Changes of Variables in Noncommutative Polynomials

Classical Questions

The following are classical questions from complex analysis.

- **When is a domain in $\mathbb{C}^n$ bianalytic to a ball in $\mathbb{C}^n$?**
- **What are the analytic maps of the unit ball in $\mathbb{C}^n$ to the unit ball in $\mathbb{C}^m$ that carry the boundary to the boundary?**

We have examined some NC analoges to these problems and the results are pretty.

NC Polynomials

For the purposes of this talk we will consider polynomials with real coefficients in the $2g$ noncommuting variables $(x_1, \ldots, x_g, x_1{}^T, \ldots, x_g{}^T)$. We call a polynomial $p(\vec{x})$ NC analytic if $p(\vec{x})$ is a polynomial in the g noncommuting variables $(x_1, \ldots, x_g)$. *I.e.* **For any j, $x_j{}^T$ cannot be a factor of any term in any NC analytic polynomial.**

NC Polynomials

Suppose that $p(\vec{x})$ is a polynomial. Then there exist monomials $m_j(\vec{x})$ for $j = 1, \ldots, K$ such that

$$p(\vec{x}) = \sum_{j=1}^{K} m_j(\vec{x}).$$

We say that $p(\vec{x})$ is **hereditary** if for each $j = 1, \ldots, K$ there exist NC analytic polynomials $a_j(\vec{x})$ and $b_j(\vec{x})$ such that $m_j(\vec{x}) = a_j^T(\vec{x})b_j(\vec{x})$. *I.e* in each term of $p(\vec{x})$ any factors ${x_j}^T$ appear to the left of all nontransposed factors x_i.

Positive Polynomials

A polynomial, $p(\vec{x})$, is symmetric iff $p(\vec{x}^T) = p^T(\vec{x})$. We say that a symmetric polynomial $p(\vec{x})$ is matrix-positive if for all $n \geq 1$ and all $\vec{X} \in (\mathbb{R}^{n \times n})^g$, $p(\vec{X})$ is positive semi-definite.

Domains

Given a symmetric polynomial $p(\vec{x})$ the positivity domain of $p(\vec{x})$, $\mathcal{D}_p$, is defined as follows.

Let $\mathcal{D}_p^n := \{\vec{X} \in (\mathbb{R}^{n\times n})^g : p(\vec{X})$ is PsD $\}$. Then

$$\mathcal{D}_p := \bigcup_{n\in\mathbb{N}} \mathcal{D}_p^n.$$

Example: The matrix ball $B_\epsilon(0) := \mathcal{D}_{p_\epsilon}$ where

$$p_\epsilon(\vec{x}) := \epsilon - ({x_1}^T x_1 + \cdots + {x_g}^T x_g).$$

Convex Polynomials

Suppose that $\mathcal{D}$ is a convex domain. A symmetric polynomial, $p(\vec{x})$, is matrix convex iff for all $\vec{X}_1, \vec{X}_2 \in \mathcal{D}$ (with the same dimensions) and all $0 \leq t \leq 1$, the matrix $tp(\vec{X}_1) + (1-t)p(\vec{X}_2) - p(t\vec{X}_1 + (1-t)\vec{X}_2)$ is PsD.

NC Analytic Functions

Consider $\vec{x} = ({x_1}^T, \ldots, {x_g}^T)^T$ a vector of NC indeterminants and a formal power series

$$f(\vec{x}) = \sum_{|w|=1}^{\infty} a_w \vec{x}^{\,w}$$

where there is some k so that $a_w \in \mathbb{R}^k$.

We say that such a series $f(\vec{x})$ is NC analytic near 0 (or simply NC analytic) if there exixts some $\epsilon > 0$ so that for all n and $\vec{X} \in (\mathbb{R}^{n\times n})^g$ such that $\|\vec{X}\| < \epsilon$, then $f(\vec{X})$ converges.

Bianalytic Map Questions

Here are some analogues to an earlier question from complex analysis. Again the question was:

- **When is a domain in $\mathbb{C}^n$ bianalytic to a ball in $\mathbb{C}^n$?**

Bianalytic Map Question

Our NC analogue:

Q1. When is the domain $\boldsymbol{\mathcal{D}_r}$ *bianalytic to a matrix ball,* $\boldsymbol{B}$*, and what is the map between the domains?*

Algorithm A

In answering these questions we use an LMI algorithm. The input for the algorithm is a matrix-positive, hereditary polynomial, $p(\vec{x})$, with the property $p(0) = 0$. The algorithm outputs a vector of analytic polynomials, $q(\vec{x}) = \begin{pmatrix} q_1(\vec{x}) \\ \vdots \\ q_K(\vec{x}) \end{pmatrix}$ for some K, such that

$$p(\vec{x}) = q^T(\vec{x})q(\vec{x}) = \sum_{j=1}^{K} q_j^T(\vec{x})q_j(\vec{x}).$$

Step 1 in Algorithm A

Write $p(\vec{x})$ as a sum of squares. This part is an LMI. In fact, NC Algebra will do this soon – Mauricio.

