions $f : \mathbb{R} \to \mathbb{R}$ are called linear statistics of the eigenvalues. Wigner’s original papers [37, 38] showed that, for the GOE$_N$, the ESD $\mu_N$ converges weakly in expectation to what is now called Wigner’s semicircle law:

$$
\sigma(dx) = \frac{1}{2\pi} \sqrt{4-x^2}_+ \, dx.
$$

To be precise: this means that $\mathbb{E}(\int f \, d\mu_N) \to \int f \, d\sigma$ for each $f \in C_b(\mathbb{R})$. This convergence was later upgraded to weak a.s. convergence. Many more results are known about the fluctuations of $\mu_N$, the spacing between eigenvalues, and the distribution and fluctuations of the largest eigenvalue. The reader may consult the book [1] and its extensive bibliography for more on these endeavors.

There is also a vast literature on band matrices. Originally referring to random matrices with entries that are 0 along many diagonals, these are a wider class of random matrix ensembles generalizing Wigner ensembles, where the upper-triangular entries are still independent, but need not be identically distributed (so long as they satisfy some form of uniform regularity). There is a vast literature on band matrices; see, for example, the expansive paper [2] which uses combinatorial and probabilistic methods to establish that a large class of band matrices have ESD converging a.s. to the semicircle law, with Gaussian fluctuations of a similar form to Wigner matrices. (Our Theorem 1.4 below improves on the main result in [2].)

There are comparatively few papers, however, dealing with random matrices with correlated entries. In [34], Shlyakhtenko realized that the tools of operator-valued free probability could be used to compute the limit in expectation of certain kinds of block matrices: ensembles $X_{kN}$ possessed of $k \times k$ blocks that have a fixed covariance structure (uniform among the blocks), where the $N^2$ blocks are independent up to symmetry. The recent papers [10, 11, 3] showed how to explicitly compute the limit ESD for a wide class of such block matrices with Gaussian entries, and used these results to give applications to quantum information theory. Additionally, in [33], a class of these block matrices was studied and proved to converge almost surely, with applications given to signal processing. (The actual ensembles studied in [33, 34, 10, 11, 3] are presented in a different form, with an overall $k \times k$ block structure with $N \times N$ blocks all whose entries are independent; this is just an orthonormal basis change from the description above, and so has the same ESD.) Note that in these block matrices, the limiting ESD is typically not semicircular. The combinatorial methods used to analyze such ensembles do not easily extend beyond the case that $k$ is fixed as $N \to \infty$.

Our main results, Theorems 1.1 and 1.2, give a significant generalization of ESD convergence for block-type matrices, both in terms of allowing $k$ to grow with $N$, and softening the rigid structure of the partition into independent blocks.

**Theorem 1.1.** Let $X_N$ be an $N \times N$ random matrix. Assume that the entries of $X_N$ satisfy the following conditions.

1. The family $\{N[X_N]_{ij}^2\}_{N \in \mathbb{N}, 1 \leq i, j \leq N}$ is uniformly integrable.
2. For each $N$, there is a set partition $\Pi_N$ of $\{(i, j) : 1 \leq i \leq j \leq N\}$ and a constant $d_N = o(\log N)$ such that each block of $\Pi_N$ has size $\leq d_N$, and the entries $[X_N]_{ij}$ and $[X_N]_{k\ell}$ are independent if $(i, j)$ and $(k, \ell)$ are not in the same block of $\Pi_N$.

Then the empirical spectral distribution $\mu_N$ of $X_N$ converges weakly in probability to its mean:

$$
\int f \, d\mu_N - \mathbb{E} \left( \int f \, d\mu_N \right) \to_P 0, \quad \text{for all } f \in \text{Lip}(\mathbb{R}).
$$

(1.2)

If we further assume that the family $\{\sqrt{N}[X_N]_{ij}\}_{N \in \mathbb{N}, 1 \leq i, j \leq N}$ is uniformly bounded, then the convergence in (1.2) is almost sure.
We use similar techniques to those used in the proof of Theorem 1.1 to prove the following stronger result in the case of Gaussian entries: under the appropriate uniform integrability conditions, the convergence of the ESD is almost sure, and guaranteed for blocks of much larger size.

**Theorem 1.2.** Let $X_N$ be an $N \times N$ random matrix ensemble whose entries are jointly Gaussian. Assume the entries of $X_N$ satisfy the following conditions.

1. The second moments $\{N\mathbb{E}((X_{N})_{i,j}^2)\}_{N \in \mathbb{N}, 1 \leq i, j \leq N}$ are uniformly bounded.
2. For each $N$, there is a set partition $\Pi_N$ of $\{(i, j) : 1 \leq i, j \leq N\}$ and a constant $d_N = o(N^2 / \log N)$ such that each block of $\Pi_N$ has size $\leq d_N$, and the entries $[X_N]_{ij}$ and $[X_N]_{k\ell}$ are independent if $(i, j)$ and $(k, \ell)$ are not in the same block of $\Pi_N$.

Then the empirical spectral distribution $\mu_N$ of $X_N$ converges weakly almost surely to its mean:

\[
\int f \, d\mu_N - \mathbb{E}\left( \int f \, d\mu_N \right) \rightarrow 0 \text{ a.s. for all } f \in \text{Lip}(\mathbb{R}).
\] (1.3)

Condition (1) in Theorems 1.1 and 1.2 is analogous to the requirement that the second moments of the entries of $\sqrt{N}X_N$ are normalized in Wigner ensembles. Condition (2) generalizes the independent block structure mentioned above; for example, in the ensembles treated in [10, 11, 3] but with $k$ allowed to grow with $N$ (with $k = o(N^2 / \log N)$), one gets convergence of the ESD weakly almost surely. In particular, Theorem 1.2 extends the results of those papers even in the case $k = O(1)$, since only convergence in expectation was known before.

**Remark 1.3.** Note that the conclusion of Theorems 1.1 and 1.2 is that the ESDs of these ensembles concentrate around their means; it is not true that all these ensembles converge in expectation. Rather, our results are that any of these ensembles does converge in expectation also converge in probability, or almost surely, as the case may be. In Section 2.3 we discuss some examples where these results can be applied.

While we are most interested in ensembles with correlated entries, one of the main achievements of our method is an improvement on the (first half of the) main result in [2].

**Theorem 1.4.** Let $\{\xi_{ij} : 1 \leq i \leq j\}$ be zero mean unit variance i.i.d. random variables. Let $g : [0, 1]^2 \rightarrow \mathbb{R}_+$ be a symmetric, continuous function. If $[X_N]_{ij} = [X_N]_{ji} = N^{-1/2} g(i/N, j/N)^{1/2} \xi_{ij}$, then the empirical spectral distribution of $X_N$ converges weakly in probability to a probability measure on $\mathbb{R}$. (The limit ESD is the semicircle law if $\int_0^1 g(x, y) \, dy = 1$ for each $x \in [0, 1]$.) Moreover, if the $\xi_{ij}$ are bounded random variables, or if the common law of the entries $\xi_{ij}$ satisfies a log-Sobolev inequality (cf. (1.4) below), then the convergence is almost sure.

