Multidimensional Linear Systems and Robust Control

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(ABSTRACT)

This dissertation contains two parts: Commutative and Noncommutative Multidimensional (d-D) Linear Systems Theory. The first part focuses on the development of the interpolation theory to solve the H^∞ control problem for d-D linear systems. We first review the classical discrete-time 1D linear system in the operator theoretical viewpoint followed by the formulations of the so-called Givone-Roesser and Fornasini-Marchesini models. Application of the d-variable Z-transform to the system of equations yields the transfer function which is a rational function of several complex variables, say $\mathbf{z} = (z_1, \ldots, z_d)$.

We then consider the output feedback stabilization problem for a plant $P(\mathbf{z})$. By assuming that $P(\mathbf{z})$ admits a double coprime factorization, then a set of stabilizing controllers $K(\mathbf{z})$ can be parametrized by the Youla parameter $Q(\mathbf{z})$. By doing so, one can convert such a problem to the model matching problem with performance index $F(\mathbf{z})$, which is affine in $Q(\mathbf{z})$. Then, with $F(\mathbf{z})$ as the design parameter rather than $Q(\mathbf{z})$, one has an interpolation problem for $F(\mathbf{z})$. Incorporation of a tolerance level on $F(\mathbf{z})$ then leads to an interpolation problem of multivariable Nevanlinna-Pick type. We also give an operator-theoretic formulation of the model matching problem which lends itself to a solution via the commutant lifting theorem on the polydisk.

The second part details a system whose time-axis is described by a free semigroup $\mathcal{F}_d$. Such a system can be represented by the so-called noncommutative Givone-Roesser, or noncommutative Fornasini-Marchesini models which are analogous to those in the first part. Application of a noncommutative d-variable Z-transform to the system of equations yields the transfer function expressed by a formal power series in noncommuting d-indeterminants, say $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$ where $z^v = z_{i_n} \cdots z_{i_1}$ if $v = g_{i_n} \cdots g_{i_1} \in \mathcal{F}_d$ and $z_i z_j \neq z_j z_i$ unless $i = j$. The concepts of reachability, controllability, observability, similarity, and stability are introduced by means of the state-space interpretation. Minimal realization problems for noncommutative Givone-Roesser or Fornasini-Marchesini systems are solved directly by a shift-realization procedure constructed from appropriate noncommutative Hankel matrices. This procedure adapts the ideas of Schützenberger and Fliess originally developed for "recognizable series" to our systems.

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Contents

List of Figures

List of Symbols

General notation

$\oplus$	direct sum
$E \otimes F$	tensor product of E and F, 188
$\overline{\sigma}(A)$	the largest singular value of A
$\mathcal{D}_n^d$	n-point interpolation data set
$\langle \cdot, \cdot \rangle$	an inner product
$\|\cdot\|$	norm (on a linear space)
$\|\cdot\|_p$	norm in $L^p(\mathbb{T})$, for $1 \le p \le \infty$
$\dim A$	dimension of A
codim A	co-dimension of A
$\ker A$	kernel of A
im$A$	image of A
$\mathbf{z}, \mathbf{n}, \mathbf{j}$	$(z_1, \dots, z_d)$, $(n_1, \dots, n_d)$, $(j_1, \dots, j_d)$
$d\mathbf{w}$	$dw_1 \cdots dw_d$, 10
$\frac{1}{\mathbf{w}-\mathbf{z}}$	$\frac{1}{(w_1-z_1)\cdots(w_d-z_d)}$, 10
$\|\mathbf{j}\|$	$j_1 + \cdots + j_d$, 10
$\mathbf{j}!$	$j_1! \cdots j_d!$, 11
$\frac{\partial^{\|\mathbf{j}\|}}{\partial \mathbf{z}^{\mathbf{j}}}$	$\frac{\partial^{j_1+\cdots+j_d}}{\partial z_1^{j_1} \cdots \partial z_d^{j_d}}$, 11
$\grave{z}_k$	$(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_d)$, 9
$z_1 \cdots \widehat{z_k} \cdots z_d$	$z_1 \cdots z_{k-1} z_{k+1} \cdots z_d$, 10

Algebra

$\mathbb{R}$	a field of real number
$\mathbb{R}[\mathbf{z}]$	the polynomial ring over $\mathbb{R}$ in d-indeterminants
$\mathbb{R}(\mathbf{z})$	the field of rational functions over $\mathbb{R}$ in d-indeterminants

$\mathbb{R}_s(\mathbf{z}) \subseteq \mathbb{R}(\mathbf{z})$	the stable rational functions
$\mathbb{R}^{m\times l}(\mathbf{z})$	a set of $m \times l$ matrices with entries in $\mathbb{R}(\mathbf{z})$
$\mathbb{R}_s^{m\times l}(\mathbf{z})$	a set of $m \times l$ matrices with entries in $\mathbb{R}_s(\mathbf{z})$
$\mathrm{Hol}(\mathbb{D}^d)$	a ring of holomorphic functions in $\mathbb{D}^d$, 9
$\mathcal{C}(\overline{\mathbb{D}^d})$	a set of continuous functions in $\overline{\mathbb{D}^d}$, 9
$\mathcal{A}$	an analytic set, 14
$\mathcal{Z}(f)$	zero variety of f, 14, 16, 68
$\mathcal{P}(f)$	polar variety of f, 16

Linear spaces and domain

$\mathbb{C}$	complex Euclidean space, 7
$\mathbb{C}^d$	d-dimensional complex Euclidean space, 7
$\mathbb{R}_+^d$	d-dimensional positive real Euclidean space, 8
$\mathbb{Z}^d$	a set of d-dimensional integer, 10
$\mathbb{D}^d(\mathbf{z}^0, \mathbf{r})$	a polydisk, 8
$\overline{\mathbb{D}^d}(\mathbf{z}^0, \mathbf{r})$	a closed polydisk, 8
$\mathbb{T}^d(\mathbf{z}^0, \mathbf{r})$	a distinguished boundary, 8
$U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$	a neighborhood around a point $\mathbf{z}^0$, 9
D	a domain, 14

Hardy spaces and Lebesgue spaces

$H^p(\mathbb{D})$	Hardy space: a set of holomorphic functions on $\mathbb{D}$, 17
$L^p(\mathbb{T})$	Lebesgue space: a set of measurable functions on $\mathbb{T}$, 17
$L^\infty(\mathbb{T})$	a set of measurable functions which are bounded a.e. on $\mathbb{T}$, 18
$L_n^2(\mathbb{T})$	a set of $n \times 1$ vector valued functions whose entries in $L^2(\mathbb{T})$, 18
$L_{m\times n}^\infty(\mathbb{T})$	a set of $m \times n$ matrix valued functions whose entries in $L^\infty(\mathbb{T})$, 18
$M_p(r, f)$	the mean of order p at radius r, $1 \leq p \leq \infty$, 16

Hilbert spaces and linear operators

$\mathcal{X}, \mathcal{Y}$	vector spaces, 19
$\mathcal{L}(\mathcal{X}, \mathcal{Y})$	a set of linear operators from $\mathcal{X}$ to $\mathcal{Y}$, 19

$\mathcal{L}(\mathcal{X})$	a set of linear operators on $\mathcal{X}$, 19
$P_{\mathcal{K}\to\mathcal{H}} : \mathcal{K} \mapsto \mathcal{H}$	the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$, 21
$P_k : \mathcal{H} \mapsto \mathcal{H}_k$	the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_k$ with image equal to $\mathcal{H}_k$, 49
$\iota_k : \mathcal{H}_k \mapsto \mathcal{H}$	the inclusion map from $\mathcal{H}_k$ to $\mathcal{H}$, 49
M_G	the multiplication operator, 84
$S_{\mathcal{M},j}$	the model operator, 86
Γ_Y	the completely positive operator, 87
$\mathcal{H}, \mathcal{U}, \mathcal{Y}$	state, input and output spaces, 33
$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$	a connecting operator from $\mathcal{H} \oplus \mathcal{U}$ to $\mathcal{H} \oplus \mathcal{Y}$, 33
Σ^*	an adjoint system of Σ, 34
$\bigvee_{n=k}^{\ell} \mathcal{M}_n$	the closed linear span of subspaces $\{\mathcal{M}_n\}$ for $n \in [k, \ell] \subset \mathbb{Z}$, 22
$\mathcal{S}_d$	Schur class, 24
$\mathcal{SA}_d$	Schur-Agler class, 24

Part 2

$*$	concatenation, 93; or star operation, 94
λ	null word
v, w	words or strings, say $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$
$w^\top$	the word with the same letters but listed in reverse order
$\|w\|$	the length of word (if $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$, then $\|w\| = n$), 92
$\mathcal{F}_d$	a free semigroup generated by letters from the set $\mathcal{F}$, 93
H_T	Hankel operator, 95
I	the identity operator, 97
$Z(z)$	the generalized structured noncommutative dynamics, 106
$Z_d(z)$	the noncommutative NCGR dynamics, 106
$Z_r(z)$	the noncommutative NCFM dynamics, 106
$\mathcal{H}, \mathcal{U}, \mathcal{Y}$	state, input and output spaces, 97
$\mathcal{C}$	controllability matrix, 145
$\mathcal{C}_w^{i_n}$	$\{w\}$-controllability matrix, 145
$\mathcal{C}_n$	length-n-controllability matrix, 144
$\mathcal{C}_n^k$	length-n-controllability matrix with respect to a letter g_k, 144
X_n^C	controllability subspace, 144
$\mathcal{O}$	observability matrix, 152
$\mathcal{O}_w^{i_1}$	$\{w\}$-observability matrix, 148

$\mathcal{O}_n$	length-n-observability matrix, 149
$\mathcal{O}_n^k$	length-n-observability matrix with respect to a letter g_k, 149
X_n^O	observability subspace, 149
$\mathcal{R}$	reachability matrix, 142
$\mathcal{R}_w^{i_n}$	$\{w\}$-reachability matrix, 139
$\mathcal{R}_n$	length-n-reachability matrix, 141
$\mathcal{R}_n^{i_n}$	length-n-reachability matrix with respect to a letter g_{i_n}, 140
X_n^R	reachability subspace, 141
$\mathcal{S}$	a set of admissible similarity transformation, 137
$\mathcal{S}$	the forward shift operator, 97
$\mathcal{T}_f$	Future time for noncommutative d-D linear system, 112
$\mathcal{T}_p$	Past time for noncommutative d-D linear system, 112
Δ	uncertainty operator, 98
$\mathbf{Q} = \text{diag}\{Q, Q, \dots\}$	an infinite diagonal matrix, 98
$\delta_{v,w}$	noncommutative Kronecker delta, 171

$\ell^2_+(\mathcal{H}) = \ell^2(\mathbb{Z}_+, \mathcal{H})$ the set of all $\mathcal{H}$-valued functions $k \in \mathbb{Z}_+ \mapsto f(k)$ where

$$\|f\|^2_{\ell^2_+(\mathcal{H})} = \sum_{k\in\mathbb{Z}_+} \|f(k)\|^2 < \infty, \quad 97$$

$\ell^2((\mathcal{F}_d \times \mathcal{F}_d), \mathcal{H})$ the set of all $\mathcal{H}$-valued function $(w, v) \in \mathcal{F}_d \times \mathcal{F}_d \mapsto f(w, v)$ where

$$\|f\|^2_{\ell^2((\mathcal{F}_d\times\mathcal{F}_d),\mathcal{H})} \triangleq \sum_{(w,v)\in\mathcal{F}_d\times\mathcal{F}_d} \|f(w, v)\|^2_{\mathcal{H}} < \infty, \quad 131$$

ℓ_{fin}	summable space on finite support, 171, 175
Γ^w	a path on tree, 109
${}^w\Gamma$	the reverse path of Γ^w, 111

List of Acronyms

d-D	d-Dimension, 7, 29
DCF	Double Coprime Factorization, 59, 61
FM	Fornasini-Marchesini, 30
FRC	Factor Right Coprime, 60
GR	Givone-Roesser, 30
ID set	Interpolation Data set, 26, 77
i/s/o	Input-State-Output, 91
LFT	Linear Fractional Transformation, 95
LMI	Linear Matrix Inequality, 3, 56, 79
LOI	Linear Operator Inequality, 3, 87, 89
MFD	Matrix Fraction Description, 57, 63
MIMO	Multi Input Multi Output, 89
MRC	Minor Right Coprime, 60
NCFM	Noncommutative Fornasini-Marchesini, 116
NCGR	Noncommutative Givone-Roesser, 116
NPIP	Nevanlinna-Pick Interpolation Problem, 26, 77
PDE	Partial Differential Equation, 31, 32
SISO	Single Input Single Output, 89
ZRC	Zero Right Coprime, 60

Chapter 1

Introduction

The theory of multidimensional linear systems has been a subject of research for over two decades after Attasi [Att73, Att75], Givone-Roesser, [GR72, GR73], and Fornasini-Marchesini [FM76, FM77] introduced two-dimensional linear models in the 1970's. Since then, these prototypes have been generalized to d-dimensional (d-D) linear models where $d > 2$. It is well-known that the model proposed by Attasi is a special case of the models established by Givone-Roesser and Fornasini-Marchesini in the sense that it can be embedded into either Givone-Roesser's or Fornasini-Marchesini's models, and hence in this dissertation, we shall focus on system models which have mathematical structures in the form of Givone-Roesser's and Fornasini-Marchesini's models

This dissertation consists of two parts: Commutative and Noncommutative Multidimensional (d-D) Linear Systems Theory; the material covered in each part is independent of each other. The first part focuses on the development of the interpolation theory to solve the H^∞ control problem for d-D linear system; whilst the second part details a linear system with evolution along the elements of a free semigroup.

1.1 Motivation

Part I

In the case of classical 1D linear systems, the H^∞ control problem can be solved via either state-space analysis in the time domain, or interpolation theory in the frequency domain. In the state-space approach, the H^∞ control and filtering problems for 2D linear systems have already been solved via an extended bounded real lemma for 2D systems in [DX02, DXZ01]; however, it is known that the 2D bounded real lemma gives only a sufficient (not necessary) condition for a system to be bounded real.

Z. Lin studied the (output) feedback stabilization problem in [Lin98, Lin00], and obtained an analogue of the well-known Youla parametrization of the set of all stabilizing controllers. However, in his work, Lin did not take the next step of seeking to find a stabilizing controller which optimizes some performance function (i.e. the H^∞ control problem). The main goal here is to develop the interpolation theory to solve the H^∞ control problem in the multidimensional setting ($d \geq 2$). We are able to obtain a solution of the H^∞ control problem in the frequency domain via the interpolation approach, which gives a necessary and sufficient condition for the existence of a solution in the 2D case.

Part II

In the robust control literature, many control problems can be formulated in a linear fractional transformation (LFT) framework which provides a mathematical paradigm to analyze and design stabilizing linear controllers for closed-loop systems in an effective way. Beck and Doyle studied uncertain systems using the LFT as a tool for modeling systems with structured perturbations on a nominal model in [BD99]. They considered each perturbation δ_j as an arbitrary time-varying operator on the square summable sequence space ℓ^2 or a real-valued parameter uncertainty, both of which can be viewed as noncommuting indeterminants z_j. By replacing δ_j with z_j, the input/output map of the original system is nothing more nor less than a transfer function of a linear system which can be expressed as a formal power series in several noncommuting indeterminants. The role of formal power series in analyzing uncertain systems (i.e., linear time-invariant plants having time-varying structured uncertainties) was investigated in [Bec01, BD99, BD97] in a more formal, but less precise way. One goal of this part is to reformulate a connection between the robust control theory using the LFT framework and the multidimensional system theory in a more precise way.

The authors in [BD99] also reviewed the realization theory and the Lyapunov stability theory for uncertain systems, proposed a necessary and sufficient condition for reducibility in terms of coupling Lyapunov inequalities, and discussed the controllability and observability of an uncertain system realization; however, they did not provide the state-space interpretation for these objects. Beck [Bec01] studied a connection between minimal realization theory results for formal power series and the concept of minimality for uncertain systems represented by the LFT framework. As in [BD99], Beck did not provide the state-space interpretation in her work.

Rather, we here introduce an input-state-output linear system with evolution along the elements of a free semigroup. The corresponding transfer function for such a system is represented by a formal power series in noncommuting indeterminants. We then formulate two linear models, namely the noncommutative Givone-Roesser model and the noncommutative Fornasini-Marchesini model, introduce various concepts of stability, reachability, controllability, observability and similarity of such systems, and establish the realization theory for noncommutative Givone-Roesser and Fornasini-Marchesini systems by adapting the noncommutative shift realization of Schützenberger and Fliess originally developed for the class of the "recognizable series".

1.2 Contributions

In the first part, we develop the interpolation theory to solve the H^∞ control problem in the frequency domain as described below:

- We use the Youla parametrization for the d-D case obtained in [Lin98, Lin00] to establish the connection between (output) feedback stabilization and interpolation conditions for d-D linear systems via a model matching form.
- We impose an optimization criteria on performance, in addition to internal stability, as a design goal (i.e. H^∞ control problem), and solve the resulting d-D matrix Nevanlinna-Pick interpolation problem using the recent work on Nevanlinna-Pick interpolation on the polydisk.
- The solution criterion for the d-D H^∞ control problem involves an infinite Linear Matrix Inequality (LMI) or Linear Operator Inequality (LOI) . This leaves open the question as how best to solve such infinite LMIs or LOIs. By working out a simply example using the MATLAB: LMI Control Toolbox (see Appendix B), we show that in general it is not possible to extend a solution corresponding to n interpolation data points to a solution corresponding to the set of $n+1$ interpolation data points obtained by adding one additional interpolation condition, as is the case for the classical one-variable case encoded in the famous Schur algorithm.
- We also present a solution based on the polydisk Commutant Lifting Theorem.

The second part of this dissertation concerns systems whose transfer functions can be expressed as formal power series. The contributions in this part are as follows:

- We provide a precise connection between the robust control framework and multidimensional linear systems whose transfer functions can be expressed by formal power series in several noncommuting variables. While Beck and Doyle pointed out the link between robust control and formal power series in noncommuting variables, part of the contribution here is to point out that these formal power series in noncommuting variables can be viewed as the transfer functions of linear systems with evolution along the elements of a free semigroup, or, equivalently, over a homogeneous tree with root. While such systems appear indirectly in [BNW94] in connection with multiscale filtering theory, their connection with robust control appears here for the first time.

- We propose various notions of stability, reachability, controllability, and observability and show how to verify whether or not the given system is stable, reachable, controllable, or observable.
- We establish the similarity theory and minimality theory for these systems.
- We solve minimal realization problems for noncommutative Givone-Roesser or Fornasini-Marchesini systems directly by shift-realization procedure constructed from appropriate noncommutative Hankel matrices. Our procedure adapts the ideas of Schützenberger and Fliess originally developed for "recognizable series" to the setting of noncommutative Givone-Roesser and Fornasini-Marchesini systems. We also show that a simple identification procedure can be used to see that the realization theorem for the case of noncommutative Fornasini-Marchesini systems can be seen to follow directly from the realization result of the Schützenberger/Fliess for recognizable series.