After step 1 we have a set of analytic polynomials, $\mathcal{S} = \{s_j(\vec{x})\}_{j=1}^{L}$, such that

$$p(\vec{x}) = \sum_{j=1}^{L} s_j^T(\vec{x}) s_j(\vec{x}).$$

Step 2 in Algorithm A

Attempt to minimize the number of squares from step 1. In this step we find a basis $\{q_j(\vec{x})\}_{j=1}^{K}$ for $span(\mathcal{S})$ such that

$$p(\vec{x}) = \sum_{j=1}^{K} q_j^T(\vec{x}) q_j(\vec{x}).$$

The output of Algorithm A is the vector of NC analytic polynomials

$$q(\vec{x}) = \begin{pmatrix} q_1(\vec{x}) \\ \vdots \\ q_K(\vec{x}) \end{pmatrix}.$$

A Minimization Problem

In step 2 of Algorithm A, we implement a procedure that will cut down the number of squares necessary for an SoS decomposition. In general finding the minimal number of squares necessary is a hard problem.

Q2. When can an NC polynomial in the $\boldsymbol{g}$*-variables* $(\boldsymbol{x_1}, \ldots, \boldsymbol{x_g})$ *be written as a sum of squares of* $\boldsymbol{g}$ *NC polynomials?*

An Answer to Q1

Q1. When is the domain $\mathcal{D}_{\boldsymbol{r}}$ bianalytic to a matrix ball, $\boldsymbol{B}$, and what is the map between the domains?

Theorem 1. *Suppose that $\boldsymbol{p}(\overrightarrow{x})$ is a NC hereditary, matrix positive polynomial such that $\boldsymbol{p}(\mathbf{0}) = \mathbf{0}$ and suppose $\boldsymbol{a} > \mathbf{0}$. Define $\boldsymbol{r}(\overrightarrow{x}) := \boldsymbol{a} - \boldsymbol{p}(\overrightarrow{x})$ and $\mathbf{d}(\overrightarrow{x}) := \boldsymbol{a} - \mathbf{c}(\overrightarrow{x})$ where*

$$\mathbf{c}(\overrightarrow{x}) := \sum_{j=1}^{g} x_j{}^T x_j.$$

Then Algorithm A will construct an NC bianalytic map between $\mathcal{D}_{\boldsymbol{r}}$ and $\mathcal{D}_{\mathbf{d}}$ or one does not exist.

An Answer to Q2

Q2. When can an NC polynomial in the g-variables $(x_1, \dots, x_g)$ be written as a sum of squares of g NC polynomials?

Theorem 2. *Suppose that $p(\vec{x})$ is matrix positive, hereditary, and $p(0) = 0$. Suppose that $p(\vec{x})$ is 0-local bianalytic to*

$$c(\vec{x}) := {x_1}^T x_1 + \dots + {x_g}^T x_g.$$

Then $p(\vec{x})$ is the sum of g squares of analytic polynomials $q_1(\vec{x}), \dots, q_g(\vec{x})$. I.e.

$$p(\vec{x}) = q_1^T(\vec{x}) q_1(\vec{x}) + \dots + q_g^T(\vec{x}) q_g(\vec{x}).$$

NC Ball Maps

Recall the classical question from complex analysis

- **What are the analytic maps of the unit ball in $\mathbb{C}^n$ to the unit ball in $\mathbb{C}^m$ that carry the boundary to the boundary?**

Results on this question in the classical case are fairly nice. See D'Angelo for constructions of such maps. In proving the Theorems that we just stated we needed to solve an analogue to this question.

NC Ball Maps

Suppose that $f(\vec{x})$ is an NC analytic function such that $f(0) = 0$. We say that $f(\vec{x})$ is an **NC ball map from the γ-ball to the β-ball** provided the following conditions hold for all n:

1. If $\vec{X} \in (\mathbb{R}^{n\times n})^g$ such that $\|\vec{X}\| < \gamma$, then $p(\vec{X})$ converges.

2. If $\vec{X} \in (\mathbb{R}^{n\times n})^g$ such that $\|\vec{X}\| < \gamma$, then $\|p(\vec{X})\| < \beta$.

3. If $\vec{X} \in (\mathbb{R}^{n\times n})^g$ such that $\|\vec{X}\| = \gamma$, then $\overline{\lim_{\delta\uparrow 1}} \|p(\delta\vec{X})\| = \beta$.

NC Ball Map Classification

The following Theorem classifies all NC ball maps.

Theorem 3. *A map* $\boldsymbol{p}(\overrightarrow{\boldsymbol{x}})$ *is an NC ball map from the* $\boldsymbol{\gamma}$*-ball to the* $\boldsymbol{\beta}$*-ball iff* $\boldsymbol{p}(\overrightarrow{\boldsymbol{x}}) = \frac{\boldsymbol{\beta}}{\boldsymbol{\gamma}}\boldsymbol{A}\overrightarrow{\boldsymbol{x}}$ *for some Euclidean isometry* $\boldsymbol{A}$*.*