The ensembles addressed in Theorem 1.4 are the typical formulation of band matrices, although that name only really applies when the function $g$ has the form $g(x, y) = \mathbbm{1}_{|x-y| \leq \delta}$ for some $\delta \in (0, 1)$. (In order to satisfy the stochasticity condition to get the semicircle law in the limit, one must use periodic band matrices, where $g$ is the indicator of the strip $|x-y| \leq \delta$ on all of $\mathbb{R}^2$, projected into $[0, 1]^2$ via the equivalence relation identifying two points if they differ by an element of $\mathbb{Z}^2$. See [17, 18] for details.) The central theorem in [2] is a proof of the semicircular case of Theorem 1.4, assuming that the common law of the entries $\xi_{ij}$ satisfies a Poincaré inequality (cf. 3.1 below). Our Theorem 1.4 yields the convergence in complete generality, only assuming finite second moments; moreover, a technical condition on the laws of the entries (similar to the assumption of a Poincaré inequality) yields almost sure convergence.

**Remark 1.5.** It should be noted that this is only half of the main result in [2], where the authors also show that the fluctuations of these ensembles are Gaussian with an explicit covariance determined by the function $g$. Their methods are largely combinatorial, while ours are analytic/probabilistic.

Theorems 1.1, 1.4 are proved below in Section 2. (In fact, in Section 2.3 we prove the more general Theorem 2.11, of which Theorem 1.2 is a special case.) We prove these results using concentration of measure mediated by a powerful coercive inequality: the log-Sobolev inequality. A probability measure $\mu$ on $\mathbb{R}^d$ satisfies a log-Sobolev
inequality with constant $c$ if
\begin{equation}
\text{Ent}_\mu(f^2) \leq c \int |\nabla f|^2 \, d\mu
\end{equation}
for all sufficiently integrable positive functions $f$ with $\int f^2 \, d\mu = 1$; here $\text{Ent}_\mu(g) = \int g \log g \, d\mu$ for a $\mu$-probability density $g$. The inequality \eqref{eq:Ent} first appeared in \cite{35} (in a slightly different form, written in terms of $g = f^2$, where the Dirichlet form on the right-hand-side becomes the relative Fisher information of $g$), in the context of Gaussian measures. It was later rediscovered by Gross \cite{24} who named it a log-Sobolev inequality, and used it to prove an important result in constructive quantum field theory. Over the past four decades, it has played an important role probability theory, functional analysis, and differential geometry; see, for example, \cite{5, 6, 14, 16, 20, 21, 22, 23, 27, 30, 31, 32, 36, 40, 41, 42}. There is a big industry of literature devoted to necessary and sufficient conditions for a log-Sobolev inequality to hold; cf. \cite{8, 9, 15, 26, 29}. Many of the above applications rely on uniform concentration of measure bounds that hold for measures satisfying a log-Sobolev inequality; one nice form of these concentration inequalities is called a Herbst inequality, cf. \cite{26}, which yields Gaussian concentration of Lipschitz functionals about their mean. Using the Herbst inequality, Guionnet \cite{25} gave a fundamentally new proof of Wigner’s semicircle law; this proof automatically generalized \cite{39}, which yields Gaussian concentration of Lipschitz functionals about their mean. Using the Herbst inequality, we will need a multivariate version of the mollified log-Sobolev inequality, with sufficient growth in \cite{16}, building on our techniques, the authors generalized mollified log-Sobolev inequalities to $\mathbb{R}^d$ (and with a class of measures more general than compactly-supported), using a version of the Lyapunov approach as we do. However, they gave no quantitative bounds on the log-Sobolev constant, which is crucial to our present analysis.

Remark 1.7. To further expound on the history of Theorem \ref{thm:main} following the second author’s paper \cite{43}, in \cite{21} the authors generalized mollified log-Sobolev inequalities to $\mathbb{R}^d$ (and with a class of measures more general than compactly-supported), using a version of the Lyapunov approach as we do. However, they gave no quantitative bounds on the log-Sobolev constant, which is crucial to our present analysis.

Remark 1.8. We do not know if the optimal constant grows with dimension. In \cite{12}, some evidence is given to support the conjecture that the optimal constant is independent of dimension. For our present purposes, a dimension independent bound of this form would not improve our result in Theorem \ref{thm:main}. It is the exponential dependence of the constant on $||Y||_\infty$ that forces the blocks to be of size $o(\log N)$; and this dependence is sharp, as shown below in Example ??.
The remainder of this paper is organized as follows. In Section 2.1, we discuss how the log-Sobolev inequality can be used to yield concentration results for eigenvalues of random matrices. Following this, Section 2.2 gives the proof of Theorem 1.1. Then Section 2.3 proves Theorem 1.2 and a generalization (Theorem 2.11) which allows more general entries than Gaussians, and applies these results to several random matrix models from the literature. Section 2.4 then proves Theorem 1.4 as a corollary to Theorems 1.1 and 2.11 and discusses a generalization of band matrices where these results still apply. Finally, Section 3 is devoted to the proof of Theorem 1.6.

2. Concentration Results for Ensembles with Correlated Entries

2.1. Guionnet’s Approach to Wigner’s Law. Let us fix notation as in the introduction: let $X_N$ be a symmetric random $N \times N$ matrix ensemble with eigenvalues $\lambda_1^N \leq \cdots \leq \lambda_N^N$, and let $\mu_N$ denote the empirical spectral distribution (ESD) of $X_N$; cf. (1.1). Wigner’s law [37] states that $\mu_N$ converges weakly a.s. to the semicircle law $\sigma$, in the case that $X_N$ is a GOE$_N$. Wigner’s proof proceeded by the method of moments and is fundamentally combinatorial. Analytic approaches (involving fixed point equations, complex PDEs, and orthogonal polynomials) developed over the ensuing decades. An argument based on concentration of measure was provided by Guionnet in [25, p.70, Thm. 6.6]. The result can be stated thus.

**Theorem 2.1.** (Guionnet). Let $X_N$ be a symmetric random matrix. If the joint law of entries of $\sqrt{N}X_N$ satisfies a log-Sobolev inequality with constant $c$, then for all $\epsilon > 0$ and all Lipschitz $f : \mathbb{R} \to \mathbb{R}$,

$$
\mathbb{P}\left(\left|\int f \, d\mu_N - \mathbb{E}\left(\int f \, d\mu_N\right)\right| \geq \epsilon\right) \leq 2 \exp\left(-\frac{N^2\epsilon^2}{c||f||^2_{\text{Lip}}}\right).
$$

In fact, in the Wigner ensemble setting, the i.i.d. condition means we really need only assume that the law of each entry satisfies a log-Sobolev inequality. This is due to the following result often called Segal’s lemma; for a proof, see [24, p. 1074, Rk. 3.3].

**Lemma 2.2** (Segal’s Lemma). Let $\nu_1, \nu_2$ be probability measures on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, satisfying log-Sobolev inequalities with constants $c_1, c_2$, respectively. Then the product measure $\nu_1 \otimes \nu_2$ on $\mathbb{R}^{d_1+d_2}$ satisfies a log-Sobolev inequality with constant $\max\{c_1, c_2\}$.

Theorem 2.1 explicitly gives weak convergence in probability of $\mu_N$ to its limit mean. Moreover, in the Wigner ensemble case where the constant $c$ is determined by the common law of the entries and so doesn’t depend on $N$, the rate of convergence is fast enough that a standard Borel–Cantelli argument immediately upgrades this to a.s. convergence. In [43], the second author showed that, under certain integrability conditions, the empirical law of eigenvalues $\mu_N$ converges weakly in probability to its mean, regardless of whether or not the joint laws of entries satisfy a log-Sobolev inequality. The idea is to use the mollified log-Sobolev inequality (the $d = 1$ case of Theorem 1.6) applied to a cutoff of $X_N$ with GOE$_N$ noise added in with variance $t$, and then let $t \downarrow 0$.