1.3 Dissertation Outline

This dissertation is organized as follows:

Chapter 2: We first briefly review some fundamental facts and basic notation—e.g., polydisk, holomorphic function, analytic set, variety, Hardy spaces, Schur class, and Schur-Agler class—from the function theory of several complex variables. We also summarize the main theoretical results from operator theory and Nevanlinna-Pick interpolation theory.

Chapter 3: This Chapter discusses chronologically the development of the state-space representations of the 2D linear models—Givone-Roesser (GR) model, and Fornasini-Marchesini (FM) model—including the generalized version of them; the identification between these two models is also presented. Application of the d-variable Z-transform to the system equations of these models yields the corresponding transfer functions which are rational matrix-valued functions of several complex variables.

Chapter 4: The aim of this Chapter is to solve the H^∞ control problem in the frequency domain via interpolation theory. We first summarize the results of Z. Lin [Lin88, Lin98, Lin99, Lin00] on (output) feedback stabilization problem. We then go beyond the work of Lin by establishing the connection of his work with interpolation conditions via a model matching form. If one demands a set of controllers not only internally stabilizing the closed-loop system, but also optimizing some performance function, there results the so-called H^∞ control problem. This problem can be solved by using the result of the Nevanlinna-Pick interpolation theory on the polydisk (see e.g., [Agl87, AM, AM02, BB, BT98]). We also present a solution of the H^∞ control problem based on the recent work on the polydisk Commutant Lifting Theorem [BLTT99].

Chapter 5: We briefly present some fundamental results from noncommutative algebra including the notions of formal power series and its applications. We summarize the linear fractional transformation (LFT) framework commonly used in the robust control literature in connection with the multidimensional linear system. We then introduce the notion of the "time-axis" which can be represented by a free semigroup, or equivalently in other language, by a homogeneous tree with a root. Finally we mention some examples of other classes of generalized systems in the engineering literature which have some elements in common with the notion of generalized system presented here.

Chapter 6: To get an analogue with the commutative case as in Part 1, we here formulate two linear models, namely noncommutative Givone-Roesser (NCGR) and noncommutative Fornasini-Marchesini (NCFM), each of which has mathematical structure parallel to GR and FM models, respectively; the identification between these two models is established. Application of the noncommutative d-variable Z-transform to the system of equations of these models yields the corresponding transfer function of several noncommuting variables, which can also be represented by a formal power series.

Chapter 7: We introduce several notions of reachability, controllability and observability, and establish the criteria to verify whether or not the system is reachable, controllable, or observable. We also present two similarity theorems: one is for the system described by NCGR model, and the other is for the system described by NCFM model.

Chapter 8: A minimality theorem is established and a minimal realization problem is solved using the noncommutative Hankel operator. The stability issue and the Lyapunov theorem for noncommutative systems are also provided in this Chapter.

Chapter 9: This Chapter is devoted to open problems and conclusion.

Part I

Multidimensional Linear Systems

Chapter 2

Preliminaries and Notation

This Chapter presents some fundamental facts and notation which will be used throughout the first part of this dissertation. We try to follow the conventional notation that points and variables are lower-case, whilst matrices, operators and spaces are upper-case. In Section 2.1, we describe the notions of, for instance, polydisk, holomorphic function, analytic set, and Hardy spaces, and summarize the main theoretical results from the Function theory of several complex variables. Since the Mathematical System Theory formulated in this dissertation follows an operator theoretic approach, we shall also provide some basic facts from Operator Theory in Section 2.2 followed by the definitions of Schur and Schur-Agler classes in Section 2.3. We end this Chapter with the well-known *Nevanlinna-Pick* interpolation theory in Section 2.4.

2.1 Multivariable Complex Analysis

This Section is devoted to the function theory of several complex variables which is being used as a central part to prove the interpolation theory for multidimensional (d-D) linear systems in Chapter 4. For further discussion on this subject, the reader may refer to, e.g. [FG02, HL84, Hör73, Nis96, Rud69, Sha92, Tay02].

2.1.1 General Notation

In the sequel, we denote by $\mathbf{z} = (z_1, \ldots, z_d)$, each $z_k \in \mathbb{C}$, a point in $\mathbb{C}^d$, which is the Cartesian product of d complex planes: $\mathbb{C}^d = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{d \text{ times}}$ together with the standard basis:

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \cdots, \mathbf{e}_d = (0, \ldots, 0, 1).$$

Algebraically, $\mathbb{C}^d$ is a d-dimensional vector space over $\mathbb{C}$; topologically, $\mathbb{C}^d$ is the Euclidean space of dimension $2d$. For a set A in $\mathbb{C}^d$, we shall denote by $\mathrm{conj}(A)$ the complex conjugate of A:

$$\mathrm{conj}(A) \triangleq \{\overline{\mathbf{z}} = (\overline{z}_1, \ldots, \overline{z}_d) \mid \mathbf{z} \in A\}, \qquad \text{where } \overline{z}_k = x_k - iy_k \text{ if } z_k = x_k + iy_k.$$

The space $\mathbb{C}^d$ has an inner product defined by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{k=1}^{d} z_k \overline{w}_k, \qquad \text{for any } \mathbf{z}, \mathbf{w} \in \mathbb{C}^d.$$

The space $\mathbb{C}^d$ equipped with this inner product forms a Hilbert space. Furthermore, the inner product defines a norm on $\mathbb{C}^d$ given by: $\|\mathbf{z}\| = \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle}$ and a metric on $\mathbb{C}^d$ defined by

$$d(\mathbf{z}, \mathbf{w}) \triangleq \|\mathbf{z} - \mathbf{w}\| = \sqrt{\langle \mathbf{z} - \mathbf{w}, \mathbf{z} - \mathbf{w} \rangle}.$$

Let $\mathbf{z}^0 = (z_1^0, \ldots, z_d^0) \in \mathbb{C}^d$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}_+^d$, the d-dimensional positive real Euclidean space. Then the following set

$$\mathbb{D}^d(\mathbf{z}^0, \mathbf{r}) \triangleq \{\mathbf{z} \in \mathbb{C}^d \mid |z_k - z_k^0| < r_k \quad \text{for } k = 1, \ldots, d\} \tag{2.1}$$

is called the *polydisk* with *polyradius* $\mathbf{r}$ and center at $\mathbf{z}^0$. In other words, the polydisk is the Cartesian product of d discs:

$$\mathbb{D}^d(\mathbf{z}^0, \mathbf{r}) = \underbrace{\mathbb{D}(z_1^0, r_1) \times \cdots \times \mathbb{D}(z_d^0, r_d)}_{d \text{ times}}, \quad \text{where } \mathbb{D}(z_k^0, r_k) = \{|z_k - z_k^0| < r_k\}.$$

The *distinguished boundary* of the polydisk, $\mathbb{D}^d(\mathbf{z}^0, \mathbf{r})$ is defined as a set

$$\mathbb{T}^d(\mathbf{z}^0, \mathbf{r}) \triangleq \{\mathbf{z} \in \mathbb{C}^d \mid |z_k - z_k^0| = r_k \quad \text{for } k = 1, \ldots, d\}, \tag{2.2}$$

which is also the Cartesian product of d circles:

$$\mathbb{T}^d(\mathbf{z}^0, \mathbf{r}) = \underbrace{\mathbb{T}(z_1^0, r_1) \times \cdots \times \mathbb{T}(z_d^0, r_d)}_{d \text{ times}}, \quad \text{where } \mathbb{T}(z_k^0, r_k) = \{|z_k - z_k^0| = r_k\}.$$

We denote by $\overline{\mathbb{D}^d}(\mathbf{z}^0, \mathbf{r})$ the closed polydisk which is the union $\mathbb{D}^d(\mathbf{z}^0, \mathbf{r}) \bigcup \mathbb{T}^d(\mathbf{z}^0, \mathbf{r})$. In particular, if $\mathbf{z}^0 = \mathbf{0}$, and $r_k = 1$, for all $k = 1, \ldots, d$, we write $\mathbb{D}^d$ (respectively, $\mathbb{T}^d$) rather than $\mathbb{D}^d(\mathbf{0}, \mathbf{1})$ (respectively, $\mathbb{T}^d(\mathbf{0}, \mathbf{1})$), and we shall call it a *unit polydisk* (respectively, a *unit torus*).

2.1.2 Holomorphic Functions

Definition 1 ($\mathbb{C}$-differentiable). Let $D \subset \mathbb{C}^d$ be an open set. A function $f : D \to \mathbb{C}$ is said to be $\mathbb{C}$-*differentiable* at a point $\mathbf{z}^0 \in D$ provided that there exists a map $\eta : D \to \mathbb{C}^d$ such that

1. η is continuous at $\mathbf{z}^0$, and
2. $f(\mathbf{z}) = f(\mathbf{z}^0) + (\mathbf{z} - \mathbf{z}^0) \cdot \eta(\mathbf{z})^\top$ for $\mathbf{z} \in D$.

Moreover, the *partial derivatives* of f at $\mathbf{z}^0$ is given by

$$\frac{\partial f}{\partial z_k}(\mathbf{z}^0) = f_{z_k}(\mathbf{z}^0) \triangleq \mathbf{e}_k \cdot \eta(\mathbf{z}^0)^\top$$

and the vector

$$\triangledown f(\mathbf{z}^0) \triangleq (f_{z_1}(\mathbf{z}^0), \ldots, f_{z_d}(\mathbf{z}^0)) = \eta(\mathbf{z}^0)$$

is called the *complex gradient* of f at $\mathbf{z}^0$.

Note that for f to be $\mathbb{C}$-differentiable at $\mathbf{z}^0$, it is sufficient that there is a small neighborhood $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \subset D$ such that the restriction $f|_U$ is also $\mathbb{C}$-differentiable at $\mathbf{z}^0$, where

$$U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \triangleq \{\mathbf{z} \in \mathbb{C}^d \mid |z_k - z_k^0| < \delta_k \text{ for some } \delta_k > 0 \text{ for } k = 1, \ldots, d\}.$$

Definition 2 (Holomorphic function). A function f, defined on a domain[1] $D \subset \mathbb{C}^d$, is said to be *holomorphic* at a point $\mathbf{z}^0 \in D$ provided that it is $\mathbb{C}$-differentiable in some neighborhood $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$ of such a point. If f is holomorphic at each point of D, then it is called holomorphic on D.

The sum and product of holomorphic functions at a point $\mathbf{z}^0 \in \mathbb{C}^d$ are also holomorphic. Thus the set of all holomorphic functions at the point $\mathbf{z}^0$ forms a *ring*. For any domain $D \subset \mathbb{C}^d$, we denote by $\mathrm{Hol}(D)$ the ring of holomorphic functions in the domain D.

Next we shall establish some basic properties of holomorphic functions of several complex variables which are analogous to those of functions of one complex variable. In the following, we shall denote by $\mathrm{Hol}(\mathbb{D}^d) \cap \mathcal{C}(\overline{\mathbb{D}^d})$ the set of functions that are holomorphic in $\mathbb{D}^d(\mathbf{z}^0, \mathbf{r})$ and continuous in $\overline{\mathbb{D}^d}(\mathbf{z}^0, \mathbf{r})$. For any $z_k \in \mathbb{C}$, we let $\grave{z}_k$ denote the remaining variables, i.e., $\grave{z}_k = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_d)$.

Theorem 2.1 (Cauchy's Integral Formula). *Any function $f \in Hol(\mathbb{D}^d) \cap \mathcal{C}(\overline{\mathbb{D}^d})$ at any point $\mathbf{z} \in U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$ is represented by a multivariable Cauchy integral*

$$f(\mathbf{z}) = \frac{1}{(2\pi i)^d} \int_{\mathbb{T}^d} \frac{f(\mathbf{w})}{\mathbf{w} - \mathbf{z}} d\mathbf{w}, \tag{2.3}$$

[1]A set $D \subset \mathbb{C}^d$ is called a domain if it is an open and connected subset of $\mathbb{C}^d$.

where $d\mathbf{w} = dw_1 \cdots dw_d$, *and* $\frac{1}{\mathbf{w}-\mathbf{z}} = \frac{1}{(w_1-z_1)\cdots(w_d-z_d)}$.

Proof. The proof is by induction on the dimension d. For $d = 1$, this is the classical Cauchy's integral formula. We now assume that the result is true in dimension $d-1$. For any point $\mathbf{z} \in U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$, f is holomorphic with respect to each variable z_k since by assumption f is holomorphic in $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \subset \mathbb{D}^d$. From this fact, we first consider z_k as a variable whilst the remaining variables $\grave{z}_k$ are treated as a constant. Application of the Cauchy's integral formula for one complex variable to the function $f(\mathbf{z}) = f(\grave{z}_k, z_k)$ which is holomorphic in z_k in the disk $\mathbb{D}(z_k^0, r_k)$ and continuous on its boundary $\mathbb{T}(z_k^0, r_k)$ yields

$$f(\grave{z}_k, z_k) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\grave{z}_k, w_k)}{w_k - z_k} dw_k. \tag{2.4}$$

Now fix any point $w_k \in \mathbb{T}(z_k^0, r_k)$ and for any $\grave{z}_k \in U(\grave{z}_k^0, \grave{r}_k) \in \mathbb{C}^{d-1}$, $f(\grave{z}_k, w_k)$ is holomorphic in $d-1$ variables. It follows from the inductive hypothesis that,

$$f(\grave{z}_k, w_k) = \frac{1}{(2\pi i)^{d-1}}\int_{\mathbb{T}^{d-1}} \frac{f(\grave{w}_k, w_k)}{(w_1 - z_1)\cdots\widehat{(w_k - z_k)}\cdots(w_d - z_d)} dw_k \cdots \widehat{dw_k} \cdots dw_d,$$

where $z_1 \cdots \widehat{z_k} \cdots z_d = z_1 \cdots z_{k-1} z_{k+1} \cdots z_d$. By substituting this expression into (2.4), we obtain an iterated integral. Since f is continuous in the set of all variables, the iterated integral is replaced by a multiple integral over the distinguished boundary $\mathbb{T}^d$, and hence we arrive at (2.3). This completes the proof. ■

Note that the kernel (often called the *Cauchy kernel*) of integral (2.3) can be expressed as a multivariable geometric progression:

$$\begin{aligned}
\frac{1}{\mathbf{w}-\mathbf{z}} &= \frac{1}{\mathbf{w}-\mathbf{z}^0}\cdot\frac{1}{(1-\frac{z_1-z_1^0}{w_1-z_1^0})\cdots(1-\frac{z_d-z_d^0}{w_d-z_d^0})} \\
&= \frac{1}{\mathbf{w}-\mathbf{z}^0}\sum_{|\mathbf{j}|=0}^{\infty}\left(\frac{\mathbf{z}-\mathbf{z}^0}{\mathbf{w}-\mathbf{z}^0}\right)^{\mathbf{j}} \\
&= \sum_{|\mathbf{j}|=0}^{\infty}\frac{(\mathbf{z}-\mathbf{z}^0)^{\mathbf{j}}}{(\mathbf{w}-\mathbf{z}^0)^{\mathbf{j}+1}},
\end{aligned} \tag{2.5}$$

where $\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, a set of the d-dimensional integers, and $|\mathbf{j}| = j_1 + \cdots + j_d$.

Multiplying both sides of (2.5) by $\dfrac{1}{(2\pi i)^d} f(\mathbf{w})$ and integrating over $\mathbb{T}^d$, we have

$$f(\mathbf{z}) = \frac{1}{(2\pi i)^d}\int_{\mathbb{T}^d}\frac{f(\mathbf{w})}{\mathbf{w}-\mathbf{z}}d\mathbf{w}$$

$$= \frac{1}{(2\pi i)^d} \int_{\mathbb{T}^d} f(\mathbf{w}) \sum_{|\mathbf{j}|=0}^{\infty} \frac{(\mathbf{z}-\mathbf{z}^0)^{\mathbf{j}}}{(\mathbf{w}-\mathbf{z}^0)^{\mathbf{j}+1}} d\mathbf{w}$$

$$= \sum_{|\mathbf{j}|=0}^{\infty} \left[\frac{1}{(2\pi i)^d} \int_{\mathbb{T}^d} \frac{f(\mathbf{w}) d\mathbf{w}}{(\mathbf{w}-\mathbf{z}^0)^{\mathbf{j}+1}} \right] (\mathbf{z}-\mathbf{z}^0)^{\mathbf{j}} \tag{2.6}$$

This result leads to the following Theorem.

Theorem 2.2 (Osgood's Lemma). *Every function $f \in Hol(\mathbb{D}^d) \cap \mathcal{C}(\overline{\mathbb{D}^d})$ admits a multivariable power series representation at each point $\mathbf{z} \in \mathbb{D}^d$:*

$$f(\mathbf{z}) = \sum_{|\mathbf{j}|=0}^{\infty} C_{\mathbf{j}} (\mathbf{z}-\mathbf{z}^0)^{\mathbf{j}} \tag{2.7}$$

with the coefficients

$$C_{\mathbf{j}} = \frac{1}{(2\pi i)^d} \int_{\mathbb{T}^d} \frac{f(\mathbf{w}) d\mathbf{w}}{(\mathbf{w}-\mathbf{z}^0)^{\mathbf{j}+1}}. \tag{2.8}$$

In fact, if the continuity assumption on the boundary is dropped from the condition in Theorem 2.2, then f can still be represented by a multivariable power series for each point $\mathbf{z} \in \mathbb{D}^d$ since we can choose a small neighborhood $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \subset \mathbb{D}^d$ that contains such a point $\mathbf{z}$ and apply Theorem 2.2 to $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$. From this fact, one can show that *if f is holomorphic on $\mathbb{D}^d$, then at each point $\mathbf{z} \in \mathbb{D}^d$, f has partial derivative of all orders which also are holomorphic.* Thus, one can express the coefficients $C_{\mathbf{j}}$ in terms of a partial derivatives of a holomorphic function f with respect to $\mathbf{z}$ as stated precisely in the following Theorem.

Theorem 2.3. *If a function f is holomorphic at a point $\mathbf{z}^0$, then f has a multivariable power series* (2.7) *with coefficients defined by Taylor's formulas:*

$$C_{\mathbf{j}} = \frac{1}{\mathbf{j}!} \left. \frac{\partial^{|\mathbf{j}|} f}{\partial \mathbf{z}^{\mathbf{j}}} \right|_{\mathbf{z}=\mathbf{z}^0}, \tag{2.9}$$

where $\mathbf{j}! = j_1! \cdots j_d!$ *and* $\dfrac{\partial^{|\mathbf{j}|}}{\partial \mathbf{z}^{\mathbf{j}}} = \dfrac{\partial^{j_1+\cdots+j_d}}{\partial z_1^{j_1} \cdots \partial z_d^{j_d}}$.

It is clear that if f is holomorphic at any point $\mathbf{z} \in U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \subset \mathbb{C}^d$, then it is also holomorphic with respect to each variable separately. On the other hand, if f is holomorphic at any point $\mathbf{z} \in U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \subset \mathbb{C}^d$ with respect to each variable z_k, $k = 1, \ldots, d$ separately, then the question arising here is whether or not such a function f is holomorphic in $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$. The answer is YES and it follows from the well-known *Hartog's Theorem* stated as follows:

Theorem 2.4 (Hartog's Theorem (1906)). *If a function f is holomorphic at any point $\mathbf{z} \in U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \subset \mathbb{C}^d$ with respect to each variable z_k separately, then it is also holomorphic in $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$.*

Remark 1. In order for a function f to be represented by a multivariable Cauchy's integral (2.3) it is sufficient for f to be holomorphic in each variable z_k in the unit disk $\mathbb{D}$ and continuous with respect to the set of all variables in $\overline{\mathbb{D}^d}$. Suppose such a function f is given, then Theorem 2.2 implies that f has a power series representation, and hence it is $\mathbb{C}$-differentiable, i.e., f is holomorphic in $\mathbb{D}^d$. This observation implies that for a function f to be holomorphic in the whole domain, it suffices to show only that such a function, which is holomorphic with respect to each variable separately, is continuous with respect to the set of all the variables. The proof of Hartog's Theorem can be found in, for example, [FG02, Sha92, Tay02]. ▲

The application of Hartog's Theorem leads to the alternative definitions of a holomorphic function.

Definition 3. A function f is said to be:

(R) holomorphic at a point $\mathbf{z}^0 \in \mathbb{C}^d$ *in the sense of RIEMANN* if f is holomorphic with respect to each variable z_k in some polydisk $\mathbb{D}^d(\mathbf{z}^0, \mathbf{r})$.

(W) holomorphic at a point $\mathbf{z}^0 \in \mathbb{C}^d$ *in the sense of WEIERSTRASS* if in some polydisk $\mathbb{D}^d(\mathbf{z}^0, \mathbf{r})$, f admits a power series representation

$$f(\mathbf{z}) = \sum_{|\mathbf{j}|=0}^{\infty} C_{\mathbf{j}}(\mathbf{z} - \mathbf{z}^0)^{\mathbf{j}}.$$

As mentioned above, $(W) \Rightarrow (R)$ is obvious and conversely $(R) \Rightarrow (W)$ is the content of the Hartog's Theorem. Thus, we have: *the concept of holomorphy in the sense of Riemann and holomorphy in the sense of Weierstrass are equivalent* (see, [Sha92, Chapter I Section §3]). This is not the case, however, in the $\mathbb{R}^d$ space. For instance, the function

$$f(x, y) = \frac{2xy^2}{x^3 + y^3}, \quad f(0, 0) = 0$$

is differentiable with respect to a real variable x when y is fixed and with respect to a real variable y when x is fixed; however, it is not even continuous at the point $(0, 0) \in \mathbb{R}^2$.

Suppose $f = (f_1, \ldots, f_k) : D \to \mathbb{C}^k$ where D is a domain in $\mathbb{C}^d$. We shall call the mapping f a *holomorphic mapping* (respectively, *real differentiable*) if all its components $f_i, i = 1, \ldots, k$ are holomorphic (respectively, real differentiable) in D. The following Theorem is an extended version of the so-called *Implicit Function Theorem* in the one variable case.

Theorem 2.5 (Implicit Function Theorem). *If functions $f_1, \ldots, f_k, (k < d)$ are holomorphic in a neighborhood of a point $\mathbf{z}^0 \in \mathbb{C}^d$ and also* $\det\left(\frac{\partial f_i}{\partial z_j}\right) \neq 0$ *in that neighborhood $(i, j = 1, \ldots, k)$, then the system of equations $f_1(\mathbf{z}) = \cdots = f_k(\mathbf{z}) = 0$ is locally solvable for $z_1, \ldots, z_k$ in terms* *of $z_{k+1}, \ldots, z_d$ and the solution $z_j = g_j(z_{k+1}, \ldots, z_d)$ for $j = 1, \ldots, k$ is holomorphic in a neighborhood of the point $(z^0_{k+1}, \ldots, z^0_d)$.*

Example 1. Let $\mathbf{z}^0 = (0,0) \in \mathbb{C}^2$ and consider a function $f(\mathbf{z}) = z_1 - z_2^2$. Since $\det(\frac{\partial f}{\partial z_1})\Big|_{\mathbf{z}=\mathbf{z}^0} = 1 \neq 0$, the Implicit Function Theorem implies that $f(\mathbf{z}) = 0$ is locally solvable near $\mathbf{z}^0$. Thus, we have $z_1 \triangleq g(z_2) = z_2^2$ which is holomorphic in a neighborhood of a point $\{0\}$. $\diamondsuit$

2.1.3 Analytic Sets

It is well known in the one variable case that if f is holomorphic on the disk $\mathbb{D}$, $f \not\equiv 0$ and $f(z^0) = 0$ where $z^0 \in \mathbb{D}$, then in some neighborhood $U(z^0, \delta)$ of z^0,

$$f(z) = (z - z^0)^\ell \phi(z),$$

where $\ell \geq 1$, and ϕ is holomorphic and does not vanish at z^0. This result was generalized to the higher dimensional case in 1879 which is stated as follows:

Theorem 2.6 (Weierstrass Preparation Theorem). *Suppose the function f is holomorphic in some neighborhood $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$ of a point $\mathbf{z}^0 = (\grave{z}^0_k, z^0_k) \in \mathbb{C}^d$ and $f(\grave{z}^0_k, z^0_k) = 0$ but $f(\grave{z}^0_k, z_k) \not\equiv 0$, then in some neighborhood $V(\mathbf{z}^0, \boldsymbol{\delta}')$,*

$$f(\mathbf{z}) = P(\grave{z}_k, z_k)\phi(\mathbf{z}), \tag{2.10}$$

where ϕ is holomorphic in V and does not vanish there, and P the Weierstrass polynomial *is given by*

$$P(\grave{z}_k, z_k) = (z_k - z^0_k)^\ell + \alpha_1(\grave{z}_k)(z_k - z^0_k)^{\ell-1} + \cdots + \alpha_\ell(\grave{z}_k) \tag{2.11}$$

where $\ell \geq 1$ is the order of the zero of $f(\grave{z}^0_k, z_k)$ at the point $z_k = z^0_k$, the function α_j are holomorphic in $\grave{V} := V(\grave{z}^0_k, \grave{\delta}'_k)$, and $\alpha_j(\grave{z}^0_k) = 0$.

Remark 2. It should be noted that the Implicit Function Theorem 2.5 follows directly from the Weierstrass Preparation Theorem 2.6. To see this, let us assume that for any point $\mathbf{z}^0 \in \mathbb{C}^d$, a holomorphic function $f(\mathbf{z}^0) = 0$ but $\frac{\partial f}{\partial z_k}\Big|_{\mathbf{z}^0} \neq 0$ for some k, then Theorem 2.6 is applicable with $\ell = 1$ (since if $\ell > 1$, $\frac{\partial f}{\partial z_k}\Big|_{\mathbf{z}^0} = 0$). Therefore, the equation $f(\mathbf{z}) = 0$ in a neighborhood of $\mathbf{z}^0$ is equivalent to the equation $P(\mathbf{z}) = P(\grave{z}_k, z_k) = z_k - z^0_k + \alpha_1(\grave{z}_k) = 0$, which is solvable relative to z_k and $z_k = z^0_k - \alpha_1(\grave{z}_k)$ is holomorphic in the remaining variables, $\grave{z}_k$. ▲

Definition 4 (Irreducible Function). A function $f \not\equiv 0$ which is holomorphic at a point $\mathbf{z}^0 \in \mathbb{C}^d$ and $f(\mathbf{z}^0) = 0$ is said to be *irreducible* at this point if in a neighborhood of the point $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$, it cannot be represented as a product of functions holomorphic at $\mathbf{z}^0$, each of which is equal to 0 at $\mathbf{z}^0$.

Thus, for any function $f \not\equiv 0$ which is holomorphic at a point $\mathbf{z}^0 \in \mathbb{C}^d$ and $f(\mathbf{z}^0) = 0$, it can be decomposed uniquely (up to holomorphic and nonzero constants) in the form

$$f = f_1^{m_1} \cdots f_s^{m_s} \tag{2.12}$$

where $f_j^{m_j}, j = 1, \ldots, s$ are irreducible functions at the point $\mathbf{z}^0$. The following Lemma provides an important property of irreducible functions and will be used to prove the existence of a holomorphic function ψ in the proof of Theorem 4.9.

Lemma 2.7. *Suppose that the function f is holomorphic and irreducible at a point $\mathbf{z}^0 \in \mathbb{C}^d$ and $f(\mathbf{z}^0) = 0$, and that the function g is holomorphic in a neighborhood of $\mathbf{z}^0$ and is equal to 0 where $f = 0$. Then g is divisible by f in the sense that in a neighborhood of $\mathbf{z}^0$,*

$$g = fh,$$

where h is a holomorphic function.

Proof. See, e.g., [Sha92, Section §8, page 129]. ■

Next we shall consider closely the properties of sets on which holomorphic functions vanish.

Definition 5 (Analytic set). An analytic set $\mathcal{A}$ in a domain $D = \mathbb{D}^d \subset \mathbb{C}^d$ is locally defined as the set of common zeros of a finite number of holomorphic functions f_i's. In other words, for any point $\mathbf{z}^0 \in \mathbb{D}^d$ there exist a neighborhood $U = U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \subset \mathbb{D}^d$ and a finite number of functions $f_i \in \mathrm{Hol}(U)$ so that

$$\mathcal{A} \cap U = \{\mathbf{z} \in U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \mid f_1(\mathbf{z}) = \cdots = f_k(\mathbf{z}) = 0\}. \tag{2.13}$$

In particular, if $k = 1$, then we shall call the zero variety $\mathcal{Z}(f)$, rather than the analytic set, of a holomorphic function f on $\mathbb{D}^d \subset \mathbb{C}^d$:

$$\mathcal{Z}(f) = \{\mathbf{z} \in \mathbb{D}^d \mid f(\mathbf{z}) = 0\}.$$

A point $\mathbf{z}^0 \in \mathcal{A}$ is said to be a *regular point* of the set $\mathcal{A}$ if the rank of the Jacobian matrix evaluated at such a point $\left[\frac{\partial f_i}{\partial z_j}\right]\Big|_{\mathbf{z}=\mathbf{z}^0}$ for $i = 1, \ldots, k; j = 1, \ldots, d$ is equal to k. Then it follows that $\left[\frac{\partial f_i}{\partial z_j}\right]$ has rank k in a neighborhood of $\mathbf{z}^0$. The number $m \triangleq d - k$ (resp., k) is called the *local complex dimension* (resp., the *local complex co-dimension*) of $\mathcal{A}$ at a point $\mathbf{z}^0$ and denoted by $\dim_{\mathbf{z}^0} \mathcal{A}$ (resp., $\mathrm{codim}_{\mathbf{z}^0}\mathcal{A}$). The *dimension of the set* $\mathcal{A}$ is defined as $\dim \mathcal{A} = \sup_{\mathbf{z}^0 \in \mathcal{A}} \dim_{\mathbf{z}^0} \mathcal{A}$. The number $d - \dim \mathcal{A}$ is called the *co-dimension* of $\mathcal{A}$ and denoted by $\mathrm{codim}\mathcal{A}$. If $\dim_{\mathbf{z}^0} = m$ for all $\mathbf{z}^0 \in \mathcal{A}$, then we call that $\mathcal{A}$ is a *pure m-complex dimensional* analytic set in $\mathbb{D}^d$.

An analytic set $\mathcal{A} \subset \mathbb{C}^d$ of co-dimension 1 (i.e., $\dim \mathcal{A} = d - 1$) is called a *complex hypersurface*, and an analytic set of dimension 1 is called a *complex curve*. If an analytic set $\mathcal{A}$ contains only isolated points with no accumulation point in the domain D, then $\mathcal{A}$ is said to have dimension zero; while the whole domain $D = \mathbb{D}^d$ itself is an analytic set of dimension d (in this case, $f_i \equiv 0$ for all i). A point $\mathbf{z}^0$ is called a *singular point* in $\mathcal{A}$ if it is not a regular point. Any analytic set without singular points is said to be *smooth*.

Given a domain $D \subset \mathbb{C}^d$. Then an analytic set $\mathcal{A}$ is said to be *irreducible* in D if it cannot be represented as a union of analytic sets in D different from $\mathcal{A}$ itself. The set $\mathcal{A}$ is said to be *irreducible at a point* $\mathbf{z}^0 \in \mathcal{A}$ if in any sufficiently small neighborhood $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$ of $\mathbf{z}^0$, the set $\mathcal{A} \cap U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$ is irreducible.

Example 2. Let $f(\mathbf{z}) = z_1^2 z_2^2 - z_3^2$. Then the zero variety (or an analytic set) of f is given by:

$$\mathcal{Z}(f) = \{\mathbf{z} \in \mathbb{C}^3 \mid z_1^2 z_2^2 - z_3^2 = 0\}. \tag{2.14}$$

Since f can be factored as $f(\mathbf{z}) = f_1(\mathbf{z}) f_2(\mathbf{z})$ where $f_1(\mathbf{z}) = (z_1 z_2 - z_3)$ and $f_2(\mathbf{z}) = (z_1 z_2 + z_3)$, the variety (2.14) is split into a union of two sets: $\mathcal{Z}(f) = \mathcal{Z}(f_1) \cup \mathcal{Z}(f_2)$ where

$$\mathcal{Z}(f_1) = \{\mathbf{z} \in \mathbb{C}^3 \mid z_1 z_2 - z_3 = 0\} \qquad \text{and} \qquad \mathcal{Z}(f_2) = \{\mathbf{z} \in \mathbb{C}^3 \mid z_1 z_2 + z_3 = 0\}.$$

Thus, $\mathcal{Z}(f)$ is reducible in $\mathbb{C}^3$. It is also reducible on the lines $(z_1, 0, 0)$ and $(0, z_2, 0)$ which is a set of the intersection $\mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)$. For the remaining points (i.e., $\mathbf{z} \in \mathbb{C}^3 \setminus \{\mathcal{Z}(f_1) \cap \mathcal{Z}(f_2)\}$), $\mathcal{Z}(f)$ is irreducible. For instance, $\mathbf{a} = (1, 1, 1) \in \mathcal{Z}(f)$ but $\mathbf{a} \notin \mathcal{Z}(f_2)$. $\diamondsuit$

Example 3. Let $g(\mathbf{z}) = z_1 z_2^2 - z_3^2$. The zero variety of g is the set

$$\mathcal{Z}(g) = \{\mathbf{z} \in \mathbb{C}^3 \mid z_1 z_2^2 - z_3^2 = 0\}. \tag{2.15}$$

For $z_1 \neq 0$, $g(\mathbf{z}) = g(z_1, \grave{z}_1) = z_1(z_2^2 - z_1^{-1} z_3^2)$. Thus the only factorization of $(z_2^2 - z_1^{-1} z_3^2)$ in $\mathbb{C}[z_2, z_3]$ (the polynomial ring over $\mathbb{C}$ in 2 variables: z_2, z_3) is $(z_2 - \sqrt{z_1^{-1}} z_3)(z_2 + \sqrt{z_1^{-1}} z_3)$ for a fixed nonzero value of z_1. Therefore, in $\mathbb{C}^3 \setminus \{z_1 = 0\}$, g can be factored as

$$g(\mathbf{z}) = z_1(z_2 - \sqrt{z_1^{-1}} z_3)(z_2 + \sqrt{z_1^{-1}} z_3) = (\sqrt{z_1} z_2 - z_3)(\sqrt{z_1} z_2 + z_3) := g_1(\mathbf{z}) g_2(\mathbf{z}),$$

where $\sqrt{z_1}$ denotes one of the branches of the square root function. Consequently, the zero variety of g in $\mathbb{C}^3 \setminus \{z_1 = 0\}$ can be decomposed as a union of two sets $\mathcal{Z}(g) = \mathcal{Z}(g_1) \cup \mathcal{Z}(g_2)$ where

$$\mathcal{Z}(g_1) = \{\sqrt{z_1} z_2 - z_3 = 0\} \quad \text{and} \quad \mathcal{Z}(g_2) = \{\sqrt{z_1} z_2 + z_3 = 0\}. \tag{2.16}$$

Thus, for any points of the form $(a, 0, 0), a \neq 0$, the zero variety of g, $\mathcal{Z}(g)$, is locally reducible;

it is not locally reducible, however, at the point $(0,0,0)$ since $z_1 \to \sqrt{z_1}$ is not holomorphic at $z_1 = 0$. In addition, $\mathcal{Z}(g)$ is locally irreducible at any points of the form $(a,b,c) \neq (a,0,0)$, where $a \neq 0$ since there are no such points belonging to $\mathcal{Z}(g_1)$ and $\mathcal{Z}(g_2)$ in (2.16) simultaneously. $\diamondsuit$

It is easy to show that if the function f is irreducible at a point $\mathbf{z}^0$, then the zero variety of f is also irreducible at such a point; however, the converse is false, i.e., if $\mathcal{Z}(f)$ is irreducible at a point $\mathbf{z}^0$, the function f may or may not be irreducible at such a point. For instance, let g be an irreducible function at a point $\mathbf{z}^0$ and define the function $f := g^2$. Thus, f is reducible at such a point. However, the zero variety of f ($\mathcal{Z}(f) = \{g^2 = 0\}$) and the zero variety of g ($\mathcal{Z}(g) = \{g = 0\}$) coincide and this implies that $\mathcal{Z}(f)$ is irreducible.

Lemma 2.8. *If the function f is irreducible at a point $\mathbf{z}$, then so is $\mathcal{Z}(f)$.*

The last Theorem that we need here is the so-called *Riemann Extension Theorem* which is stated as follows:

Theorem 2.9 (Riemann Extension Theorem). *If f is holomorphic in $D \setminus \mathcal{A}$, where D is a domain in $\mathbb{C}^d$ for $d > 1$ and $\mathcal{A}$ is an analytic set of co-dimension at least 2, then f extends in a unique way to a function holomorphic in D.*

We shall also have occasion to need meromorphic functions (see e.g.,[FG02, Sha92]); if $h = f/g$ where f and g are holomorphic with no nontrivial common holomorphic factors, we let $\mathcal{Z}(h) = \mathcal{Z}(f)$ be the zero variety of h (defined to be the zero variety of the numerator f), and we let $\mathcal{P}(h) = \mathcal{Z}(g)$ be the polar variety of h (defined to be the zero variety of the denominator g). If F is a meromorphic matrix function on $\mathbb{D}^d$, let $\mathcal{P}(F)$ denote the union of the polar varieties of the matrix entries.

Any subset V' of a variety V of the form $V' = V \setminus A$ where A is a subvariety of V of lower dimension is said to be a *generic* subset of V.

2.1.4 Hardy spaces

The purpose of this Subsection is to give a brief review on the function theory of *Hardy classes* on the disk. For a more detailed treatment on this subject, the reader may refer to [Dur70, Hof62] for scalar valued holomorphic functions; [RR85] for vector and operator valued holomorphic functions; and [Rud69] for holomorphic functions on the polydisk.