For our present purposes, where we no longer assume independence or identical distribution of the entries of $X_N$, it will not suffice to assume each entry satisfies a (mollified) log-Sobolev inequality, which is why we state Guionnet’s result as such in Theorem 2.1. Guionnet proved the theorem from the Herbst concentration inequality [26], which shows that Lipschitz functionals of a random variable whose law satisfies a log-Sobolev inequality have sub-Gaussian tails (with dimension-independent bounds determined by the Lipschitz norm of the functional). Theorem 2.1 is then proved by combining this with Lipschitz functional calculus, together with the following lemma from matrix theory (see [28, p.37, Thm. 1, and p.39, Rk. 2]).

**Lemma 2.3.** (Hoffman, Wielandt). Let $A, B$ be symmetric $N \times N$ matrices with eigenvalues $\lambda_1^A \leq \lambda_2^A \leq \cdots \leq \lambda_N^A$ and $\lambda_1^B \leq \lambda_2^B \leq \cdots \leq \lambda_N^B$. Then

$$
\sum_{j=1}^{N} (\lambda_j^A - \lambda_j^B)^2 \leq \text{Tr}[(A - B)^2].
$$
2.2. The Proof of Theorem 1.1 We now prove to proceed Theorem 1.1 using Theorem 1.6. We first prove the second statement of the theorem: let \( X_N \) be the matrix ensemble satisfying conditions (1) and (2) of Theorem 1.1 together with the assumption that the entries of \( \sqrt{N}X_N \) are bounded by some uniform constant \( R \), \( \|\sqrt{N}[X_N]_{ij}\|_{\infty} \leq R \) for all \( N \) and all \( 1 \leq i, j \leq N \). (This latter assumption subsumes (1).) Denote the blocks of the partition in assumption (2) as \( \Pi_N = \{P_1, \ldots, P_r\} \).

Now, let \( t = t_N > 0 \) (to be chosen later), and let \( G_N \) be a GOE (with entries of variance \( \frac{1}{N} \)) independent from \( X_N \). Set

\[
\widetilde{X}_N = X_N + tG_N.
\]

For \( 1 \leq k \leq r \), let \( Y_k \) denote the random vector in \( \mathbb{R}^{|P_k|} \) given by the entries \( [X_N]_{ij} \) with \( (i, j) \in P_k \); similarly, let \( Z_k \) be the corresponding entries of \( G_N \). Notice that \( \sqrt{N}Y_k \) is a bounded random vector: by assumption, all of its entries have \( L^\infty \)-norm \( \leq R \), and so \( ||N|Y_k||_\infty \leq R|P_k|^{1/2} \leq R^3N^{1/2} \). The vector \( \sqrt{N}Z_k \) is a standard normal random vector in \( \mathbb{R}^{|P_k|} \). Thus, by Theorem 1.6 the law of \( \sqrt{N}(Y_k + tZ_k) \) satisfies a log-Sobolev inequality with constant

\[
c(t) \leq \left( K_1d_N + K_2\frac{R^2d_N}{t} \right) R^2d_N \exp \left( \frac{4R^2d_N}{t} \right) \leq K_3R^4\frac{d_N^2}{t} \exp \left( \frac{4R^2d_N}{t} \right)
\]

where \( K_3 = \max\{K_1, K_2\} \), and we have assumed that \( t \leq 1 \) and \( R \geq 1 \). By assumption, the random variables \( \{Y_k\}_{k=1}^r \) are independent, as are \( \{Z_k\}_{k=1}^r \). Hence \( \{\sqrt{N}(Y_k + tZ_k)\}_{k=1}^r \) are independent. Thus, the joint law of entries of \( \sqrt{N}\widetilde{X}_N \) is the product measure of the laws of these random variables. As all their laws satisfy log-Sobolev inequalities with the same constant \( c(t) \) in (2.2), Segal’s Lemma 2.2 shows that:

**Corollary 2.4.** The joint law of entries of \( \sqrt{N}\widetilde{X}_N \) satisfies a log-Sobolev inequality with constant \( c(t) \) of (2.2).

In particular, Guionnet’s Theorem 2.1 shows that the (Lipschitz) linear statistics of the ensemble \( \widetilde{X}_N \) are highly concentrated around their means (for fixed \( t \)).

Our goal is now to compare the linear statistics of \( X_N \) to those of \( \widetilde{X}_N \). As usual, let \( \mu_N \) denote the ESD of \( X_N \), and let \( \widetilde{\mu}_N \) denote the ESD of \( \widetilde{X}_N \). Then, for each \( \epsilon > 0 \), and each test function \( f \), we have the following standard triangle inequality estimate.

\[
\left| \int f \, d\mu_N - \mathbb{E} \left( \int f \, d\mu_N \right) \right| \leq \left| \int f \, d\mu_N - \int f \, d\widetilde{\mu}_N \right| \quad (2.3)
\]

\[
+ \left| \int f \, d\widetilde{\mu}_N - \mathbb{E} \left( \int f \, d\widetilde{\mu}_N \right) \right| \quad (2.4)
\]

\[
+ \left| \mathbb{E} \left( \int f \, d\mu_N \right) - \mathbb{E} \left( \int f \, d\mu_N \right) \right| \quad (2.5)
\]

We will now show that, with a judicious choice of \( t = t_N \), each of the quantities (2.3)-(2.5) converges to 0 a.s. We do this in the following three lemmas.

**Lemma 2.5.** Let \( t = t_N > 0 \) be a sequence tending to 0. Then for each \( f \in \text{Lip}(\mathbb{R}) \),

\[
\left| \int f \, d\mu_N - \int f \, d\widetilde{\mu}_N \right| \to 0 \text{ a.s. as } N \to \infty.
\]
Proof. Let $\lambda_1^N \leq \lambda_2^N \leq \ldots \leq \lambda_N^N$ and $\tilde{\lambda}_1^N \leq \tilde{\lambda}_2^N \leq \ldots \leq \tilde{\lambda}_N^N$ be the eigenvalues of $X_N$ and $\tilde{X}_N$. Then by the Cauchy-Schwarz inequality and Lemma 1.3, we have

$$\left| \int f \, d\mu_N - \int f \, d\tilde{\mu}_N \right| = \frac{1}{N} \left| \sum_{j=1}^{N} \left[ f(\lambda_j^N) - f(\tilde{\lambda}_j^N) \right] \right| \leq \frac{1}{N} \sum_{i=1}^{N} \|f\|_{\text{Lip}} \left| \lambda_i^N - \tilde{\lambda}_i^N \right| \leq \frac{\|f\|_{\text{Lip}}}{\sqrt{N}} \left( \sum_{i=1}^{N} (\lambda_i^N - \tilde{\lambda}_i^N)^2 \right)^{1/2} \leq \frac{\|f\|_{\text{Lip}}}{\sqrt{N}} \left( \text{Tr}[(X_N - \tilde{X}_N)^2] \right)^{1/2}.$$ 

Now, for any symmetric $N \times N$ matrix $A$, $(\frac{1}{N} \text{Tr}(A^2))^{1/2}$ is the non-commutative $L^2$-norm of $A$ with respect to the faithful normal state $\frac{1}{N} \text{Tr}$; it is bounded above by the operator norm of $A$. Applying this to $A = X_N - \tilde{X}_N = t_N G_N$, we therefore have

$$\left| \int f \, d\mu_N - \int f \, d\tilde{\mu}_N \right| \leq \|f\|_{\text{Lip}}\|X_N - \tilde{X}_N\|_{\text{op}} = t_N \|f\|_{\text{Lip}}\|G_N\|_{\text{op}} \text{ a.s.}$$

According to [4], the largest eigenvalue $\|G_N\|_{\text{op}}$ of the GOE$_N$ is $a.s. \leq 3$ for all sufficiently large $N$. This proves the result. 

Lemma 2.6. Let $f \in \text{Lip}(\mathbb{R})$, and suppose $t_N$ is chosen such that $c(t_N) = o\left( \frac{N^2}{\log N} \right)$, where $c(t)$ denote the log-Sobolev constant in (2.2). Then

$$\int f \, d\mu_N - \mathbb{E} \left( \int f \, d\tilde{\mu}_N \right) \to 0 \text{ a.s. as } N \to \infty.$$ 

Proof. Theorem 2.1 and Corollary 2.4 yield that, for any $\epsilon > 0$ ans $N \in \mathbb{N}$,

$$\mathbb{P} \left( \left| \int f \, d\tilde{\mu}_N - \mathbb{E} \left( \int f \, d\tilde{\mu}_N \right) \right| \geq \epsilon \right) \leq 2 \exp \left( - \frac{N^2 \epsilon^2}{c(t_N) \|f\|_{\text{Lip}}^2} \right).$$

By assumption, there is a sequence $s_N \to 0$ so that $c(t_N) = \frac{N^2}{\log N} s_N$. Thus

$$\exp \left( - \frac{N^2 \epsilon^2}{c(t_N) \|f\|_{\text{Lip}}^2} \right) = \exp \left( - \frac{\epsilon^2}{\|f\|_{\text{Lip}}^2} \frac{\log N}{s_N} \right) = N^{-\epsilon^2 \frac{1}{\|f\|_{\text{Lip}}^2} \frac{1}{s_N}}.$$ 

Since $\frac{1}{s_N} \to \infty$, for all sufficiently large $N$ this is $\leq \frac{1}{N^2}$. The result now follows from the Borel–Cantelli lemma. 

Lemma 2.7. Let $t = t_N > 0$ be a sequence tending to 0. Then for each $f \in \text{Lip}(\mathbb{R})$,

$$\mathbb{E} \left( \int f \, d\tilde{\mu}_N \right) - \mathbb{E} \left( \int f \, d\mu_N \right) \to 0 \text{ as } N \to \infty.$$ 

Proof. In Lemma 2.5 we showed that $\int f \, d\tilde{\mu}_N - \int f \, d\mu_N \to 0$ a.s. Hence, to show that the expectation goes to 0, it suffices to show that these random variables have finite $L^1$-norm for all large $N$. This follows from estimates like the ones in the proof of Lemma 2.5

$$\mathbb{E} \left( \left| \int f \, d\tilde{\mu}_N - \int f \, d\mu_N \right| \right) \leq \mathbb{E} \left( \frac{\|f\|_{\text{Lip}}}{\sqrt{N}} \left( \text{Tr}[(X_N - \tilde{X}_N)^2] \right)^{1/2} \right) \leq \frac{\|f\|_{\text{Lip}}}{\sqrt{N}} \left( \mathbb{E} \left( \text{Tr}[(X_N - \tilde{X}_N)^2] \right)^{1/2} \right)^{1/2} = \|f\|_{\text{Lip}} \frac{1}{N} < \infty$$

where we applied Jensen’s inequality in the second step. The result follows.
We can now prove the theorem under the boundedness assumption.

Proof of Theorem 1.1. Assuming \( \sqrt{N} X_N \) has entries uniformly bounded by \( R \). In light of Lemma 2.5 and 2.7, it suffices to show that there is a sequence \( t_N \) with \( t_N \to 0 \) and \( c(t_N) = o(\frac{N^2}{\log N}) \). For \( N \) sufficiently large, we define

\[
t_N := \frac{5R^2 d_N}{\log \frac{N}{K_3R^2}}.
\]

By Assumption (2) of Theorem 1.1, \( d_N = o(\log N) \), and hence \( t_N \to 0 \) as \( N \to \infty \). As such, \( \frac{R^2}{t_N} > 2 \) for all large \( N \), and it follows (from elementary calculus) that \( \frac{R^2}{t_N} d_N^2 \leq \exp(\frac{R^2}{t_N} d_N) \). Thus, (2.2) yields

\[
c(t_N) \leq K_3 R^2 \cdot \frac{R^2}{t_N} d_N^2 \exp \left( \frac{4R^2 d_N}{t_N} \right) \leq K_3 R^2 \exp \left( \frac{5R^2 d_N}{t_N} \right) = N = o \left( \frac{N^2}{\log N} \right).
\]

This concludes the proof. \( \square \)

Remark 2.8. We could have arranged for \( c(t_N) \) to be of larger order but still \( o(N^2/\log N) \), but this would only have resulted in the ratio \( d_N/t_N \) being a constant factor larger, and thus would still require \( d_N = o(\log N) \) in order for it to be possible for \( t_N \to 0 \). Moreover, even if the blunt estimate \( \frac{R^2}{t_N} d_N^2 \leq \exp(\frac{R^2}{t_N} d_N) \) had not been employed, or even if (2.2) were known to hold without the prefactor (as might be true if the sharp form Theorem 1.6 held with a constant independent of dimension), it would still be impossible to arrange for \( t_N \to 0 \) while \( c(t_N) = o(\frac{N^2}{\log N}) \) unless \( d_N = o(\log N) \). That is: the result of Theorem 1.1 cannot be improved using the approach of this paper.

To conclude the proof, it remains only to remove the boundedness assumption on the entries of \( \sqrt{N} X_N \) (at the expense of a downgrade from almost sure convergence to convergence in probability). This is where the uniform integrability comes in, via a standard cutoff argument that we briefly outline. Let \( \epsilon, \eta > 0 \). Let \( f \in \text{Lip}(\mathbb{R}) \). By uniform integrability, there exists some \( R \geq 0 \) such that

\[
\mathbb{E} \left( N[X_{N}]_{ij}^2 \cdot 1_{\{\sqrt{N}||X_N||_{ij} > R\}} \right) < \min(1, \eta) \cdot \epsilon^2 / (9\|f\|_{\text{Lip}}^2)
\]

for all \( i, j, N \). Let \( \hat{X}_N \) be the matrix whose entries are the appropriate cutoffs of \( X_N \):

\[
[\hat{X}_N]_{ij} = [X_N]_{ij} \cdot 1_{\{\sqrt{N}||X_N||_{ij} \leq R\}}.
\]

Then \( ||\sqrt{N} \hat{X}_{ij}||_\infty \leq R \) for all \( N, i, j \). Let \( \hat{\mu}_N \) denote the ESD of \( \hat{X}_N \). The preceding proof shows that \( \int f \, d\mu_N \) converge to its mean almost surely, and hence in probability. We now compare the linear statistics of \( \mu_N \) and \( \hat{\mu}_N \). This is similar to the preceding analysis. We make the standard \( \epsilon/3 \)-decomposition:

\[
P \left( \left| \int f \, d\mu_N - \mathbb{E} \left( \int f \, d\mu_N \right) \right| \geq \epsilon \right) \leq P \left( \left| \int f \, d\mu_N - \int f \, d\hat{\mu}_N \right| \geq \frac{\epsilon}{3} \right) + P \left( \left| \int f \, d\hat{\mu}_N - \mathbb{E} \left( \int f \, d\hat{\mu}_N \right) \right| \geq \frac{\epsilon}{3} \right) + P \left( \left| \mathbb{E} \left( \int f \, d\hat{\mu}_N \right) - \mathbb{E} \left( \int f \, d\mu_N \right) \right| \geq \frac{\epsilon}{3} \right). \tag{2.6}
\]

The above proof in the uniform bounded case shows that the second term in (2.6) converges to 0 as \( N \to \infty \). The first term on the right hand side of (2.6) is bounded using the same reasoning as done in the proof of Lemma 2.6.
Since $\eta > 0$, thus the push-forward measure $T_\mu$ by a change of variables, (2.7) therefore shows that $P$.