Definition 6. For functions f holomorphic in a unit disk $f : \mathbb{D} \mapsto \mathbb{C}$, the mean of order p at radius r ($0 < r < 1$) is defined by

$$M_p(r, f) := \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right]^{1/p}, \quad \text{for } 1 \le p < \infty,$$
$$M_\infty(r, f) := \sup_{\theta\in[0,2\pi]} |f(re^{i\theta})|, \quad \text{for } p = \infty.$$

Definition 7 (Hardy Spaces). The Hardy space $H^p(\mathbb{D})$ is the set of holomorphic functions f on a unit disk $\mathbb{D}$ whose means of order p, $M_p(r, f)$, remain bounded as $r \to 1$, i.e.

$$H^p(\mathbb{D}) = \{f \in \mathrm{Hol}(\mathbb{D}) \mid \|f\|_p := \sup_{0<r<1} M_p(r, f) < \infty\}.$$

Let us consider the case when $p = 2$. If f is holomorphic on the unit disk $\mathbb{D}$, then it admits a power series representation, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1.$$

Thus, the mean of order 2 is

$$\begin{aligned} M_2^2(r, f) &= \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |\sum_{n=0}^{\infty} a_n r^n e^{i\theta n}|^2 d\theta \\ &= \sum_{n=0}^{\infty} |a_n|^2 r^{2n}, \end{aligned}$$

and hence $f \in H^2(\mathbb{D})$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. In addition, this also shows that $M_2(r, f)$ increases with r.

When $p = \infty$, it follows from the maximum modulus principle that $M_\infty(r, f)$ is an increasing function of r, and $H^\infty(\mathbb{D})$ is the class of bounded holomorphic functions in the disk. In general, for any $p \ge 1$, the means $M_p(r, f)$ of any holomorphic function f are increasing functions of r. As a result, the p-norm of a holomorphic function f on $\mathbb{D}$ is a radial limit of the mean of order p:

$$\|f\|_p = \lim_{r\uparrow 1} M_p(r, f) = \lim_{r\uparrow 1} \|f_r\|_{L^p(\mathbb{T})},$$

where f_r denotes the function $f_r(z) = f(rz)$, and $L^p(\mathbb{T})$ is the space of (equivalence[2] classes of) all measurable functions f on the circle $\mathbb{T}$ with $\int_0^{2\pi} |f(e^{i\theta})|^p d\theta < \infty$.

[2] Two functions f and g defined on the same measurable set are said to be *equivalent* if $\mu\{x : f(x) \neq g(x)\} = 0$.

Note that every $H^p(\mathbb{D})$ function $f(re^{i\theta})$ converges almost everywhere[3] to an $L^p(\mathbb{T})$ boundary function $f(e^{i\theta})$. Thus $H^p(\mathbb{D})$ is a normed linear space with the norm defined as the $L^p(\mathbb{T})$ norm of the boundary function, i.e.

$$\|f\|_p = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right]^{1/p} = \lim_{r\uparrow 1} M_p(r,f) \quad \text{for } 1 \le p < \infty$$
$$\|f\|_\infty = \sup_{z:|z|<1} |f(z)| = \text{ess} \sup_{\theta\in[0,2\pi]} |f(e^{i\theta})|,$$

where $\text{ess}\sup f(e^{i\theta})$ is the infimum of $\sup g(e^{i\theta})$ as g ranges over all functions which are equal to f almost everywhere. Thus,

$$\begin{aligned}\|f\|_\infty = \text{ess} \sup_{\theta\in[0,2\pi]} f(e^{i\theta}) &= \inf_{g=f \text{ a.e.}} \{ \sup_{\theta\in[0,2\pi]} |g(e^{i\theta})|\} \\ &= \inf\{M \mid \mu\{\theta : f(e^{i\theta}) > M\} = 0\}.\end{aligned}$$

We denote by $L^2_n(\mathbb{T})$ the set of $n \times 1$ vector valued functions with entries in $L^2(\mathbb{T})$ and define the inner product as

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})^* f(e^{i\theta}) d\theta,$$

for any $f, g \in L^2_n(\mathbb{T})$. Thus, $L^2_n(\mathbb{T})$ equipped with this inner product forms a Hilbert spaces. Furthermore, such an inner product on $L^2_n(\mathbb{T})$ defines a norm on $L^2_n(\mathbb{T})$ given by:

$$\|f\|_2 = \sqrt{\langle f, f\rangle}.$$

More explicitly, the 2-norm of a vector valued function $f \in L^2_n(\mathbb{T})$ is defined as

$$\|f\|_2 = \left[\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})^* f(e^{i\theta}) d\theta\right]^{1/2}.$$

Let $L^\infty_{m\times n}(\mathbb{T})$ denote the set of $m \times n$ matrix valued functions whose entries belong to $L^\infty(\mathbb{T})$, where $L^\infty(\mathbb{T})$ is the space of (equivalence classes of) all measurable functions which are bounded except possibly on a subset of measure zero. For any $F \in L^\infty_{m\times n}(\mathbb{T})$, its ∞-norm is defined as

$$\|F\|_\infty = \text{ess} \sup_{\theta\in[0,2\pi]} \overline{\sigma}\left(F(e^{i\theta})\right),$$

where $\overline{\sigma}(A)$ is the largest singular value of A.

[3] A property is said to hold *almost everywhere (abbreviated a.e.)* if the set of points where it fails to hold is a set of measure zero. In particular, we say that two functions f and g are equivalent if $f = g$ almost everywhere.

2.2 Operator theory

This Section is devoted to the fundamental facts on the operator theory and functional calculus which will be used throughout this dissertation. For more details, readers should consult, e.g. [AM02, FF90, FFGK98, Kre78, RSN55, Roy88, SNF70].

2.2.1 Basic facts in Operator Theory

Definition 8 (Linear operator). Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces. An operator $T : \mathcal{X} \to \mathcal{Y}$ is called a *linear* operator provided that the superposition property holds; i.e., for any $x_1, x_2 \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$),

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2) \tag{2.17}$$

If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces, we can define boundedness of a linear operator T as follows:

Definition 9 (Bounded linear operator). Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces. An operator $T : \mathcal{X} \to \mathcal{Y}$ is said to be a *bounded linear operator* provided that

1. T is linear, and
2. there is a constant c such that for all $x \in \mathcal{X}$,

$$\|Tx\| \leq c\|x\|. \tag{2.18}$$

Remark 3. From this point on, by an *operator* from $\mathcal{X}$ to $\mathcal{Y}$ we shall always mean a bounded linear operator and denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of operators from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{X} = \mathcal{Y}$, we shall write $\mathcal{L}(\mathcal{X})$ for $\mathcal{L}(\mathcal{X}, \mathcal{X})$ and call such an operator as an *operator on* $\mathcal{X}$. By a *subspace* of a vector space, we always mean a closed linear subspace. ▲

For any operator, it is of interest to find the smallest constant c such that (2.18) holds and this leads to the definition of the operator norm.

Definition 10 (Operator norm). The operator norm of the operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, denoted by $\|T\|_{\mathrm{Op}}$, is the smallest possible c such that (2.18) holds, i.e.

$$\|T\|_{\mathrm{Op}} = \inf\{c : \|Tx\| \leq c\|x\| \text{ for all } x \in \mathcal{X}\}. \tag{2.19}$$

Note that $Tx = 0$ whenever $x = 0$. Thus we can leave out the case that $x = 0$ and divide both sides of (2.18) by $\|x\|$,

$$\frac{\|Tx\|}{\|x\|} \leq c \qquad (\text{when } x \neq 0).$$

Since the operator norm of T is the infimum of all possible c satisfying (2.19), this leads to an alternative definition of the operator norm, i.e. for any $x \in \mathcal{X}$,

$$\|T\|_{\mathrm{Op}} = \sup_{x \in \mathcal{X}: x \neq 0} \frac{\|Tx\|}{\|x\|}. \tag{2.20}$$

For notational convenience, we shall use $\|\cdot\|$ for the operator norm rather than $\|\cdot\|_{\mathrm{Op}}$.

Proposition 2.10. *Let T be the operator as defined in Definition 9. Then the operator norm of T in* (2.20) *satisfies the following:*

$$\|T\| = \sup_{x \in \mathcal{X}: x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \in \mathcal{X}: \|x\| = 1} \|Tx\| = \sup_{x \in \mathcal{X}: \|x\| < 1} \|Tx\| \tag{2.21}$$

Proof. For any $x \neq 0$, define $y = \frac{x}{\|x\|}$ and due to the linearity of T, we have

$$\sup_{x \in \mathcal{X}: x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \in \mathcal{X}: x \neq 0} \left\| T \frac{x}{\|x\|} \right\| = \sup_{y \in \mathcal{X}: \|y\| = 1} \|Ty\|.$$

By writing x for y, we get $\|T\| = \sup_{x \in \mathcal{X}: \|x\| = 1} \|Tx\|$. Since $\|T\| = \sup_{x \in \mathcal{X}: x \neq 0} \frac{\|Tx\|}{\|x\|}$, this implies that $\|T\| \cdot \|x\| \geq \|Tx\|$ for all $x \in \mathcal{X}$. It also follows that, if $x \in \mathcal{X}$ and $\|x\| < 1$, then $\|T\| > \|Tx\|$. Therefore the operator norm $\|T\|$ can be expressed as $\|T\| = \sup_{x \in \mathcal{X}: \|x\| < 1} \|Tx\|$. ∎

An adjoint operator T^* of T is the operator acting from $\mathcal{Y}$ into $\mathcal{X}$, defined by

$$\langle Tx, y \rangle_{\mathcal{Y}} = \langle x, T^* y \rangle_{\mathcal{X}} \qquad \text{for all } x \in \mathcal{X}, y \in \mathcal{Y}. \tag{2.22}$$

To prove the existence and uniqueness property of the adjoint operator, we need Riesz representation theorem and the notion of *sesquilinear functional*; for complete details on this, we refer to [Kre78, RSN55].

2.2.2 Isometry, Unitary, and Contraction

Definition 11 (Isometry and Unitary). A linear operator T from $\mathcal{X}$ into $\mathcal{Y}$ is said to be an *isometry* provided that

$$\langle Tx_1, Tx_2 \rangle_{\mathcal{Y}} = \langle x_1, x_2 \rangle_{\mathcal{X}} \qquad \text{for all } x_1, x_2 \in \mathcal{X}. \tag{2.23}$$

Equivalently, $T^*T = I_{\mathcal{X}}$, where $I_{\mathcal{X}}$ is an identity operator on a Hilbert space $\mathcal{X}$, indicating such a space by a subscript if it is necessary. In addition, if T is an isometry mapping from $\mathcal{X}$ onto $\mathcal{Y}$ (i.e. $T^*T = I_{\mathcal{X}}$ and $T\mathcal{X} = \mathcal{Y}$), then T is called a *unitary*. T is called a *co-isometry* if T^* is isometry.

Proposition 2.11. *T is a unitary operator if and only if $T^*T = I_{\mathcal{X}}$ and $TT^* = I_{\mathcal{Y}}$.*

Proof. Since T is isometric, $T^*T = I_{\mathcal{X}}$. Then this implies that $T(T^*T) = (TT^*)T = T$. Then for any $y \in \mathcal{Y}$, there exists $x \in \mathcal{X}$ such that $Tx = y$ since T is onto. This implies that $(TT^*)y = y$, and hence $TT^* = I_{\mathcal{Y}}$. Conversely, suppose T is such that $T^*T = I_{\mathcal{X}}$ and $TT^* = I_{\mathcal{Y}}$. This implies evidently T is onto. ■

Definition 12 (Contraction). An operator T is called a *contraction* provided that for each $x \in \mathcal{X}$,

$$\|Tx\|_{\mathcal{Y}} \leq \|x\|_{\mathcal{X}}, \tag{2.24}$$

i.e., $\|T\| \leq 1$. If $\|T\| < 1$, then T is said to be *strictly contractive*.

The condition (2.24) implies that $\langle Tx, Tx\rangle_{\mathcal{Y}} = \langle T^*Tx, x\rangle_{\mathcal{X}} \leq \langle x, x\rangle_{\mathcal{X}}$ for all $x \in \mathcal{X}$. Since $\|T\| = \|T^*\|$, T^* is also a contraction from $\mathcal{Y}$ into $\mathcal{X}$. Thus, by analogous argument, $\langle TT^*y, y\rangle_{\mathcal{Y}} \leq \langle y, y\rangle_{\mathcal{Y}}$ for all $y \in \mathcal{Y}$. Therefore, for any contraction operator T of $\mathcal{X}$ into $\mathcal{Y}$, $T^*T \leq I_{\mathcal{X}}$ and $TT^* \leq I_{\mathcal{Y}}$. Obviously, if T is a unitary, then it is also a contraction.

Besides the unitary operator, there is another type of contractions on a Hilbert space, which is called a *completely nonunitary* contraction.

Definition 13 (Completely Nonunitary Contraction). A contraction T on $\mathcal{H}$ is called a *completely nonunitary* (c.n.u.) if there is no nonzero reducing subspace[4] $\mathcal{M}$ such that the restriction of the operator T to the subspace $\mathcal{M}$, denoted by $T|_{\mathcal{M}}$, is unitary on $\mathcal{M}$.

2.2.3 Unitary Dilation and Lifting Theorem

Definition 14 (Dilation). An operator U on a Hilbert space $\mathcal{K}$, (i.e. $U \in \mathcal{L}(\mathcal{K})$) is called a *(weak or Halmos) dilation* of an operator $T \in \mathcal{L}(\mathcal{H})$ if $\mathcal{H}$ is a subspace of $\mathcal{K}$ and $T = P_{\mathcal{K}\to\mathcal{H}}U|_{\mathcal{H}}$, where $P_{\mathcal{K}\to\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. If, in addition, $T^n = P_{\mathcal{K}\to\mathcal{H}}U^n|_{\mathcal{H}}$ for all $n \geq 1$, then U is called a *dilation* of T. Moreover, U is called a *unitary dilation* of T if U itself is unitary. If U is a dilation of T, then T is called a *compression* of U.

It is easy to see that U is a weak dilation of T if and only if U admits a matrix representation of the form

$$U = \begin{bmatrix} T & * \\ * & * \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{bmatrix},$$

where $*$ denotes an arbitrary entry of U.

[4]A subspace $\mathcal{M} \subset \mathcal{H}$ is said to be a *reducing subspace* of an operator T on $\mathcal{H}$ if $\mathcal{M}$ is invariant to both T and T^*.

In 1965, D. Sarason [Sar65] showed that a necessary and sufficient condition such that a weak dilation is in fact a dilation is that the subspace $\mathcal{H}$ be the difference of two invariant subspaces of $\mathcal{K}$.

Theorem 2.12 (Sarason's Lemma). *Let $\mathcal{H}$ be a subspace of a Hilbert space $\mathcal{K}$ and an operator $U \in \mathcal{L}(\mathcal{K})$. Suppose that $T \in \mathcal{L}(\mathcal{H})$ defined by $T = P_{\mathcal{K}\to\mathcal{H}}U|_{\mathcal{H}}$. Then U is a dilation of T ($T^n = P_{\mathcal{K}\to\mathcal{H}}U^n|_{\mathcal{H}}$ for $n \geq 1$) if and only if there exist nested invariant subspaces $\mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{K}$ for U with $\mathcal{H} = \mathcal{H}_1 \ominus \mathcal{H}_2$.*

Thus, by Sarason's Lemma, U is a dilation of T if and only if it has a matrix representation of the form

$$U = \begin{bmatrix} * & * & * \\ 0 & T & * \\ 0 & 0 & * \end{bmatrix} : \begin{bmatrix} \mathcal{H}_2 \\ \mathcal{H} \\ \mathcal{K}\ominus\mathcal{H}_1 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_2 \\ \mathcal{H} \\ \mathcal{K}\ominus\mathcal{H}_1 \end{bmatrix} \tag{2.25}$$

There is another notion which is somewhat stronger than being a dilation.

Definition 15 (Extension). An operator U on a Hilbert space $\mathcal{K}$ is called an *extension* of an operator $T \in \mathcal{L}(\mathcal{H})$ if $\mathcal{H}$ is a subspace of $\mathcal{K}$ which is invariant for U, and $T = U|_{\mathcal{H}}$. If U is an extension of T, then T is called a *part* of U.

Obviously U is an extension of T if and only if U admits a matrix representation of the form

$$U = \begin{bmatrix} T & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{bmatrix}.$$

Definition 16 (Minimal Unitary Dilation). Let T be any contraction on the Hilbert space $\mathcal{H}$. Then U on $\mathcal{K} \supseteq \mathcal{H}$ is said to be a *minimal unitary dilation* of T if U itself is a unitary dilation of T and satisfies the following minimality condition

$$\mathcal{K} = \bigvee_{n=-\infty}^{\infty} U^n\mathcal{H}, \tag{2.26}$$

where $\bigvee_{n=k}^{\ell} \mathcal{M}_n$ denotes the closed linear span of a set of subspaces $\{\mathcal{M}_n\}$ for $n \in [k,\ell] \subset \mathbb{Z}$.

Theorem 2.13 (Sz.-Nagy's Theorem). *Any contraction T on Hilbert space $\mathcal{H}$ admits a minimal unitary dilation. Moreover, all minimal unitary dilations of T are uniquely determined up to isomorphism.*

Definition 17 (Lifting). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be any Hilbert spaces and $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. An operator $U \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ is said to be *lifting* of T if $\mathcal{H}_1$ and $\mathcal{H}_2$ are subspaces of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, and

$$P_{\mathcal{K}_2\to\mathcal{H}_2}U = TP_{\mathcal{K}_1\to\mathcal{H}_1} \tag{2.27}$$

It is clear that U is a lifting of T if and only if U admits a matrix representation of the form

$$U = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_1^\perp \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{H}_2 \\ \mathcal{H}_2^\perp \end{bmatrix}. \tag{2.28}$$

In particular, if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{H}$, then an operator U is a lifting of T if and only if it has a matrix representation of the form

$$U = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{H}^\perp \end{bmatrix}. \tag{2.29}$$

Obviously, if U is a lifting of T, then U is a (weak) dilation of T.