Finally, the third term is bounded as in Lemma 2.7:

$$
\left| E \left( \int f \, d\hat{\mu}_N \right) - E \left( \int f \, d\mu_N \right) \right| 
\leq \frac{\|f\|_{\text{Lip}}}{\sqrt{N}} \left( \sum_{i,j} E \left( \left| X_N \right|^2 \mathbb{1}_{\{\|X_N\| > R\}} \right) \right)^{1/2}
$$

so $P \left( \left| E \left( \int f \, d\hat{\mu}_N \right) - E \left( \int f \, d\mu_N \right) \right| \geq \frac{\epsilon}{3} \right) = 0$. Therefore

$$
\limsup_{N \to \infty} P \left( \left| \int f \, d\mu_N - E \left( \int f \, d\mu_N \right) \right| \geq \epsilon \right) \leq \eta.
$$

Since $\eta > 0$ was arbitrary, we have $P \left( \left| \int f \, d\mu_N - E \left( \int f \, d\mu_N \right) \right| \geq \epsilon \right) \to 0$ as $N \to \infty$, giving convergence in probability. This concludes the proof.

Remark 2.9. Instead of doing the Gaussian mollification and then the cutoff argument, we could combine the two in the hopes of proving almost sure convergence in the general case. The obstruction to this is Lemma 2.5, where we used the fact (proved in [4]) that the GOE has no asymptotic outlier eigenvalues above 2: with probability 1, all eigenvalues are eventually $\leq 3$, for example. If we were to combine the cutoff argument with the mollification argument, in this lemma $X_N - \bar{X}_N$ would not be $t_N G_N$ but rather $t_N G_N + (X_N - \bar{X}_N)$; i.e. Gaussian noise plus a matrix whose entries are of the form $[X_N]_{ij} \mathbb{1}_{\{\|X_N\| > R\}}$. If the entries of $X_N$ were independent, then additional moment growth assumptions would imply the necessary lack of outlier eigenvalues following [4]; however, in our case where the entries may be correlated, the behavior of the largest eigenvalue is, at present, unknown.

2.3. Theorem 1.2, a Generalization, and Applications. We begin with a lemma which appeared in the second author’s paper [44 Prop. 6]. We reproduce the simple proof here, for completeness.

**Lemma 2.10.** Let $X$ be a random vector in $\mathbb{R}^d$ whose law satisfies a log-Sobolev inequality (1.4) with constant $c$. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz map. Then the law of $T(X)$ satisfies a log-Sobolev inequality with constant $c\|T\|_{\text{Lip}}^2$.

**Proof.** Let $\mu$ denote the law of $X$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally-Lipschitz non-negative function. Then $f \circ T$ is locally-Lipschitz and non-negative. Since $\mu$ satisfies the LSI with constant $c$, it follows that

$$
\int (f \circ T)^2 \log \frac{(f \circ T)^2}{(f \circ T)^2} \, d\mu \leq c \int |\nabla (f \circ T)|^2 \, d\mu.
$$

Since $T$ is Lipschitz, we also have the pointwise estimate

$$
|\nabla (f \circ T)| \leq (|\nabla f| \circ T) \|T\|_{\text{Lip}}.
$$

By a change of variables, (2.7) therefore shows that

$$
\int f^2 \log \frac{f^2}{f^2} dT_* \mu \leq c\|T\|_{\text{Lip}}^2 \int |\nabla f|^2 \, dT_* \mu.
$$

Thus, the push-forward measure $T_* \mu$ satisfies the LSI with constant $c\|T\|_{\text{Lip}}^2$. Since $T_* \mu$ is the law of $T(X)$, this concludes the proof. \qed
The following theorem covers a wide range of examples of correlated random matrix ensembles. We use the notation $M_N^{\text{sym}}$ to denote the vector space of real $N \times N$ symmetric matrices, equipped with the Hilbert–Schmidt inner product.

**Theorem 2.11.** Let $\{\xi_{ij} : 1 \leq i \leq j\}$ be a triangular array of i.i.d. random variables whose common law satisfies a log-Sobolev inequality \((1.4)\). Let $\Xi_N$ be the $N \times N$ symmetric random matrix with entries $[\Xi_N]_{ij} = \xi_{ij}$ for $1 \leq i \leq j \leq N$. Let $T_N : M_N^{\text{sym}} \to M_N^{\text{sym}}$ be a Lipschitz function, with $\|T_N\|_{\text{Lip}} = o\left(\frac{N}{\log N}\right)$.

Let $X_N = T_N(N^{-1/2}\Xi_N)$, and let $\mu_N$ denote the ESD of $X_N$. Then $\mu_N$ converges to its mean almost surely:

$$
\int f \, d\mu_N - \mathbb{E}\left(\int f \, d\mu_N\right) \to 0 \ a.s. \quad \text{for all } f \in \text{Lip}(\mathbb{R}).
$$

**Proof.** Let $S_N : M_N^{\text{sym}} \to M_N^{\text{sym}}$ be the conjugate-scaled map $S_N(A) = N^{1/2}T_N(N^{-1/2}A)$. Then $S_N$ is also Lipschitz with $\|S_N\|_{\text{Lip}} = \|T_N\|_{\text{Lip}}$. By assumption, the entries $\xi_{ij}$ satisfy a LSI with some constant $c$; by Lemma 2.2, the joint law of $\Xi_N$ therefore satisfies the LSI with constant $c$. Hence, by Lemma 2.10, $S_N(\Xi_N) = \sqrt{N}X_N$ satisfies the LSI with constant $c\|S_N\|^2_{\text{Lip}} = c\|T_N\|^2_{\text{Lip}}$. By Theorem 2.1, it therefore follows that, for any $\epsilon > 0$ and $f \in \text{Lip}(\mathbb{R})$,

$$
\mathbb{P}\left(\left| \int f \, d\mu_N - \mathbb{E}\left(\int f \, d\mu_N\right) \right| \geq \epsilon \right) \leq 2 \exp\left(-\frac{\epsilon^2 N^2}{c\|T_N\|^2_{\text{Lip}}\|f\|^2_{\text{Lip}}}\right).
$$

By assumption, $\|T_N\|^2_{\text{Lip}} = o\left(\frac{N^2}{\log N}\right)$. The result now follows exactly as in the proof of Lemma 2.6. □

We now prove Theorem 1.2 essentially as a Corollary to Theorem 2.11 (although we really prove it as a corollary to the proof of Theorem 2.11 to most easily deduce the optimal result).