Theorem 2.14 (Sz.-Nagy-Foiaş Lifting Theorem for Unitary Dilation). *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Hilbert spaces, $\mathcal{H}_1 \subset \mathcal{K}_1, \mathcal{H}_2 \subset \mathcal{K}_2$ be closed subspaces, $U_1 \in \mathcal{L}(\mathcal{K}_1), U_2 \in \mathcal{L}(\mathcal{K}_2)$ be unitary operators, and $T_1 \in \mathcal{L}(\mathcal{H}_1), T_2 \in \mathcal{L}(\mathcal{H}_2)$ be contraction operators such that U_i is a dilation of T_i for $i = 1, 2$. If $X : \mathcal{H}_1 \to \mathcal{H}_2$ is a contraction intertwining T_1, T_2, i.e. $XT_1 = T_2X$ and $\|X\| \leq 1$, then there exists an operator $Y : \mathcal{K}_1 \to \mathcal{K}_2$ such that*

1. *Y is contraction, i.e. $\|Y\| \leq 1$,*
2. *Y intertwines U_1, U_2, i.e. $YU_1 = U_2Y$, and*
3. *$X = P_{\mathcal{K}_2 \to \mathcal{H}_2} Y|_{\mathcal{H}_1}$.*

$$\begin{array}{ccc} \mathcal{K}_1 & \xrightarrow{\;Y\;} & \mathcal{K}_2 \\ {\scriptstyle P_{\mathcal{K}_1 \to \mathcal{H}_1}}\downarrow & & \downarrow{\scriptstyle P_{\mathcal{K}_2 \to \mathcal{H}_2}} \\ \mathcal{H}_1 & \xrightarrow{\;X\;} & \mathcal{H}_2 \end{array}$$

Figure 2.1: The lifting diagram

2.2.4 Andô's Theorem and von Neumann's Inequality

In 1953, B. Sz.-Nagy [SN53] discovered the fact that every contraction has an extension that is a co-isometry, and a unitary dilation. Ten years later, T. Andô [And63] generalized the previous fact from one contraction to a pair of commuting contractions; the result is stated precisely as follows:

Theorem 2.15 (Andô's Theorem [And63]). *Let T_1, T_2 be a pair of commuting contractions on the Hilbert space $\mathcal{H}$. Then there exists a pair of commuting unitary operators U_1, U_2 on a*

Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace such that for any $f \in \mathcal{H}$,

$$T_1^{n_1} T_2^{n_2} f = P_{\mathcal{K}\to\mathcal{H}} U_1^{n_1} U_2^{n_2} f \qquad \text{for } n_1, n_2 = 1, 2, \ldots$$

Theorem 2.16 (von Neumann's Inequality for $d = 1$). *Let T be a contraction on a Hilbert space $\mathcal{H}$. Then for any matrix-valued polynomial $p(z)$ on the disk, we have*

$$\|p(T)\| \le \sup_{z\in\mathbb{D}} |p(z)|. \tag{2.30}$$

Theorem 2.17 (von Neumann's Inequality for $d = 2$). *Let T_1, T_2 be commuting contractions on a Hilbert space $\mathcal{H}$. Then for any matrix-valued polynomial $p(z_1, z_2)$ on the bidisk, we have*

$$\|p(T_1, T_2)\| \le \sup_{(z_1,z_2)\in\mathbb{D}^2} |p(z_1, z_2)|. \tag{2.31}$$

Remark 4. For three or more commuting contractions ($d \ge 3$), Andô's theorem and the corresponding von Neumann's inequality fail in general. S. Parrott [Par70] showed that there exist three commuting contractions which do not have commuting unitary dilations. M. J. Crabb and A. M. Davie [CD75], and N. Th. Varopoulos [Var74] independently discovered the failure of von Neumann's inequality for three commuting contractions. For more details, see e.g. [AM02, Chapter 10] or [Pau86, Chapter 4 and 6.9]. ▲

2.3 Schur Class versus Schur-Agler Class

The class of all contractive valued functions holomorphic on $\mathbb{D}^d$ is often called the *Schur class*, denoted by $\mathcal{S}_d$. To be more precise, let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and an operator-valued function $F : \mathbb{D}^d \mapsto \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then the Schur class is defined as

$$\mathcal{S}_d(\mathcal{X}, \mathcal{Y}) \triangleq \left\{ F : \mathbb{D}^d \mapsto \mathcal{L}(\mathcal{X}, \mathcal{Y}) \text{ holomorphic } \mid \sup_{(z_1,\ldots,z_d)\in\mathbb{D}^d} \|F(z_1, \ldots, z_d)\|_{\mathrm{Op}} \le 1 \right\}. \tag{2.32}$$

However, there is a closely related class of functions introduced by J. Agler (see [Agl87, Agl90]) which we shall call the *Schur-Agler class*, $\mathcal{SA}_d$, defined as follows:

$$\mathcal{SA}_d(\mathcal{X}, \mathcal{Y}) \triangleq \left\{ F : \mathbb{D}^d \mapsto \mathcal{L}(\mathcal{X}, \mathcal{Y}) \text{ holomorphic } \mid \sup\{\|F(T_1, \ldots, T_d)\|\} \le 1 \right\} \tag{2.33}$$

for any d-tuple of commuting strict contractions $T = (T_1, \ldots, T_d)$ (i.e., $\|T_j\| < 1$ and $T_iT_j = T_jT_i$ for $i, j = 1, \ldots, d$) where $T_i \in \mathcal{L}(\mathcal{H})$ for any auxiliary Hilbert space $\mathcal{H}$. If $F(\mathbf{z}) = \sum_{|\mathbf{j}|=0}^{\infty} C_{\mathbf{j}} \mathbf{z}^{\mathbf{j}}$, then

$F(T) = \sum_{|\mathbf{j}|=0}^{\infty} C_{\mathbf{j}} \otimes T^{\mathbf{j}} \in \mathcal{L}(\mathcal{X} \otimes \mathcal{H}, \mathcal{Y} \otimes \mathcal{H})$ where $T^{\mathbf{j}} = T_1^{j_1} \cdots T_d^{j_d}$ if $\mathbf{j} = (j_1, \ldots, j_d)$. (See definition of tensor $\otimes$ in Appendix A on page 188).

In particular if $\mathcal{X}$ and $\mathcal{Y}$ are linear vector spaces, say $\mathcal{X} = \mathbb{C}^m$ and $\mathcal{Y} = \mathbb{C}^l$, then F admits a matrix representation, $F \in \mathbb{C}^{l \times m}$. In this case, the Schur class, denoted by $\mathcal{S}_d(\mathbb{C}^m, \mathbb{C}^l)$, is the class of all $l \times m$ matrix-valued functions F holomorphic on $\mathbb{D}^d$ and satisfying the norm constraint in (2.32). Likewise, the class of all $l \times m$ matrix-valued functions F holomorphic on $\mathbb{D}^d$ and satisfying the norm constraint (2.33) is denoted by $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l)$.

In the following Theorem, we shall show that for $d \leq 2$, $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l) = \mathcal{S}_d(\mathbb{C}^m, \mathbb{C}^l)$; otherwise, $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l) \underset{\neq}{\subset} \mathcal{S}_d(\mathbb{C}^m, \mathbb{C}^l)$.

Theorem 2.18. *$\mathcal{S}_d \equiv \mathcal{SA}_d$ for $d \leq 2$.*

Proof—Sketch. Obviously, the Schur-Agler class is a subclass of the Schur class, $\mathcal{SA}_d \subset \mathcal{S}_d$. We only need to show that for $d \leq 2$, $\mathcal{S}_d \subset \mathcal{SA}_d$.

For $d = 1$, let $F(z) \in \mathcal{S}$. Then, $F^*(z) = \lim_{r \uparrow 1} F(rz)$ exists for almost every $z \in \mathbb{T}$. This implies that for any unitary operator U with spectral measure absolutely continuous with respect to Lebesgue measure (i.e., U absolutely continuous unitary operator), $F(U) = \int_{\mathbb{T}} F(z) dE(z)$ is well defined where $E(\sigma)$ is the spectral measure defined for the Borel subsets σ on the distinguished boundary $\mathbb{T}$. Then, we have

$$\|F(U)\| \leq \int_{\mathbb{T}} |F(z)| \|dE(z)\| \leq \|F\|_\infty = \sup_{z \in \mathbb{T}} |F(z)| = \sup_{z \in \mathbb{D}} |F(z)|. \tag{2.34}$$

Note that the last equality in the above expression is due to the Maximum Modulus Principle. Now from Sz.-Nagy's theorem, every contraction T has a minimal unitary dilation, say U, i.e., $T = P_{\mathcal{K} \to \mathcal{H}} U|_{\mathcal{H}}$. If T is a strict contraction, i.e., $\|T\| < 1$ (or in the more general class called *completely nonunitary (c.n.u.) contraction*), then the minimal unitary dilation U of T is absolutely continuous. Thus, $F(T) = P_{\mathcal{K} \to \mathcal{H}} F(U)|_{\mathcal{H}}$ where $F(U) = \sum a_n U^n$ if $F(z) = \sum a_n z^n$. This expression together with (2.34) imply that

$$\|F(T)\| \leq \|F(U)\| \leq \sup_{z \in \mathbb{D}} |F(z)| \leq 1, \qquad \text{(since } F \in \mathcal{S}\text{)}$$

i.e., $F \in \mathcal{SA}$ and hence, $\mathcal{S} \subset \mathcal{SA}$. But always $\mathcal{SA} \subset \mathcal{S}$. Then we have $\mathcal{S} = \mathcal{SA}$.

Now for $d = 2$, let us assume that $F \in \mathcal{S}_2$. By Andô's theorem, we know that any two commuting contractions T_1, T_2 have commuting unitary dilations (see Theorem 2.15). As before, the joint spectral measure for U_1, U_2 is absolutely continuous if T_1, T_2 are strict contractions.

By using the similar argument as for the case when $d = 1$, we have

$$\|F(T_1, T_2)\| = \| P_{\mathcal{K}\to\mathcal{H}} F(U_1, U_2)|_{\mathcal{H}} \| \leq \|F(U_1, U_2)\| \leq \sup_{(z_1,z_2)\in\mathbb{D}^2} |F(z_1, z_2)| \leq 1.$$

Hence, $\mathcal{S}_2 \subset \mathcal{SA}_2$. This implies that $\mathcal{S}_2 = \mathcal{SA}_2$. For complete proof, we refer the reader to [FF90, SNF70]. ∎

From the definitions, we see that the Schur class function F is also in the Schur-Agler class if we can show that $\sup \|F(T_1, \ldots, T_d)\| \leq 1$. But in this step, we need the famous van Neumann's inequality which is applicable only for $d \leq 2$ due to the fact that the von Neumann's inequality and Andô's theorem fail in general for three or more contractions. Varopoulos [Var74], and Crabb-Davie [CD75] constructed independently the examples of an F which is in the Schur class but not in the Schur-Agler class, $F \in \mathcal{S}_3 \setminus \mathcal{SA}_3$ (see also Remark 4). So, we can identify the Schur class and the Schur-Agler class only when $d \leq 2$. For $d > 2$, the Schur-Agler class is only a subclass of the Schur class.

2.4 Nevanlinna-Pick Interpolation Theory

A classical interpolation problem is stated as follows: *given n distinct points, say $z^1, \ldots, z^n$ in the unit disk $\mathbb{D}$ (resp., in the right half plane Π^+) and a collection of any complex numbers $w_1, \ldots, w_n$ in $\mathbb{C}$, determine a scalar-valued holomorphic function f on $\mathbb{D}$ (resp., Π^+) with*

$$\sup_{z\in\mathbb{D}\,(resp.,\Pi^+)} |f(z)| \leq 1 \tag{2.35}$$

such that $f(z^k) = w_k$ for all $k = 1, \ldots, n$.

This problem was studied independently by G. Pick and R. Nevanlinna in 1910's, and hence we shall refer to such a problem as the *n-point Nevanlinna-Pick Interpolation Problem* (or *n-point* NPIP, for short). G. Pick discovered a necessary and sufficient condition to solve such a problem in 1916, and also proved that in the extremal case[5], the solution is unique and is given by a Blaschke product[6]. He defined an $n \times n$ matrix $P = \left[P^{k,\ell}\right]_{k,\ell=1}^n$, which is now called the *Pick Matrix*, by using the *n-point Interpolation Data Set $\mathcal{D}_n$* (or *n-point* ID *set*, for short):

$$\mathcal{D}_n = \{(z^k, w_k) \in (\mathbb{D} \times \mathbb{C}) : k = 1, \ldots, n\}.$$

[5]The extremal case is the case when there is no solution that maps into a disk of radius less than one.

[6]A function of the form

$$B(z) = z^n \prod_k \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a}_k z}$$

is called a *Blaschke product*. Here n is a nonnegative integer and $\sum(1 - |a_k|) < \infty$. The set $\{a_k\}$ may be finite, or even empty. If $\{a_k\}$ is empty, it is understood that $B(z) = z^n$.

The Pick's theorem is stated precisely as follows:

Theorem 2.19 (Pick's Theorem). *Given the* ID *set* $\mathcal{D}_n$ *as above, the solution of the Pick problem for the unit disk case (resp., for the right half plane case) exists if and only if the associated Pick matrix,*

$$P = \left[\frac{1 - w_k\overline{w}_\ell}{1 - z_k\overline{z}_\ell}\right]_{k,\ell=1}^n \qquad \left(\textit{resp., } P = \left[\frac{1 - \overline{w}_k w_\ell}{\overline{z}_k + z_\ell}\right]_{k,\ell=1}^n\right) \tag{2.36}$$

is positive semidefinite. Moreover, the function f *is unique if and only if the Pick matrix has rank* $M < n$*. In this case,* f *is a Blaschke product of degree* M*.*

Unaware of Pick's work[7], R. Nevanlinna considered the same problem in 1919, and proposed an algorithm to solve such a problem using an induction step together with the *Schwarz's lemma*. Ten years later, he also parametrized all solutions of non-extremal problems.

J. Agler [Agl87] studied the extension of the classical NPIP, which concerns only one variable, to the several complex variables case; i.e., an NPIP on the unit polydisk $\mathbb{D}^d$ with the given n-point ID set

$$\mathcal{D}_n^d = \{(\mathbf{z}^k, w_k) \in (\mathbb{D}^d \times \mathbb{C}) : k = 1, \ldots, n\}.$$

Theorem 2.20 (Agler's Theorem). *Given interpolation nodes* $\mathbf{z}^1, \ldots, \mathbf{z}^n \in \mathbb{D}^d$*, and interpolation values* $w_1, \ldots, w_n \in \mathbb{C}$*, there exists a scalar-valued function* $f \in \mathcal{SA}_d$ *satisfying the interpolation conditions*

$$f(\mathbf{z}^k) = w_k \quad \textit{for } k = 1, \ldots, n \tag{2.37}$$

if and only if there exist d *positive-semidefinite* $n \times n$ *matrices* $P_1, \ldots, P_d$ *satisfying the following* Agler's condition *on the polydisk* $\mathbb{D}^d$*:*

$$1 - w_k\overline{w}_\ell = \sum_{j=1}^d \left(1 - z_j^k \overline{z_j^\ell}\right) P_j^{k,\ell} \quad \textit{for } k, \ell = 1, \ldots, n. \tag{2.38}$$

Particularly, the interpolation problem on the Bidisk $\mathbb{D}^2$ is: *given a* n*-point* ID *set*

$$\mathcal{D}_n^2 = \{(\mathbf{z}^k, w_k) \in (\mathbb{D}^2 \times \mathbb{C}) : k = 1, \ldots, n\},$$

find a function $f \in \mathcal{S}_2 (= \mathcal{SA}_2)$ *such that* $f(\mathbf{z}^k) = f(\alpha^k, \beta^k) = w_k$ *for* $k = 1, \ldots, n$.

Application of Agler's Theorem 2.20 to this particular case can be restated as follows:

[7] Communication across Europe was disrupted due to the World War I.

Theorem 2.21. *The n-point* NPIP *on the Bidisk has a solution $f \in \mathcal{S}_2$ if and only if there exist two positive-semidefinite $n \times n$ matrices, P_1 and P_2 so that the Agler's condition*

$$1 - w_k \overline{w}_\ell = \left(1 - \alpha^k \overline{\alpha^\ell}\right) P_1^{k,\ell} + \left(1 - \beta^k \overline{\beta^\ell}\right) P_2^{k,\ell}$$

holds.

Chapter 3

Multidimensional Linear Models

Multidimensional linear system theory has drawn considerable attention in the control literature recently. The two-dimensional linear models were first introduced in the 1970's by a group of researchers, such as Attasi [Att73, Att75], Givone-Roesser [GR72, GR73, Roe75], and Fornasini-Marchesini [FM76, FM77, FM78], and have been generalized to the multidimensional (d-D) models later on. There are only two models discussed in the first part of dissertation, namely Givone-Roesser (GR) and Fornasini-Marchesini (FM) models. Both models are not independent of each other; in fact, the GR model can be embedded into the FM model and vice versa. However, embedding the FM model into the GR model in general cannot be accomplished without increasing the dimension of the state-space. We shall show that under some proper assumptions, both models are equivalent.

3.1 Introduction

The mathematical system theory has been well developed and applied in the control applications for decades after the work of Kalman [Kal63] in 1963. The transfer function description and the state-space representation are powerful tools to analyze and design a linear controller. Recently, the field of multidimensional (d-D) digital signal processing has been growing rapidly, and this motivates mathematicians and system engineers to study the area of d-D systems broadly and intensively. In fact, the theory of d-D linear systems has been a subject for research for over two decades after D. Givone and R. Roesser [GR72] introduced the system equations for linear iterative circuits in 1972. They formulated the state-space representation in such a way that the state variable at each point on the grid (2D state-space) is split into two state components, say x_h, and x_v, which propagate the information in two independent directions, namely horizontal and vertical directions; each state component is a function of two independent time-shift variables. This is an elegant idea and can be easily extended to the general d-D linear system where the state vector is split into d state components, each of which is a function of d independent time-shift variables. In the following, we shall refer to this state-space representation as a *Givone-Roesser (*GR*) model*.

A few years later, E. Fornasini and G. Marchesini [FM78] proposed an alternative model based on the Nerode equivalence classes. They first studied a function of two complex variables and then showed that if such a function is rational, it admits a state-space realization. We shall call this representation as a *Fornasini-Marchesini (*FM*) model*. They dictated that such a realization is the most general one since the other can be embedded into this model. However, Ball-Sadosky-Vinnikov [BSV] showed that for conservative systems, both models are equivalent.

Even though the 2D linear model proposed by D. Givone and R. Roesser was originated from the network point of view, it is applicable to many physical systems, data analysis procedures, learning algorithms, and 2D digital filters as well. The typical problems that might require 2D linear models are medical X-ray image enhancement, the analysis of satellite weather photos, the enhancement and analysis of aerial photographs for agriculture, and sonar array processing (see [Gał96, Rob96, RU91, VA91] for more examples).

From the physical viewpoint, a d-D signal is a continuous function of d independent variables. For instance, the light intensity in the case of a photograph or image is considered as a function of distances in the x and y directions. Thus, the mathematical expression for this type of signal deals with a partial differential equation. A sampled version of a continuous d-D signal is a discrete d-D signal which is normally in the form of a d-D array of numbers; thus, it is described by a difference equation which is much easier for analysis than its partial differential equation counterpart.

Based on which type of signal we are considering, the d-D system can be classified into two categories: continuous and discrete d-D systems. For the discrete d-D systems, they can also be characterized in terms of state-space representation in d independent variables, and in terms of transfer function description. Under assumption that the input and output spaces are finite-dimensional, the transfer function for the discrete d-D system is a rational function of d complex variables, say $\mathbf{z} = (z_1, \ldots, z_d)$; whilst the transfer function of the classical discrete-time linear system is a rational function of one complex variable, say z, which corresponds to the time-shift operator in the time domain.

Since we are focusing on the d-D linear systems, the "time axis" in this case is multidimensional, say d copies of the integers $\mathbb{Z}^d, d > 1$, and we shall refer to it as an *integer lattice* rather than "time" as in the classical discrete-time system. Thus, the time variable $\mathbf{n}$ in d-D system is a d-tuple of integers, say $\mathbf{n} = (n_1, n_2, \ldots, n_d)$. The goal of the present chapter is two-fold: (1) to discuss chronologically the development of the state-space representations for the 2D linear systems including the generalized version of these models, and (2) to explain the correspondences between the GR and the FM formalisms.