**Proof of Theorem 1.2** To begin, we clarify what is meant by “jointly Gaussian”. We say a random vector $X \in \mathbb{R}^d$ has jointly Gaussian entries if there is an affine map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $X = T(G)$, where $G$ has i.i.d. normal entries. A more standard definition of “jointly Gaussian” — that the joint law of the centered entries should have a density of the form $\exp(x \cdot C^{-1}x)$ for a positive definite matrix $C$ and a normalization constant $c$ — is a special case: it is easy to check $\sqrt{C^{-1}}(X)$ has i.i.d. standard normal entries, and so $T = \sqrt{C}$ will suffice.

Let $\Pi_N = \{P_1, \ldots, P_r\}$ denote the partition of $\{(i, j) : 1 \leq i \leq j \leq N\}$ in the theorem, and for $1 \leq k \leq r$ let $X_k$ denote the random vector given by the entries of $X_N$ with indices in $P_k$. By assumption, the random variables $X_1, \ldots, X_r$ are independent; it follows that there are affine maps $T_1, \ldots, T_r$ with $T_k : \mathbb{R}^{|P_k|} \to \mathbb{R}^{|P_k|}$, such that $N^{1/2}X_k = T_k(G_k)$, where $G_k$ is a standard Gaussian random vector in $\mathbb{R}^{|P_k|}$. The law of $G_k$ satisfies a log-Sobolev inequality with constant 1 (cf. [24]), and therefore by Lemma 2.10 the law of $N^{1/2}X_k$ satisfies a log-Sobolev inequality with constant $\|T_k\|^2_{\text{Lip}}$.

Now, $T_k$ has the form $T_k = \tilde{T}_k + N^{1/2}E(X_k)$ for some linear map $\tilde{T}_k$, and $\|T_k\|_{\text{Lip}} = \|\tilde{T}_k\|_{\text{op}} \leq \|\tilde{T}_k\|_{\text{HS}}$, where $\|\cdot\|_{\text{HS}}$ is the (un-normalized) Hilbert–Schmidt norm. Thus, we have

$$
\|T_k\|^2_{\text{Lip}} \leq \sum_{a,b=1}^{|P_k|} [\tilde{T}_k]^2_{ab}
$$

(2.9)

where we use the indices $a, b$ to enumerate the entries of $X_k$. Now, note that

$$
\text{Var}(N^{1/2}[X_k]_a) = \text{Var}([\tilde{T}_k(G_k)]_a) = \text{Var} \left( \sum_b [\tilde{T}_k]_{ab}[G_k]_b \right) = \sum_b [\tilde{T}_k]^2_{ab}
$$

because $\text{Cov}([G_k]_b,[G_k]_c) = \delta_{bc}$. By assumption, there is a uniform bound $R$ so that $\text{Var}(N^{1/2}[X_k]_a) \leq N\mathbb{E}([X_k]_a^2) \leq R^2$ for all $k$ and $a$. Thus (2.9) yields

$$
\|T_k\|^2_{\text{Lip}} \leq \sum_{a=1}^{|P_k|} \text{Var}(N^{1/2}[X_k]_a) \leq R^2|P_k| \leq R^2d_N.
$$
We have thus shown that the law of $X_k^{1/2}$ satisfies a log-Sobolev inequality with constant $R^2d_N$, for each $k$. Since the random variables $X_k$ are independent, Lemma 2.2 shows that the joint law of entries of $X_N^{1/2}$ satisfies a log-Sobolev inequality with constant $R^2d_N$. Since $d_N = o\left(\frac{N}{\log N}\right)$, the almost sure convergence now follows precisely as in the proof of Theorem 2.11 above.

□

Remark 2.12. (1) Theorem 1.2 is really a special case of Theorem 2.11, what the preceding proof essentially does is show that the affine function which is “block diagonal” combining all the block $T_k$ maps has Lipschitz norm $= o\left(\frac{N}{\sqrt{\log N}}\right)$.

(2) The proof shows that it is really enough to assume, in the statement of Theorem 1.2, that the scaled variances of the entries are uniformly bounded. However, in any instance we wish to apply the theorem, we must have the expectations of empirical integrals converging, and hence there is no loss in making the nominally stronger assumption that the scaled second moments of the entries are uniformly bounded.

2.4. Theorem 1.4 and Generalizations. In this section, we begin by showing how to prove Theorem 1.4 as a straightforward corollary to Theorems 1.1 and 2.11. To begin, we note that the topic of the paper [34] is the convergence of the ESD, it suffices by Theorem 2.11 to show that

\[ \|T_N\|_{Lip} = O\left(\frac{N}{\sqrt{\log N}}\right). \]

But since $T_N$ is linear and diagonal, its Lipschitz norm (i.e. operator norm) is simply the maximum modulus of the entries, $\|T_N\|_{op} = \max_{i,j} |g(i/N, j/N)| \leq \|g\|_{\infty} = O(1) = o\left(\frac{N}{\sqrt{\log N}}\right)$. This concludes the proof.

□

3. Mollified Log-Sobolev Inequalities on $\mathbb{R}^d$

In this section we will prove Theorem 1.6. For convenience, we restate it below as Theorem 3.1 in measure theoretic language.
Theorem 3.1. Let $\mu$ be a probability measure on $\mathbb{R}^d$ whose support is contained in a ball of radius $R$, and let $\gamma_t$ be the centered Gaussian of variance $t$ with $0 < t \leq R^2$, i.e., $\gamma_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right) dx$. Then for some absolute constant $K$, the optimal log-Sobolev constant $c(t)$ for the convolution $\mu * \gamma_t$ satisfies
\[
c(t) \leq K R^2 \exp\left(20d + \frac{5R^2}{t}\right).
\]
$K$ can be taken above to be 289.

Remark 3.2. Theorem 3.1 is slightly more general than Theorem 1.6 since it only requires the support to be contained in some ball of radius $R$; by contrast, in Theorem 1.6 $R$ is the radius of a ball centered at 0 containing $\text{supp} \, \mu$. If we could use the theorem in this form, we could actually improve Theorem 1.1 by softening the requirement that the entires be uniformly square integrable, only requiring their centered versions $\sqrt{N}(\langle X_N \rangle_{ij} - \mathbb{E}(\langle X_N \rangle_{ij}))$ to be uniformly square integrable. However, since any ensembles we wish to apply Theorem 1.1 to must converge in expectation, this does not give any practical improvement.

3.1. The Proof of Theorem 3.1 To prove Theorem 3.1, we use the following theorem (see [19, p.288, Thm. 1.4]):

Theorem 3.3. (Cattiaux, Guillin, Wu). Let $\mu$ be a probability measure on $\mathbb{R}^d$ with $d\mu(x) = e^{-V(x)} dx$ for some $V \in C^2(\mathbb{R}^d)$. Suppose the following:

(1) There exists a constant $K \leq 0$ such that $\text{Hess}(V) \geq K I$.

(2) There exists a $W \in C^2(\mathbb{R}^d)$ with $W \geq 1$ and constants $b, c > 0$ such that
\[
tW(x) - \langle \nabla V, \nabla W \rangle(x) \leq (b - c|x|^2)W(x)
\]
for all $x \in \mathbb{R}^d$.