By means of introduction, let us consider some examples that may be solved via the d-D system theory. The first example is the thermal process in chemical reactors, heat exchangers and pipe furnaces which is described by the partial differential equation (PDE) [Kac85]:

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - T(x,t) + U(t) \tag{3.1}$$

with initial and boundary conditions

$$T(x,0) = f_1(x), \quad T(0,t) = f_2(t)$$

where $T(x,t)$ is usually the temperature at x(space) $\in [0, x_f]$ and t(time) $\in \mathbb{Z}_+$, and $U(t)$ is a given forcing function.

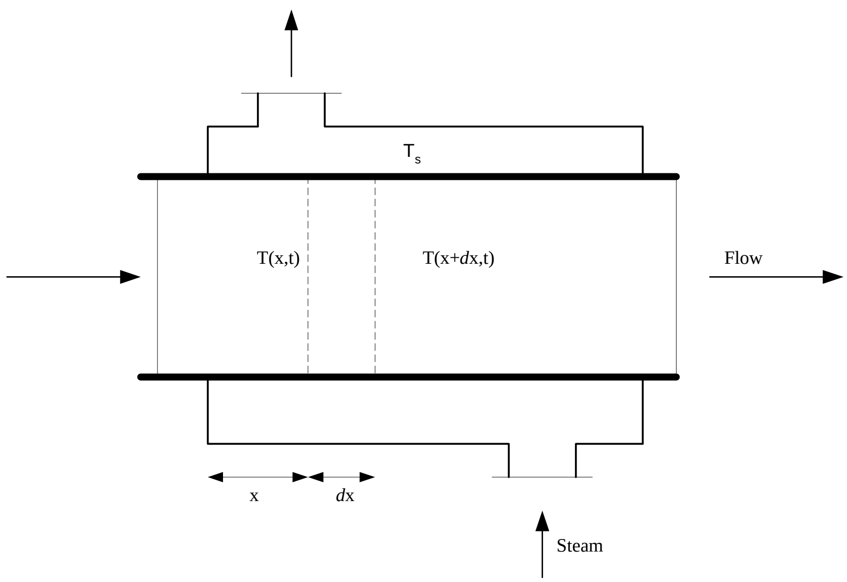

Figure 3.1: Heat exchanger

One way to solve this problem is by the discretization method. Taking $T(i,j) = T(i\triangle x, j\triangle t)$ and $U(j) = U(j\triangle t)$ and approximating the partial derivative as

$$\frac{\partial T(x,t)}{\partial t} \cong \frac{T(i,j+1) - T(i,j)}{\triangle t}, \quad \frac{\partial T(x,t)}{\partial x} \cong \frac{T(i,j) - T(i-1,j)}{\triangle x},$$

we approximate the PDE system (3.1) as

$$T(i,j+1) = (\tfrac{\triangle t}{\triangle x})T(i-1,j) + (1 - \tfrac{\triangle t}{\triangle x} - \triangle t)T(i,j) + (\triangle t)U(j).$$

Define $x_h(i,j) := T(i-1,j), x_v(i,j) := T(i,j)$, and let $a_1 = 1 - \frac{\triangle t}{\triangle x} - \triangle t, a_2 = \frac{\triangle t}{\triangle x}$, and $b = \triangle t$, the PDE system (3.1) can be converted into a GR model

$$\begin{bmatrix} x_h(i+1,j) \\ x_v(i,j+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} x_h(i,j) \\ x_v(i,j) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} U(j). \qquad \diamondsuit$$

The next example is the dynamical process in gas absorption, water stream heating and air drying which can be described by the Darboux equation [Kac85, DX02]:

$$\frac{\partial^2 s(x,t)}{\partial x \partial t} = a_1 \frac{\partial s(x,t)}{\partial t} + a_2 \frac{\partial s(x,t)}{\partial x} + a_0 s(x,t) + b f(x,t) \tag{3.2}$$

with the initial and boundary conditions

$$s(x,0) = S_1(x), \quad s(0,t) = S_2(t)$$

where $s(x,t)$ is an unknown function at $x(\text{space}) \in [0, x_f]$ and $t(\text{time}) \in \mathbb{Z}_+$, a_0, a_1, a_2 and b are real coefficients, $f(x,t)$ is the given input function and $S_1(x), S_2(t)$ are given.

Define $r(x,t) = \frac{\partial s(x,t)}{\partial t} - a_2 s(x,t)$ and hence the PDE system (3.2) can be transformed into an equivalent system of first order differential equations of the form

$$\begin{bmatrix} \frac{\partial r(x,t)}{\partial x} \\ \frac{\partial s(x,t)}{\partial t} \end{bmatrix} = \begin{bmatrix} a_1 & a_0 + a_1 a_2 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} r(x,t) \\ s(x,t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} f(x,t) \tag{3.3}$$

with initial condition

$$r(0,t) = \left. \frac{\partial s(x,t)}{\partial t} \right|_{x=0} - a_2 s(0,t) = \frac{d}{dt} S_2(t) - a_2 S_2(t) \triangleq R(t).$$

Taking $r(i,j) = r(i\triangle x, j\triangle t) \triangleq x_h(i,j)$, $s(i,j) = s(i\triangle x, j\triangle t) \triangleq x_v(i,j)$, and approximating the partial derivative as

$$\frac{\partial r(x,t)}{\partial x} \cong \frac{r(i+1,j) - r(i,j)}{\triangle x}, \qquad \frac{\partial s(x,t)}{\partial t} \cong \frac{s(i,j+1) - s(i,j)}{\triangle t},$$

we obtain the GR model from (3.3) as

$$\begin{bmatrix} x_h(i+1,j) \\ x_v(i,j+1) \end{bmatrix} = \begin{bmatrix} 1 + a_1 \triangle x & (a_0 + a_1 a_2)\triangle x \\ \triangle t & 1 + a_2 \triangle t \end{bmatrix} \begin{bmatrix} x_h(i,j) \\ x_v(i,j) \end{bmatrix} + \begin{bmatrix} b\triangle x \\ 0 \end{bmatrix} f(i,j) \tag{3.4}$$

with boundary conditions

$$x_h(0,j) = R(j\triangle t), \quad x_v(i,0) = S_1(i\triangle x). \qquad \diamondsuit$$

This Chapter is organized as follows: The classical discrete-time linear system is briefly reviewed in Section 3.2. The Givone-Roesser (GR) model is provided in Section 3.3 followed by the Fornasini-Marchesini (FM) model in Section 3.4. In Section 3.5, we present the connection between the FM and the GR models and we devote Section 3.6 for the conclusion.

3.2 Preliminaries

In this Section, we review the theory of the classical discrete-time linear 1D system in systematic way using the operator theoretical viewpoint. Let $\mathcal{H}, \mathcal{U}$, and $\mathcal{Y}$ be Hilbert spaces and an operator U from $\mathcal{H} \oplus \mathcal{U}$ to $\mathcal{H} \oplus \mathcal{Y}$ be a block-matrix given by

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}. \tag{3.5}$$

where $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}), B \in \mathcal{L}(\mathcal{U}, \mathcal{H}), C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$.

A quadruple $\Sigma = (\mathcal{H}, \mathcal{U}, \mathcal{Y}, U)$ where U is given in the form (3.5) is often called an *operator colligation* in the Russian literature (see e.g. [Bro71, Liv73]). Associated with this form of colligation is the classical discrete-time linear system described by

$$\Sigma \triangleq \begin{cases} x(n+1) & = Ax(n) + Bu(n) \\ \quad y(n) & = Cx(n) + Du(n), \end{cases} \tag{3.6}$$

or in the matrix form,

$$\begin{bmatrix} x(n+1) \\ y(n) \end{bmatrix} = U \begin{bmatrix} x(n) \\ u(n) \end{bmatrix}. \tag{3.7}$$

Here $x(n), u(n)$, and $y(n)$ take values in the *state space* $\mathcal{H}$, the *input space* $\mathcal{U}$, and the *output space* $\mathcal{Y}$, respectively. Furthermore, the operators A, B, C, and D are called respectively the *state operator*, the *input operator*, the *output operator*, and the *feedforward operator*. The colligation Σ is said to be *unitary, isometric, coisometric,* or *contractive* if the connecting operator U is respectively *unitary, isometric, coisometric,* or *contractive.*

It is of interest to find an adjoint system Σ^* with input space $\mathcal{U}^*$, state space $\mathcal{H}$, and output space $\mathcal{Y}^*$ so that its trajectories (u_*, x_*, y_*) of Σ^* are characterized as those $(\mathcal{U}^* \times \mathcal{H} \times \mathcal{Y}^*)$-valued functions on $\mathbb{Z}$ satisfying the adjoint pairing relation:

$$\langle x(n+1), x_*(n+1)\rangle + \langle y(n), u_*(n)\rangle = \langle x(n), x_*(n)\rangle + \langle u(n), y_*(n)\rangle \tag{3.8}$$

which is equivalent to

$$\left\langle \begin{bmatrix} x(n+1) \\ y(n) \end{bmatrix}, \begin{bmatrix} x_*(n+1) \\ u_*(n) \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x(n) \\ u(n) \end{bmatrix}, \begin{bmatrix} x_*(n) \\ y_*(n) \end{bmatrix} \right\rangle. \tag{3.9}$$

By substituting (3.7) into (3.9), we have

$$\left\langle \begin{bmatrix} x(n) \\ u(n) \end{bmatrix}, U^* \begin{bmatrix} x_*(n+1) \\ u_*(n) \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x(n) \\ u(n) \end{bmatrix}, \begin{bmatrix} x_*(n) \\ y_*(n) \end{bmatrix} \right\rangle \tag{3.10}$$

in which we deduce that trajectories (u_*, x_*, y_*) for the adjoint system Σ^* are characterized by

$$\begin{bmatrix} x_*(n) \\ y_*(n) \end{bmatrix} = U^* \begin{bmatrix} x_*(n+1) \\ u_*(n) \end{bmatrix}. \tag{3.11}$$

Thus, the adjoint system Σ^* is given by

$$\Sigma^* \triangleq \begin{cases} x_*(n) &= A^* x_*(n+1) + C^* u_*(n) \\ y_*(n) &= B^* x_*(n+1) + D^* u_*(n). \end{cases} \tag{3.12}$$

If we demand that a $(\mathcal{U}, \mathcal{H}, \mathcal{Y})$-valued function (u, x, y) is a trajectory of the original system if and only if (y, x, u) is a trajectory of the adjoint system (i.e., let $(u_*, x_*, y_*) = (y, x, u)$), then from (3.7), we have

$$\begin{bmatrix} x_*(n+1) \\ u_*(n) \end{bmatrix} = U \begin{bmatrix} x_*(n) \\ y_*(n) \end{bmatrix} = UU^* \begin{bmatrix} x_*(n+1) \\ u_*(n) \end{bmatrix}, \tag{3.13}$$

and also from (3.11),

$$\begin{bmatrix} x(n) \\ u(n) \end{bmatrix} = U^* \begin{bmatrix} x(n+1) \\ y(n) \end{bmatrix} = U^* U \begin{bmatrix} x(n) \\ u(n) \end{bmatrix}. \tag{3.14}$$

This implies that U is a unitary operator and the system Σ is said to be *conservative* since the adjoint pairing (3.8) collapses to the energy balance relation:

$$\|x(n+1)\|^2 + \|y(n)\|^2 = \|x(n)\|^2 + \|u(n)\|^2 \qquad \text{for } \Sigma \tag{3.15}$$

$$\|x_*(n+1)\|^2 + \|u_*(n)\|^2 = \|x_*(n)\|^2 + \|y_*(n)\|^2 \qquad \text{for } \Sigma^*. \tag{3.16}$$

Remark 5. If we assume that U is unitary, then from (3.12) we have equivalent backward-time system equations:

$$\Sigma_b \triangleq \begin{cases} x(n) &= A^* x(n+1) + C^* y(n) \\ u(n) &= B^* x(n+1) + D^* y(n) \end{cases} \tag{3.17}$$

and we shall call the original system equations Σ as the forward-time system. ▲

If we impose an initial condition at time $n_0 \in \mathbb{Z}$ and are given an input sequence $\{u(n)\}$ for $n \geq n_0$, and an output sequence $\{y(n)\}$ (which is an input sequence for the backward-time system) for $n < n_0$, then we are able to solve the system equations (3.6) and (3.17) recursively, i.e.

$$x(n) = A^{n-n_0}x(n_0) + \sum_{k=n_0}^{n-1} A^{n-1-k}Bu(k) \quad \text{for } n \geq n_0 \tag{3.18}$$

$$y(n) = CA^{n-n_0}x(n_0) + \sum_{k=n_0}^{n-1} CA^{n-1-k}Bu(k) + Du(n) \quad \text{for } n \geq n_0 \tag{3.19}$$

$$x(n) = (A^*)^{n_0-n}x(n_0) + \sum_{k=n}^{n_0-1} (A^*)^{k-n}C^*y(k) \quad \text{for } n < n_0 \tag{3.20}$$

$$u(n) = B^*(A^*)^{n_0-n}x(n_0) + \sum_{k=n}^{n_0-1} B^*(A^*)^{k-n}C^*y(k) + D^*y(n) \quad \text{for } n < n_0 \tag{3.21}$$

These equations are in the form of the convolution sum which is hard to compute by hand. It is well known that by virtue of the Z-transform (the Fourier transformation for this discrete setting), one is able to convert the convolution operator to a multiplication operator which is in the form of an algebraic equation.

For an arbitrary subset Ω of $\mathbb{Z}$, the Z-transform is defined as follows:

$$\{h(n)\}_{n\in\Omega} \mapsto h^{\wedge\Omega}(z) \triangleq \sum_{n\in\Omega} h(n)z^n \tag{3.22}$$

Now by applying the Z-transform to the forward-time system equations (3.6) and the backward-time system equations (3.17) where $\Omega = [n_0, \infty)$ and $(-\infty, n_0)$, respectively, we have:

$$x^{\wedge[n_0,\infty)}(z) = (I - zA)^{-1}z^{n_0}x(n_0) + z(I - zA)^{-1}Bu^{\wedge[n_0,\infty)}(z) \tag{3.23}$$

$$y^{\wedge[n_0,\infty)}(z) = C(I - zA)^{-1}z^{n_0}x(n_0) + T_{\Sigma(U)}(z) \cdot u^{\wedge[n_0,\infty)}(z) \tag{3.24}$$

where

$$T_{\Sigma(U)}(z) = zC(I - zA)^{-1}B + D = \sum_{n=1}^{\infty} \left(CA^{n-1}B\right) z^n + D \tag{3.25}$$

is called the *transfer function of the forward-time linear system* (3.6), and

$$x^{\wedge(-\infty,n_0)}(z) = z^{n_0-1}(I - z^{-1}A^*)^{-1}A^*x(n_0) + (I - z^{-1}A^*)^{-1}C^*y^{\wedge(-\infty,n_0)}(z) \tag{3.26}$$

$$u^{\wedge(-\infty,n_0)}(z) = z^{n_0-1}B^*(I - z^{-1}A^*)^{-1}x(n_0) + T_{\Sigma(U^*)}(z) \cdot y^{\wedge(-\infty,n_0)}(z) \tag{3.27}$$

where

$$T_{\Sigma_b(U^*)}(z) = z^{-1}B^*(I - z^{-1}A^*)^{-1}C^* + D^* = \sum_{n=1}^{\infty} \left(B^*(A^*)^{n-1}C^*\right) z^{-n} + D^* \tag{3.28}$$

is called the *transfer function of the backward-time linear system* (3.17).

Let us now consider the energy balance relation of the forward-time system for the moment. We may rearrange (3.15) in the form

$$\|x(n+1)\|^2 - \|x(n)\|^2 = \|u(n)\|^2 - \|y(n)\|^2$$

and then iterate from $n = M$ to $n = N$ to get

$$\|x(N+1)\|^2 - \|x(M)\|^2 = \sum_{n=M}^{N} \left[\|u(n)\|^2 - \|y(n)\|^2\right].$$

If we set $M = n_0$ and assume that $x(n_0) = 0$, then we get

$$0 \leq \|x(N+1)\|^2 = \sum_{n=n_0}^{N} \left[\|u(n)\|^2 - \|y(n)\|^2\right].$$

Hence, if the input sequence $\{u(n)\}_{n \geq n_0} \in \ell^2([n_0, \infty), \mathcal{U})$, then we may take $N \to \infty$ to get

$$0 \leq \sum_{n=n_0}^{\infty} \left[\|u(n)\|^2 - \|y(n)\|^2\right],$$

and conclude that

$$\|y\|^2_{\ell^2([n_0,\infty),\mathcal{Y})} \leq \|u\|^2_{\ell^2([n_0,\infty),\mathcal{U})} \tag{3.29}$$

Upon applying the Z-transform to (3.29) and recalling that $y^{\wedge[n_0,\infty)}(z) = T_{\Sigma(U)}(z) \cdot u^{\wedge[n_0,\infty)}(z)$ if we assume that $x(n_0) = 0$, we have

$$\|T_{\Sigma(U)} \cdot u^{\wedge[n_0,\infty)}\|_{z^{n_0}H^2(\mathbb{D},\mathcal{Y})} \leq \|u^{\wedge[n_0,\infty)}\|_{z^{n_0}H^2(\mathbb{D},\mathcal{U})}$$

As $u^{\wedge[n_0,\infty)}$ is an arbitrary element of $z^{n_0}H^2(\mathbb{D},\mathcal{U})$ for $u \in \ell^2([n_0,\infty),\mathcal{U})$, we conclude that multiplication by $T_{\Sigma(U)}$ defines a contraction operator from $z^{n_0}H^2(\mathbb{D},\mathcal{U})$ into $z^{n_0}H^2(\mathbb{D},\mathcal{Y})$. Since $\bigcup_{n_0=-\infty}^{0} z^{n_0}H^2(\mathbb{D},\mathcal{U})$ is dense in $L^2(\mathbb{D},\mathcal{U})$, one can see that multiplication by $T_{\Sigma(U)}$ extends uniquely by continuity to define a contraction operator from $L^2(\mathbb{D},\mathcal{U})$ into $L^2(\mathbb{D},\mathcal{Y})$. In other words, the transfer function of a forward-time *conservative* linear system $T_{\Sigma(U)}$ is in the *Schur class* $\mathcal{S}(\mathcal{U},\mathcal{Y})$. This analysis is applicable similarly to the backward-time system as well.

For a discussion, see e.g., [BSV]. All results discussed here can be generalized to the d-D case as we shall see in the following sections.

3.3 Givone-Roesser Model

In 1972, D. Givone and R. Roesser introduced the state-space formalism for a linear iterative circuit, which is considered as a spatial system rather than a temporal system. An iterative circuit is the combination of individual cells, each of which is identical, in a regular pattern. This type of circuit is used widely in automata and logical circuit theory. An iterative circuit is said to be *linear* if the inputs and outputs to each cell are in the form of vectors in linear vector spaces over a common finite field and each cell performs a linear transformation, see [GR72]. It is worth noting that in the image processing case, a real field is applied instead of a finite one. In practical viewpoint, the iterative circuit may be used in encoding, decoding and image processing.

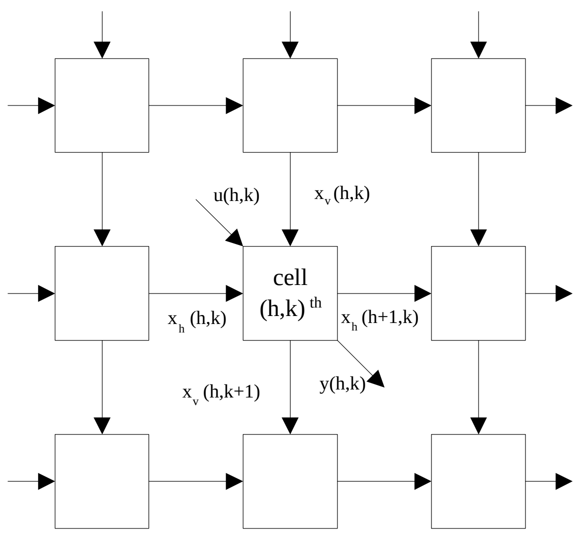

Figure 3.2: Two-dimensional unilateral iterative circuit

Based on the direction of flowing signals through each cell, the iterative circuit may be classified into two categories: unilateral and bilateral iterative circuits. The 2D unilateral circuit is depicted in Figure 3.2. In this figure, the state variable at the location (h, k) is split into horizontal and vertical components, say x_h and x_v, respectively. Each component in the next time step is a function of both state components and input at the present time, i.e.

$$\begin{bmatrix} x_h(h+1,k) \\ x_v(h,k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_h(h,k) \\ x_v(h,k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(h,k) \tag{3.30}$$

together with the initial conditions $x_h(0,0), x_v(0,0)$ and the boundary conditions

$$X(0) \triangleq \{x_h(0,j), x_v(i,0)\} \text{ for all } i,j \geq 1\ , \tag{3.31}$$

and the output equation is described by

$$y(h,k) = C_1 x_h(h,k) + C_2 x_v(h,k) + Du(h,k). \tag{3.32}$$

The generalized version of the system described above is the so-called *d-D Givone-Roesser system*, or GR system for short, which is given by:

$$\Sigma^{GR} \triangleq \begin{cases} x_1(\mathbf{n}+\mathbf{e}_1) & = \sum_{k=1}^{d} A_{1,k}^{GR} x_k(\mathbf{n}) + B_1^{GR} u(\mathbf{n}) \\ & \vdots \\ x_d(\mathbf{n}+\mathbf{e}_d) & = \sum_{k=1}^{d} A_{d,k}^{GR} x_k(\mathbf{n}) + B_d^{GR} u(\mathbf{n}) \\ y(\mathbf{n}) & = \sum_{k=1}^{d} C_k^{GR} x_k(\mathbf{n}) + D^{GR} u(\mathbf{n}) \end{cases} \tag{3.33}$$

where $\mathbf{e}_k$ denotes the standard basis in $\mathbb{C}^d$, i.e. $\mathbf{e}_1 \triangleq (1,0,\ldots,0), \ldots, \mathbf{e}_d \triangleq (0,\ldots,0,1)$ and $\mathbf{n} := (n_1,\ldots,n_d)$.

The system Σ^{GR} can be rewritten in the matrix form as:

$$\begin{bmatrix} x_1(\mathbf{n}+\mathbf{e}_1) \\ \vdots \\ x_d(\mathbf{n}+\mathbf{e}_d) \\ y(\mathbf{n}) \end{bmatrix} = U^{GR} \begin{bmatrix} x_1(\mathbf{n}) \\ \vdots \\ x_d(\mathbf{n}) \\ u(\mathbf{n}) \end{bmatrix} \tag{3.34}$$

where the connecting operator U^{GR} associated with this system is of the form

$$U^{GR} \triangleq \begin{bmatrix} A^{GR} & B^{GR} \\ C^{GR} & D^{GR} \end{bmatrix} = \begin{bmatrix} A_{1,1}^{GR} & \cdots & A_{1,d}^{GR} & B_1^{GR} \\ \vdots & \ddots & \vdots & \vdots \\ A_{d,1}^{GR} & \cdots & A_{d,d}^{GR} & B_d^{GR} \\ C_1^{GR} & \cdots & C_d^{GR} & D^{GR} \end{bmatrix} : \begin{bmatrix} \bigoplus_{i=1}^{d} \mathcal{H}_i \\ \mathcal{U} \end{bmatrix} \mapsto \begin{bmatrix} \bigoplus_{i=1}^{d} \mathcal{H}_i \\ \mathcal{Y} \end{bmatrix}. \tag{3.35}$$

Here the state space $\mathcal{H}$ is decomposed into a fixed d-fold orthogonal direct-sum

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d.$$

To get an analogue of the 1D discrete-time linear system, let us first seek an adjoint system Σ^{*GR} so that its trajectories (u_*, x_*, y_*) of Σ^{*GR} are characterized as those $(\mathcal{U}^* \times \mathcal{H} \times \mathcal{Y}^*)$-valued functions on $\mathbb{Z}^d$ satisfying the adjoint pairing relation:

$$\begin{aligned}\langle x_1(\mathbf{n}+\mathbf{e}_1), x_{*1}(\mathbf{n}+\mathbf{e}_1)\rangle + \cdots + \langle x_d(\mathbf{n}+\mathbf{e}_d), x_{*d}(\mathbf{n}+\mathbf{e}_d)\rangle + \langle y(\mathbf{n}), u_*(\mathbf{n})\rangle \\ = \langle x_1(\mathbf{n}), x_{*1}(\mathbf{n})\rangle + \cdots + \langle x_d(\mathbf{n}), x_{*d}(\mathbf{n})\rangle + \langle u(\mathbf{n}), y_*(\mathbf{n})\rangle \end{aligned} \quad (3.36)$$

which is equivalent to

$$\left\langle \begin{bmatrix} x_1(\mathbf{n}+\mathbf{e}_1) \\ \vdots \\ x_d(\mathbf{n}+\mathbf{e}_d) \\ y(\mathbf{n}) \end{bmatrix}, \begin{bmatrix} x_{*1}(\mathbf{n}+\mathbf{e}_1) \\ \vdots \\ x_{*d}(\mathbf{n}+\mathbf{e}_d) \\ u_*(\mathbf{n}) \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x_1(\mathbf{n}) \\ \vdots \\ x_d(\mathbf{n}) \\ u(\mathbf{n}) \end{bmatrix}, \begin{bmatrix} x_{*1}(\mathbf{n}) \\ \vdots \\ x_{*d}(\mathbf{n}) \\ y_*(\mathbf{n}) \end{bmatrix} \right\rangle. \quad (3.37)$$

By substituting (3.34) into (3.37), we have

$$\left\langle \begin{bmatrix} x_1(\mathbf{n}) \\ \vdots \\ x_d(\mathbf{n}) \\ u(\mathbf{n}) \end{bmatrix}, U^{*GR} \begin{bmatrix} x_{*1}(\mathbf{n}+\mathbf{e}_1) \\ \vdots \\ x_{*d}(\mathbf{n}+\mathbf{e}_d) \\ u_*(\mathbf{n}) \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x_1(\mathbf{n}) \\ \vdots \\ x_d(\mathbf{n}) \\ u(\mathbf{n}) \end{bmatrix}, \begin{bmatrix} x_{*1}(\mathbf{n}) \\ \vdots \\ x_{*d}(\mathbf{n}) \\ y_*(\mathbf{n}) \end{bmatrix} \right\rangle \quad (3.38)$$

in which we deduce that trajectories (u_*, x_*, y_*) for the adjoint system Σ^{*GR} are characterized by

$$\begin{bmatrix} x_{*1}(\mathbf{n}) \\ \vdots \\ x_{*d}(\mathbf{n}) \\ y_*(\mathbf{n}) \end{bmatrix} = U^{*GR} \begin{bmatrix} x_{*1}(\mathbf{n}+\mathbf{e}_1) \\ \vdots \\ x_{*d}(\mathbf{n}+\mathbf{e}_d) \\ u_*(\mathbf{n}) \end{bmatrix}. \quad (3.39)$$

Thus, the adjoint system Σ^{*GR} is described by

$$\Sigma^{*GR} \triangleq \begin{cases} x_{*1}(\mathbf{n}) &= \sum_{j=1}^{d} (A_{j,1}^{GR})^* x_{*j}(\mathbf{n}+\mathbf{e}_j) + (C_1^{GR})^* u_*(\mathbf{n}) \\ & \vdots \\ x_{*d}(\mathbf{n}) &= \sum_{j=1}^{d} (A_{j,d}^{GR})^* x_{*j}(\mathbf{n}+\mathbf{e}_j) + (C_d^{GR})^* u_*(\mathbf{n}) \\ y_*(\mathbf{n}) &= \sum_{j=1}^{d} (B_j^{GR})^* x_{*j}(\mathbf{n}+\mathbf{e}_j) + (D^{GR})^* u_*(\mathbf{n}). \end{cases} \quad (3.40)$$

The GR system Σ^{GR} is said to be *conservative* provided that a $(\mathcal{U}, \mathcal{H}, \mathcal{Y})$-valued function (u, x, y) is a trajectory of the GR system if and only if (y, x, u) is a trajectory of the adjoint system Σ^{*GR}. Now we shall verify that a necessary and sufficient condition so that the GR system Σ^{GR} is conservative is that the connecting operator U^{GR} is unitary. To see this, let us consider the system equations (3.34) and substitute (y, x, u) by (u_*, x_*, y_*). Thus, we have

$$\begin{bmatrix} x_{*1}(\mathbf{n}+\mathbf{e}_1) \\ \vdots \\ x_{*d}(\mathbf{n}+\mathbf{e}_d) \\ u_*(\mathbf{n}) \end{bmatrix} = U^{GR} \begin{bmatrix} x_{*1}(\mathbf{n}) \\ \vdots \\ x_{*d}(\mathbf{n}) \\ y_*(\mathbf{n}) \end{bmatrix} = U^{GR}U^{*GR} \begin{bmatrix} x_{*1}(\mathbf{n}+\mathbf{e}_1) \\ \vdots \\ x_{*d}(\mathbf{n}+\mathbf{e}_d) \\ u_*(\mathbf{n}) \end{bmatrix}. \tag{3.41}$$

Also, from (3.39),

$$\begin{bmatrix} x_1(\mathbf{n}) \\ \vdots \\ x_d(\mathbf{n}) \\ u(\mathbf{n}) \end{bmatrix} = U^{*GR} \begin{bmatrix} x_1(\mathbf{n}+\mathbf{e}_1) \\ \vdots \\ x_d(\mathbf{n}+\mathbf{e}_d) \\ y(\mathbf{n}) \end{bmatrix} = U^{*GR}U^{GR} \begin{bmatrix} x_1(\mathbf{n}) \\ \vdots \\ x_d(\mathbf{n}) \\ u(\mathbf{n}) \end{bmatrix}. \tag{3.42}$$

Clearly, the connecting operator U^{GR} is unitary since $U^{GR}U^{*GR} = U^{*GR}U^{GR} = I$. If the system is conservative, the adjoint pairing (3.36) collapses to the energy balance relation:

$$\sum_{k=1}^{d} \|x_k(\mathbf{n}+\mathbf{e}_k)\|^2 + \|y(\mathbf{n})\|^2 = \sum_{k=1}^{d} \|x_k(\mathbf{n})\|^2 + \|u(\mathbf{n})\|^2 \quad \text{for } \Sigma^{GR} \tag{3.43}$$

$$\sum_{k=1}^{d} \|x_{*k}(\mathbf{n}+\mathbf{e}_k)\|^2 + \|u_*(\mathbf{n})\|^2 = \sum_{k=1}^{d} \|x_{*k}(\mathbf{n})\|^2 + \|y_*(\mathbf{n})\|^2 \quad \text{for } \Sigma^{*GR}. \tag{3.44}$$

For the frequency domain analysis, let $\Omega = \{\mathbf{n} \in \mathbb{Z}^d : |\mathbf{n}| \triangleq \sum_{k=1}^{d} n_k \geq 0\}$, and define the Z-transform of the sequence $\{h(\mathbf{n})\}_{\mathbf{n}\in\Omega}$ as follows:

$$\{h(\mathbf{n})\}_{\mathbf{n}\in\Omega} \mapsto h^{\wedge\Omega}(\mathbf{z}) = \sum_{\mathbf{n}\in\Omega} h(\mathbf{n})\mathbf{z}^{\mathbf{n}}$$

where $\mathbf{z}^{\mathbf{n}} = z_1^{n_1} \cdots z_d^{n_d}$ if $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\mathbf{z} = (z_1, \ldots, z_d)$ is a d-tuple of commuting complex variables. Application of this Z-transform to the system equations (3.33), we have

$$\{x_k(\mathbf{n}+\mathbf{e}_k)\} \mapsto x_k^{\wedge\Omega}(\mathbf{z}) = \sum_{\mathbf{n}\in\Omega} x_k(\mathbf{n}+\mathbf{e}_k)\mathbf{z}^{\mathbf{n}} = z_k^{-1} \left[x_k^{\wedge\Omega}(\mathbf{z}) - \sum_{\mathbf{n}:|\mathbf{n}|=0} x_k(\mathbf{n})\mathbf{z}^{\mathbf{n}} \right]. \tag{3.45}$$

By assuming that the system has zero initial condition $x_k(\mathbf{n}) = 0$ for $\mathbf{n} \in \Omega_0 \triangleq \{\mathbf{n} \in \mathbb{Z}^d : |\mathbf{n}| = 0\}$ and for $k = 1, \ldots, d$, (3.45) collapses to

$$\{x_k(\mathbf{n}+\mathbf{e}_k)\} \mapsto x_k^{\wedge\Omega}(\mathbf{z}) = z_k^{-1} \left[x_k^{\wedge\Omega}(\mathbf{z}) \right]$$

and the state equations in (3.33) becomes

$$\begin{bmatrix} z_1^{-1}x_1^{\wedge\Omega}(\mathbf{z}) \\ \vdots \\ z_d^{-1}x_d^{\wedge\Omega}(\mathbf{z}) \end{bmatrix} = A^{GR} \begin{bmatrix} x_1^{\wedge\Omega}(\mathbf{z}) \\ \vdots \\ x_d^{\wedge\Omega}(\mathbf{z}) \end{bmatrix} + B^{GR}u^{\wedge\Omega}(\mathbf{z}). \tag{3.46}$$

More compactly, let us define $Z_d(\mathbf{z}) = \begin{bmatrix} z_1 I_{\mathcal{H}_1} & & \\ & \ddots & \\ & & z_d I_{\mathcal{H}_d} \end{bmatrix}$, and hence, (3.46) becomes

$$x^{\wedge\Omega}(\mathbf{z}) \triangleq \begin{bmatrix} x_1^{\wedge\Omega}(\mathbf{z}) \\ \vdots \\ x_d^{\wedge\Omega}(\mathbf{z}) \end{bmatrix} = (I - Z_d(\mathbf{z})A^{GR})^{-1}Z_d(\mathbf{z})B^{GR}u^{\wedge\Omega}(\mathbf{z}).$$

Then, by applying the Z-transform to the output equation of (3.33) and substituting $x^{\wedge\Omega}(\mathbf{z})$ from the above expression, we get

$$\begin{aligned} y^{\wedge\Omega}(\mathbf{z}) &= \left[C^{GR}(I - Z_d(\mathbf{z})A^{GR})^{-1}Z_d(\mathbf{z})B^{GR} + D^{GR}\right] \cdot u^{\wedge\Omega}(\mathbf{z}) \\ &\triangleq T_{\Sigma^{GR}}(\mathbf{z}) \cdot u^{\wedge\Omega}(\mathbf{z}) \end{aligned} \tag{3.47}$$

where $T_{\Sigma^{GR}}(\mathbf{z})$ is called the *Givone-Roesser (*GR*) transfer function.*

3.4 Fornasini-Marchesini Model

E. Fornasini and G. Marchesini [FM76, FM77] proposed an alternative formalism of 2D transfer functions, or digital filters, with the different state-space realization. While Givone-Roesser formulated the state-space representation based on the *iterative circuit*, Fornasini and Marchesini obtained the state-space realization from the factorization of the 2D input-output map based on the algebraic viewpoint of Nerode equivalence classes[1].

The state at a given time (h, k) in the sense of Nerode is the minimal amount of past information required to completely determine the mapping from future input to future output, where the notions of the past and the future are defined as follows: Let $T = \mathbb{Z} \times \mathbb{Z}$ denote the

[1]Let u and v be input-strings and $*$ denote the concatenation. Then the inputs u and v are said to be equivalent in the sense of Nerode, denoted by $u \sim v$, if u and v are concatenated with the same arbitrary input-string w and the resulting outputs are the same regardless the value of w. More precisely, let f be an input/output map $f : \mathcal{U} \mapsto \mathcal{Y}$. Then for all $u, v \in \mathcal{U}$,

$$u \sim v \text{ if and only if } f(u * w) = f(v * w) \quad \forall w \in \mathcal{U}.$$

grid (integer lattice) for 2D linear system equipped with the partial ordering of integer:

- $(i,j) \geq (h,k)$ if and only if $i \geq h$ and $j \geq k$,
- $(i,j) = (h,k)$ if and only if $i = h$ and $j = k$,
- $(i,j) > (h,k)$ if and only if $(i,j) \geq (h,k)$ and $(i,j) \neq (h,k)$.

Then for any $t \in T$, the time τ is said to be the *past* with respect to t if τ is *not* $\geq t$; the time τ is said to be the *future* with respect to t if $\tau > t$. Thus, for 2D systems, this Nerode state-space is usually infinite dimensional. This was the motivation for E. Fornasini and G. Marchesini to introduce the idea of "local state" which is distinguishable from the "global state" (or, Nerode state), and this observation is the major difference between 1D and 2D systems. The local state contains information that will only be used to compute any state of interest at each step of recursions, and hence one can achieve system realizations with finite dimensional local state; while the global state (or Nerode state) at each time (h,k) provides all past information. Generally, the global state is of infinite dimension (see [FM76, KLMK77]).