Then $\mu$ satisfies a LSI. In particular, let $r_0, b', \lambda > 0$ be such that
\[
tW(x) - \langle \nabla V, \nabla W \rangle(x) \leq -\lambda W(x) + b' 1_{B_{r_0}}
\]
where $B_{r_0}$ denotes the ball centered at 0 of radius $r_0$ (the existence of such $r_0, b', \lambda$ is implied by Assumption 2). By [7, p.61, Thm. 1.4], $\mu$ satisfies a Poincaré inequality with constant $C_P$; that is, for every sufficiently smooth $g$ with $\int g \, d\mu = 0$,
\[
\int g^2 d\mu \leq C_P \int |\nabla g|^2 d\mu;
\]
$C_P$ can be taken to be $(1 + b' \kappa_{r_0})/\lambda$, where $\kappa_{r_0}$ is the Poincaré constant of $\mu$ restricted to $B_{r_0}$. A bound for $\kappa_{r_0}$ is
\[
\kappa_{r_0} \leq Dr_0^2 \sup_{x \in B_{r_0}} p(x) \frac{\inf_{x \in B_{r_0}} p(x)}{p(x)},
\]
where $p(x) = e^{-V(x)}$ and $D$ is some absolute constant that can be taken to be $4/\pi^2$. Let
\[
A = \frac{2}{c} \left(1 - \frac{K}{2}\right) + \epsilon
\]
\[
B = \frac{2}{c} \left(1 - \frac{K}{2}\right) \left(b + c \int |x|^2 d\mu(x)\right),
\]
where $\epsilon$ is an arbitrarily chosen parameter. Then $\mu$ satisfies a LSI with constant $A + (B + 2)C_P$.

We remark that the statement of Theorem 3.3 is given in [19] in the more general context of Riemannian manifolds. Also, the constants given above are derived in [19] but not presented there; for our purposes we have collected those constants and presented them here.

With the above, we now prove Theorem 3.1, which we restate here for the reader’s convenience.
Theorem 3.4. Let \( \mu \) be a probability measure on \( \mathbb{R}^d \) whose support is contained in a ball of radius \( R \), and let \( \gamma_t \) be the centered Gaussian of variance \( t \) with \( 0 < t \leq R^2 \), i.e., \( d\gamma_t(x) = (2\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{2t}\right) \) \( dx \). Then for some absolute constant \( K \), the optimal log-Sobolev constant \( c(t) \) for \( \mu \ast \gamma_t \) satisfies

\[
c(t) \leq K R^2 \exp \left( 20n + \frac{5R^2}{t} \right).
\]

\( K \) can be taken above to be \( 289 \).

Proof. By translation invariance of LSI, we will assume that \( \mu \) is supported in \( B_R \). We will apply Theorem 3.3 to \( \mu \) and compute the appropriate bounds and expressions for \( K, W, b, c, r_0, \lambda, \kappa, f, \int |x|^2 d\mu_\mu(x), A, \) and \( B \).

To find \( K, b, \) and \( c \), we follow the computations as done in [39, pp. 7-8]. Let

\[
V_t(x) = \frac{x^2}{2t} \quad \text{and} \quad V_t^*(x) = -\log(p_t(x)),
\]

so

\[
d\mu_t(x) = e^{-V_t(x)} dx = d(e^{-V} \ast \mu)(x).
\]

Also let

\[
d\mu_x(z) = \frac{1}{p_t(x)} e^{-V(x-z)} d\mu(z),
\]

so \( \mu_x \) is a probability measure for each \( x \in \mathbb{R}^d \). Then for \( X \in \mathbb{R}^d \) with \( |X| = 1 \),

\[
\text{Hess}(V_t)(X,X)(x) = \left( \int_{B_R} \nabla_X V(x-z) d\mu_x(z) \right)^2 - \int_{B_R} (|\nabla_X V(x-z)|^2 - \text{Hess}(V)(X,X)(x-z)) d\mu_x(z)
\]

\[
= \frac{1}{t} - \left( \int_{B_R} |\nabla_X V(x-z)|^2 d\mu_x(z) - \left( \int_{B_R} \nabla_X V(x-z) d\mu_x(z) \right)^2 \right)
\]

since \( \text{Hess}(V) = \frac{1}{t} I \).

But for any \( C^1 \) function \( f \),

\[
\int_{B_R} f^2 d\mu_x(z) - \left( \int_{B_R} f d\mu_x(z) \right)^2 = \frac{1}{2} \int_{B_R \times B_R} (f(z) - f(y))^2 d\mu_x(z) d\mu_x(y)
\]

\[
\leq 2R^2 \sup |\nabla f|^2,
\]

so for \( f = \nabla_X V \), we get

\[
\text{Hess}(V_t)(X,X)(x) \geq \frac{1}{t} - 2R^2 \sup |\nabla (\nabla_X V)|^2 = \frac{1}{t} - \frac{2R^2}{t^2}.
\]

So we take

\[
K = \frac{1}{t} - \frac{2R^2}{t^2}.
\]

Note \( K \leq 0 \) since \( t \leq R^2 \).

Let

\[
W(x) = \exp \left( \frac{|x|^2}{16t} \right).
\]
We claim that
\[ \kappa \leq 1, \] as desired.

We have
\[ \langle x, \nabla V(x - z) \rangle d\mu_x(z) \]
so we take
\[ \frac{tW - \langle \nabla V_t, \nabla W \rangle}{W} \]
Using \( |x| \leq |x|^2/2R + R/2 \) above, we get
\[ \frac{tW - \langle \nabla V_t, \nabla W \rangle}{W} \leq \frac{n}{8t} - \frac{3|x|^2}{64t^2} + \frac{R}{16t} + \frac{R^2}{32t^2} - \frac{1}{64t^2} |x|^2, \]
so we take
\[ b = \frac{n}{8t} + \frac{R^2}{32t^2}, \]
\[ c = \frac{1}{64t^2}. \]

Now let
\[ r_0 = \sqrt{16nt + 2R^2}, \]
\[ b' = \frac{1}{4t} \exp \left( n + \frac{R^2}{8t} - 1 \right), \]
\[ \lambda = \frac{n}{8t}. \]

We claim that
\[ b - c|x|^2 \leq -\lambda + b' \exp \left( -\frac{|x|^2}{16t} \right) \mathbb{1}_{B_{r_0}}, \quad \text{i.e.,} \quad \frac{b + \lambda - c|x|^2}{b'} \exp \left( \frac{|x|^2}{16t} \right) \leq \mathbb{1}_{B_{r_0}}, \]
so that
\[ tW(x) - \langle \nabla V, \nabla W \rangle(x) \leq -\lambda W(x) + b' \mathbb{1}_{B_{r_0}}. \]

We have
\[ \frac{b + \lambda - c|x|^2}{b'} \exp \left( \frac{|x|^2}{16t} \right) = 4t \exp \left( -n - \frac{R^2}{8t} + 1 \right) \left( \frac{n}{8t} + \frac{R^2}{32t^2} + \frac{n}{8t} - \frac{|x|^2}{64t^2} \right) \exp \left( \frac{|x|^2}{16t} \right) \]
\[ = \left( n + \frac{R^2}{8t} - \frac{|x|^2}{16t} \right) \exp \left( -\left( n + \frac{R^2}{8t} - \frac{|x|^2}{16t} \right) + 1 \right). \]

For \( |x| \geq r_0 \), the above expression is nonpositive, and for \( |x| \leq r_0 \), the above expression is of the form \( ue^{-u+1} \), which has a maximum value of 1, as desired.

Now we estimate \( \kappa_{r_0} \) by estimating \( \sup_{x \in B_{r_0}} p_t(x) \) and \( \inf_{x \in B_{r_0}} p_t(x) \). For \( x \in B_{r_0} \), we have
\[ p_t(x) = \int_{B_R} (2\pi t)^{-n/2} \exp \left( -\frac{|x - y|^2}{2t} \right) d\mu(y) \leq \int_{B_R} (2\pi t)^{-n/2} d\mu(y) = (2\pi t)^{-n/2} \]
14
and

\[ p_t(x) = \int_{B_R} (2\pi t)^{-n/2} \exp \left( -\frac{|x-y|^2}{2t} \right) \, d\mu(y) \geq \int_{B_R} (2\pi t)^{-n/2} \exp \left( -\frac{(r_0 + R)^2}{2t} \right) \, d\mu(y) \]

\[ = (2\pi t)^{-n/2} \exp \left( -\frac{(r_0 + R)^2}{2t} \right), \]

so

\[ \kappa_{r_0} \leq Dr_0^2 \sup_{x \in B_{r_0}} \frac{p(x)}{\inf_{x \in B_{r_0}} p(x)} \leq Dr_0^2 \exp \left( \frac{(r_0 + R)^2}{2t} \right). \]

We then take

\[ C_P = \frac{1 + \beta \kappa_{r_0}}{\lambda} \]

\[ \leq \frac{8t}{n} \left( 1 + \frac{1}{4t} \exp \left( n + \frac{R^2}{8t} - 1 \right) \cdot Dr_0^2 \exp \left( \frac{(r_0 + R)^2}{2t} \right) \right) \]

\[ \leq \frac{8t}{n} + \frac{D}{e} \left( 32t + \frac{4R^2}{n} \right) \exp \left( n + \frac{R^2}{8t} + \frac{(\sqrt{16nt + 2R^2 + R})}{2t} \right). \]

Using \( \sqrt{a} + \sqrt{b} \leq \sqrt{2(a + b)} \) and the assumptions \( t \leq R^2 \) and \( n \geq 1 \) above, we get

\[ C_P \leq \frac{8R^2}{1} + \frac{D}{e} \left( 32R^2 + \frac{4R^2}{1} \right) \exp \left( n + \frac{R^2}{8t} + \frac{\sqrt{2(16nt + 2R^2 + R^2)}}{2t} \right) \]

\[ = 8R^2 + \frac{36D}{e} R^2 \exp \left( 17n + \frac{25R^2}{8t} \right) \]

\[ \leq \left( 8 + \frac{36D}{e} \right) R^2 \exp \left( 17n + \frac{25R^2}{8t} \right). \]

Next, we estimate \( \int |x|^2 \, d\mu_t(x) \):

\[ \int_{\mathbb{R}^d} |x|^2 \, d\mu_t(x) = \int_{\mathbb{R}^d} \int_{B_R} |x|^2 (2\pi t)^{-n/2} \exp \left( -\frac{|x-y|^2}{2t} \right) \, d\mu(y) \, dx \]

\[ = (2\pi t)^{-n/2} \int_{B_R} \int_{\mathbb{R}^d} |x+y|^2 \, d\mu(y) \, dx \]

by replacing \( x \rightarrow x + y \)

\[ = (2\pi t)^{-n/2} \int_{B_R} \int_{\mathbb{R}^d} (|x|^2 + |y|^2) \exp \left( -\frac{|x|^2}{2t} \right) \, dx \, d\mu(y) \]

\[ + (2\pi t)^{-n/2} \int_{B_R} \int_{\mathbb{R}^d} 2(x, y) \exp \left( -\frac{|x|^2}{2t} \right) \, dx \, d\mu(y). \]

The second integral in the last expression above equals 0 since the integrand is an odd function of \( x \). So

\[ \int_{\mathbb{R}^d} |x|^2 \, d\mu_t(x) = (2\pi t)^{-n/2} \int_{B_R} \int_{\mathbb{R}^d} (|x|^2 + |y|^2) \exp \left( -\frac{|x|^2}{2t} \right) \, dx \, d\mu(y) \]

\[ \leq (2\pi t)^{-n/2} \int_{\mathbb{R}^d} \int_{B_R} (|x|^2 + R^2) \exp \left( -\frac{|x|^2}{2t} \right) \, d\mu(y) \, dx \]

\[ = (2\pi t)^{-n/2} \int_{\mathbb{R}^d} (|x|^2 + R^2) \exp \left( -\frac{|x|^2}{2t} \right) \, dx \]

\[ = nt + R^2. \]
the last integral computed using polar coordinates.

To get expressions for $A, B$, we choose $\epsilon = 16t$; then $A, B$ satisfy

$$A = \frac{2}{c} \left( \frac{1}{\epsilon} - \frac{K}{2} \right) + \epsilon = 128t^2 \left( \frac{1}{16t} - \left( \frac{1}{2t} - \frac{R^2}{t^2} \right) \right) + 16t = 128R^2 - 40t \leq 128R^2$$

and

$$B = \frac{2}{c} \left( \frac{1}{\epsilon} - \frac{K}{2} \right) \left( b + c \int |x|^2 d\mu_t(x) \right) \leq 128t^2 \left( \frac{1}{16t} - \left( \frac{1}{2t} - \frac{R^2}{t^2} \right) \right) \left( \frac{n}{8t} + \frac{R^2}{32t^2} + \frac{1}{64t^2} (nt + R^2) \right)$$

$$= \frac{18nR^2}{t} + \frac{6R^4}{t^2} - \frac{63n}{8} - \frac{21R^2}{8}$$

$$\leq \frac{18nR^2}{t} + \frac{6R^4}{t^2} - 2.$$

Putting everything together, we get that the optimal log-Sobolev constant $c(t)$ for $\mu_t$ satisfies

$$c(t) \leq A + (B + 2) C_P$$

$$\leq 128R^2 + \left( \frac{18nR^2}{t} + \frac{6R^4}{t^2} - 2 + 2 \right) \left( 8 + \frac{36D}{e} \right) \frac{R^2}{t^2} \exp \left( 17n + \frac{25R^2}{8t} \right)$$

$$= 128R^2 + 12 \cdot \frac{R^2}{2t} \left( 3n + \frac{R^2}{t} \right) \left( 8 + \frac{36D}{e} \right) \frac{R^2}{t^2} \exp \left( 17n + \frac{25R^2}{8t} \right).$$

Applying $u \leq e^u$ to two of the terms in the expression above, we get

$$c(t) \leq 128R^2 + 12 \exp \left( \frac{R^2}{2t} \right) \exp \left( 3n + \frac{R^2}{t} \right) \left( 8 + \frac{36D}{e} \right) \frac{R^2}{t^2} \exp \left( 17n + \frac{25R^2}{8t} \right)$$

$$= 128R^2 + \left( 96 + \frac{432D}{e} \right) \frac{R^2}{2t} \exp \left( 20n + \frac{37R^2}{8t} \right)$$

$$\leq \left( 128 + 96 + \frac{432D}{e} \right) \frac{R^2}{2t} \exp \left( 20n + \frac{5R^2}{t} \right)$$

$$\leq 289R^2 \exp \left( 20n + \frac{5R^2}{t} \right).$$

This concludes the proof of Theorem 3.1. □

3.2. Remarks on the Optimal Log-Sobolev Constant.

Example 3.5.

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