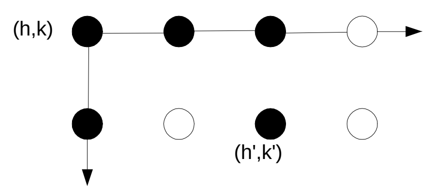

Figure 3.3: Two-dimensional time diagram

For any point $(h',k') \geq (h,k)$, a local state $x(h',k')$ depends not only on the state $x(h,k)$ but also on local states $x(h+1,k), \ldots, x(h',k)$ and $x(h,k+1), \ldots, x(h,k')$ as shown in Figure 3.3. Since (h',k') is an arbitrary point on the 2-dimensional state space, it is of interest to introduce the concept of a global state space as follows: *The global state space,* X^{Glo}*, is a 2D state space consisting of all local state spaces on the horizontal and vertical axes.*

In particular, if $h' = h+1$ and $k' = k+1$, then the local state $x(h',k')$ can be written by the updating equations:

$$x(h',k') = x(h+1,k+1) = A_0 x(h,k) + A_1 x(h,k+1) + A_2 x(h+1,k) + Bu(h,k) \tag{3.48}$$

$$y(h,k) = Cx(h,k) \tag{3.49}$$

together with the initial conditions $x(0,0)$ and the boundary conditions

$$X(0) \triangleq \{x(0,j), x(i,0) \mid 1 \leq i \leq M, 1 \leq j \leq N \text{ for some } M,N > 1\} \tag{3.50}$$

It is clear from (3.48) that the state $x(h+1,k+1)$ depends on the state $x(h,k)$ and hence it is not the first-order difference equation. Consequently, one can define a new state η as:

$$\eta(h,k) = x(h,k+1) - A_2 x(h,k)$$

Then,

$$\begin{aligned}\eta(h+1,k) &= x(h+1,k+1) - A_2 x(h+1,k) \\ &= A_0 x(h,k) + A_1 x(h,k+1) + Bu(h,k) \\ &= A_0 x(h,k) + A_1 \left[\eta(h,k) + A_2 x(h,k)\right] + Bu(h,k) \\ &= A_1 \eta(h,k) + \left[A_0 + A_1 A_2\right] x(h,k) + Bu(h,k) \qquad (3.51)\end{aligned}$$

Hence,

$$\begin{aligned}\begin{bmatrix}\eta(h+1,k) \\ x(h,k+1)\end{bmatrix} &= \begin{bmatrix}A_1 & A_0 + A_1 A_2 \\ I & A_2\end{bmatrix}\begin{bmatrix}\eta(h,k) \\ x(h,k)\end{bmatrix} + \begin{bmatrix}B \\ 0\end{bmatrix} u(h,k) \\ y(h,k) &= \begin{bmatrix}0 & C\end{bmatrix}\begin{bmatrix}\eta(h,k) \\ x(h,k)\end{bmatrix} \qquad (3.52)\end{aligned}$$

which is identical to the GR model, where $x_h = \eta$ and $x_v = x$.

Since the results in the papers of Givone-Roesser [GR72, GR73], Roesser [Roe75], and Fornasini-Marchesini [FM76, FM77] were not complete, Kung et al. [KLMK77] extended the results in the aforementioned references, provided a comparison between FM and GR models, and also constructed a hardware realization (circuit point of view). The authors in [KLMK77] believed that the GR model was the most satisfactory from the practical viewpoint and yielded the most general results since the model proposed by Fornasini-Marchesini as in (3.48) can be embedded in the GR model.

The authors in [KLMK77] also asserted that Fornasini and Marchesini failed to fully exploit the structure of the global state and its relation to the local state, so that the state-space model they introduced was unsatisfactory in the sense that what they introduced as the state is really only a *partial state*, which can be clearly observed from (3.52). More precisely, the state $x(h,k)$ is a partial state; while the full state is $\begin{bmatrix}\eta(h,k) \\ x(h,k)\end{bmatrix}$.

In 1978, one year after [KLMK77] was published, Fornasini and Marchesini proposed a new state-space representation which is the first-order difference equation and defined as follows:

$$x(h+1,k+1) = A_1 x(h,k+1) + A_2 x(h+1,k) + B_1 u(h,k+1) + B_2 u(h+1,k) \qquad (3.53)$$

$$y(h,k) = Cx(h,k). \qquad (3.54)$$

We shall call this representation a *Fornasini-Marchesini (FM) model*. According to this representation, the authors in [FM78] showed that the models defined by Attasi [Att73], Givone-Roesser [GR72, GR73], and Fornasini-Marchesini [FM76, FM77] can be embedded in the form of (3.53). They also noted that in the GR model the local state is the direct sum of the horizontal and vertical states, so that the embedding does not require any increasing of dimension. On the other hand, embedding the FM model (3.53) into the GR model (3.30) cannot be accomplished in general without increasing the dimension of the state space.

Example 4. Consider the scalar transfer function $\frac{2z_1+z_2}{1-z_1-2z_2}$. It is obvious that a state-space realization of the given function in the FM model is

$$x(h+1,k+1) = x(h,k+1) + 2x(h+1,k) + 2u(h,k+1) + u(h+1,k), \quad y(h,k) = x(h,k).$$

However, to obtain a realization in the GR formalism, the dimension of the state-space must be at least two for this case. Note that one GR realization could be

$$\begin{aligned} x_1(h+1,k) &= x_1(h,k) + x_2(h,k) + 2u(h,k) \\ x_2(h,k+1) &= 2x_1(h,k) + 2x_2(h,k) + u(h,k) \\ y(h,k) &= x_1(h,k) + x_2(h,k), \end{aligned}$$

which is obvious that the dimension of the state-space is equal to 2. ♢

Remark 6. Due to the fact that the local state space X in the GR model is split into the horizontal and vertical components, this implies that the structure of the updating equations is not invariant under similarity transformations in X. To see this, let S be the structured similarity transformation defined by

$$\begin{bmatrix} \tilde{x}_h \\ \tilde{x}_v \end{bmatrix} = \underbrace{\begin{bmatrix} S_h & 0 \\ 0 & S_v \end{bmatrix}}_{S} \begin{bmatrix} x_h \\ x_v \end{bmatrix}. \tag{3.55}$$

Then, if $\tilde{B} = SB, \tilde{A} = SAS^{-1}$, and $\tilde{C} = CS^{-1}$,

$$\tilde{C}(I - Z_d(\mathbf{z})\tilde{A})^{-1} Z_d(\mathbf{z})\tilde{B} = C(I - Z_d(\mathbf{z})A)^{-1} Z_d(\mathbf{z})B \tag{3.56}$$

so that the transfer function is invariant; while this would not be the case for a general similarity transformation

$$S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}.$$

However, the state space model (3.53)–(3.54) is invariant under such a transformation. ▲

The generalized version of the system described above is the so-called *d-D Fornasini-Marchesini system*, or FM system for short, which is given by:

$$\Sigma^{FM} \triangleq \begin{cases} x(\mathbf{n}) = \sum_{k=1}^{d} A_k^{FM} x(\mathbf{n}-\mathbf{e}_k) + \sum_{k=1}^{d} B_k^{FM} u(\mathbf{n}-\mathbf{e}_k) \\ y(\mathbf{n}) = C^{FM} x(\mathbf{n}) + D^{FM} u(\mathbf{n}) \end{cases} \tag{3.57}$$

where $\mathbf{e}_k$ denotes the standard basis in $\mathbb{C}^d$.

The connecting operator U^{FM} associated with this system is of the form

$$U^{FM} \triangleq \begin{bmatrix} A^{FM} & B^{FM} \\ C^{FM} & D^{FM} \end{bmatrix} = \begin{bmatrix} A_1^{FM} & B_1^{FM} \\ \vdots & \vdots \\ A_d^{FM} & B_d^{FM} \\ C^{FM} & D^{FM} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \bigoplus_1^d \mathcal{H} \\ \mathcal{Y} \end{bmatrix}. \tag{3.58}$$

Next we shall seek an adjoint system Σ^{*FM} so that trajectories (u_*, x_*, y_*) of Σ^{*FM} are characterized as those $(\mathcal{U}^* \times \mathcal{H} \times \mathcal{Y}^*)$-valued functions on $\mathbb{Z}^d$ satisfying the adjoint pairing relation:

$$\sum_{\mathbf{n}:|\mathbf{n}|=s+1} \langle x(\mathbf{n}), x_*(\mathbf{n})\rangle + \sum_{\mathbf{n}:|\mathbf{n}|=s} \langle y(\mathbf{n}), u_*(\mathbf{n})\rangle = \sum_{\mathbf{n}:|\mathbf{n}|=s} \langle x(\mathbf{n}), x_*(\mathbf{n})\rangle + \sum_{\mathbf{n}:|\mathbf{n}|=s} \langle u(\mathbf{n}), y_*(\mathbf{n})\rangle \tag{3.59}$$

By substituting the expression of $x(\mathbf{n})$ from the system equations (3.57), the left hand side of (3.59) can be expressed as

$$\begin{aligned}
&\sum_{\mathbf{n}:|\mathbf{n}|=s+1} \langle x(\mathbf{n}), x_*(\mathbf{n})\rangle + \sum_{\mathbf{n}:|\mathbf{n}|=s} \langle y(\mathbf{n}), u_*(\mathbf{n})\rangle \\
&= \sum_{\mathbf{n}:|\mathbf{n}|=s+1} \sum_{k=1}^{d} \left\langle A_k^{FM} x(\mathbf{n}-\mathbf{e}_k) + B_k^{FM} u(\mathbf{n}-\mathbf{e}_k), x_*(\mathbf{n})\right\rangle + \sum_{\mathbf{n}:|\mathbf{n}|=s} \left\langle C^{FM} x(\mathbf{n}) + D^{FM} u(\mathbf{n}), u_*(\mathbf{n})\right\rangle \\
&= \sum_{\mathbf{n}:|\mathbf{n}|=s} \sum_{k=1}^{d} \left\langle A_k^{FM} x(\mathbf{n}) + B_k^{FM} u(\mathbf{n}), x_*(\mathbf{n}+\mathbf{e}_k)\right\rangle + \sum_{\mathbf{n}:|\mathbf{n}|=s} \left\langle C^{FM} x(\mathbf{n}) + D^{FM} u(\mathbf{n}), u_*(\mathbf{n})\right\rangle \\
&= \sum_{\mathbf{n}:|\mathbf{n}|=s} \left\langle x(\mathbf{n}), \sum_{k=1}^{d} (A_k^{FM})^* x_*(\mathbf{n}+\mathbf{e}_k)\right\rangle + \sum_{\mathbf{n}:|\mathbf{n}|=s} \left\langle u(\mathbf{n}), \sum_{k=1}^{d} (B_k^{FM})^* x_*(\mathbf{n}+\mathbf{e}_k)\right\rangle \\
&\quad + \sum_{\mathbf{n}:|\mathbf{n}|=s} \left\langle x(\mathbf{n}), (C^{FM})^* u_*(\mathbf{n})\right\rangle + \sum_{\mathbf{n}:|\mathbf{n}|=s} \left\langle u(\mathbf{n}), (D^{FM})^* u_*(\mathbf{n})\right\rangle \\
&= \sum_{\mathbf{n}:|\mathbf{n}|=s} \left\langle x(\mathbf{n}), \sum_{k=1}^{d} (A_k^{FM})^* x_*(\mathbf{n}+\mathbf{e}_k) + (C^{FM})^* u_*(\mathbf{n})\right\rangle \\
&\quad + \sum_{\mathbf{n}:|\mathbf{n}|=s} \left\langle u(\mathbf{n}), \sum_{k=1}^{d} (B_k^{FM})^* x_*(\mathbf{n}+\mathbf{e}_k) + (D^{FM})^* u_*(\mathbf{n})\right\rangle.
\end{aligned} \tag{3.60}$$

Thus, from the above expression together with (3.59), we can deduce that the adjoint system Σ^{*FM} is given by

$$\Sigma^{*FM} \triangleq \begin{cases} x_*(\mathbf{n}) = \sum_{k=1}^{d} (A_k^{FM})^* x_*(\mathbf{n}+\mathbf{e}_k) + (C^{FM})^* u_*(\mathbf{n}) \\ y_*(\mathbf{n}) = \sum_{k=1}^{d} (B_k^{FM})^* x_*(\mathbf{n}+\mathbf{e}_k) + (D^{FM})^* u_*(\mathbf{n}). \end{cases} \tag{3.61}$$

The FM system Σ^{FM} is said to be *conservative* provided that a $(\mathcal{U},\mathcal{H},\mathcal{Y})$-valued function (u,x,y) is a trajectory of the FM system if and only if (y,x,u) is a trajectory of the adjoint system Σ^{*FM}. Ball-Sadosky-Vinnikov [BSV] showed that a necessary and sufficient condition so that the FM system Σ^{GR} is conservative is that

$$U^{FM} \text{ is an isometry, (i.e. } (U^{FM})^* U^{FM} = I), \text{ and } U^{FM}(U^{FM})^* = \begin{bmatrix} x_1 & & & \\ & \ddots & & \\ & & x_d & \\ & & & I \end{bmatrix} \tag{3.62}$$

subject to the additional side conditions

$$\sum_{k=1}^{d} x_k = I, \qquad \begin{bmatrix} (A_i^{FM})^* \\ (B_i^{FM})^* \end{bmatrix} \begin{bmatrix} A_j^{FM} & B_j^{FM} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ for } i \neq j. \tag{3.63}$$

If the conditions (3.62) and (3.63) are satisfied, then we shall call Σ^{FM} a *conservative* FM *system* and U^{FM} a *unitary* FM *colligation.* If we set (u,x,y) which is a trajectory of the original system to be equal to (y,x,u) which is a trajectory of the adjoint system (i.e., set $(u_*,x_*,y_*) = (y,x,u)$), then the adjoint pairing (3.59) collapses to the energy balance relation:

$$\sum_{\mathbf{n}:|\mathbf{n}|=s+1} \|x(\mathbf{n})\|^2 + \sum_{\mathbf{n}:|\mathbf{n}|=s} \|y(\mathbf{n})\|^2 = \sum_{\mathbf{n}:|\mathbf{n}|=s} \|x(\mathbf{n})\|^2 + \sum_{\mathbf{n}:|\mathbf{n}|=s} \|u(\mathbf{n})\|^2 \quad \text{for } \Sigma^{FM} \tag{3.64}$$

$$\sum_{\mathbf{n}:|\mathbf{n}|=s+1} \|x_*(\mathbf{n})\|^2 + \sum_{\mathbf{n}:|\mathbf{n}|=s} \|u_*(\mathbf{n})\|^2 = \sum_{\mathbf{n}:|\mathbf{n}|=s} \|x_*(\mathbf{n})\|^2 + \sum_{\mathbf{n}:|\mathbf{n}|=s} \|y_*(\mathbf{n})\|^2 \quad \text{for } \Sigma^{*FM} \tag{3.65}$$

For the frequency domain analysis, let $\Omega = \{\mathbf{n} \in \mathbb{Z}^d : |\mathbf{n}| \triangleq \sum_{k=1}^{d} n_k \geq 0\}$ and we have

$$\{x(\mathbf{n}-\mathbf{e}_k)\}_{\mathbf{n}\in\Omega} \mapsto x^{\wedge\Omega}(\mathbf{z}) = \sum_{\mathbf{n}\in\Omega} x(\mathbf{n}-\mathbf{e}_k)\mathbf{z}^{\mathbf{n}} = z_k \left[x^{\wedge\Omega}(\mathbf{z}) + \sum_{\mathbf{n}:|\mathbf{n}|=-1} x(\mathbf{n})\mathbf{z}^{\mathbf{n}} \right] \tag{3.66}$$

$$\text{and} \quad \{u(\mathbf{n}-\mathbf{e}_k)\}_{\mathbf{n}\in\Omega} \mapsto u^{\wedge\Omega}(\mathbf{z}) = \sum_{\mathbf{n}\in\Omega} u(\mathbf{n}-\mathbf{e}_k)\mathbf{z}^{\mathbf{n}} = z_k \left[u^{\wedge\Omega}(\mathbf{z}) + \sum_{\mathbf{n}:|\mathbf{n}|=-1} u(\mathbf{n})\mathbf{z}^{\mathbf{n}} \right]. \tag{3.67}$$

Then, the application of the Z-transform to the system equations (3.57) yields,

$$\begin{aligned} x^{\wedge\Omega}(\mathbf{z}) &= \sum_{k=1}^{d} z_k A_k^{FM}\left[x^{\wedge\Omega}(\mathbf{z}) + \sum_{\mathbf{n}:|\mathbf{n}|=-1} x(\mathbf{n})\mathbf{z}^{\mathbf{n}}\right] + \sum_{k=1}^{d} z_k B_k^{FM}\left[u^{\wedge\Omega}(\mathbf{z}) + \sum_{\mathbf{n}:|\mathbf{n}|=-1} u(\mathbf{n})\mathbf{z}^{\mathbf{n}}\right] \\ &= Z_r(\mathbf{z})A^{FM}x^{\wedge\Omega}(\mathbf{z}) + Z_r(\mathbf{z})B^{FM}u^{\wedge\Omega}(\mathbf{z}) \\ &\quad + \sum_{\mathbf{n}:|\mathbf{n}|=-1}\sum_{k=1}^{d} z_k A_k^{FM} x(\mathbf{n})\mathbf{z}^{\mathbf{n}} + \sum_{\mathbf{n}:|\mathbf{n}|=-1}\sum_{k=1}^{d} z_k B_k^{FM} u(\mathbf{n})\mathbf{z}^{\mathbf{n}} \end{aligned} \tag{3.68}$$

where $Z_r(\mathbf{z}) \triangleq \begin{bmatrix} z_1 I_{\mathcal{H}} & \cdots & z_d I_{\mathcal{H}} \end{bmatrix}$. If we assume now that the system has zero initial condition $x(\mathbf{z}) = 0$ for $\mathbf{n} \in \Omega_0 \triangleq \{\mathbf{n} \in \mathbb{Z}^d : |\mathbf{n}| = 0\}$, then from (3.57), we have

$$\begin{aligned} 0 = \sum_{\mathbf{n}:|\mathbf{n}|=0} x(\mathbf{n})\mathbf{z}^{\mathbf{n}} &= \sum_{\mathbf{n}:|\mathbf{n}|=0}\left[\sum_{k=1}^{d} A_k^{FM} x(\mathbf{n}-\mathbf{e}_k) + \sum_{k=1}^{d} B_k^{FM} u(\mathbf{n}-\mathbf{e}_k)\right]\mathbf{z}^{\mathbf{n}} \\ &= \sum_{\mathbf{n}:|\mathbf{n}|=-1}\left[\sum_{k=1}^{d} z_k A_k^{FM} x(\mathbf{n})\mathbf{z}^{\mathbf{n}} + \sum_{k=1}^{d} z_k B_k^{FM} u(\mathbf{n})\mathbf{z}^{\mathbf{n}}\right]. \end{aligned} \tag{3.69}$$

Thus (3.68) collapses to

$$x^{\wedge\Omega}(\mathbf{z}) = Z_r(\mathbf{z})A^{FM}x^{\wedge\Omega}(\mathbf{z}) + Z_r(\mathbf{z})B^{FM}u^{\wedge\Omega}(\mathbf{z}).$$

Application of the Z-transform to the output equation of (3.57) and substitution $x^{\wedge\Omega}$ from the above expression, we get

$$\begin{aligned} y^{\wedge\Omega}(\mathbf{z}) &= \left[C^{FM}(I - Z_r(\mathbf{z})A^{FM})^{-1}Z_r(\mathbf{z})B^{FM} + D^{FM}\right]\cdot u^{\wedge\Omega}(\mathbf{z}) \\ &\triangleq T_{\Sigma^{FM}}(\mathbf{z})\cdot u^{\wedge\Omega}(\mathbf{z}) \end{aligned} \tag{3.70}$$

where $T_{\Sigma^{FM}}(\mathbf{z})$ is called the *Fornasini-Marchesini (*FM*) transfer function.*

3.5 Identification between GR and FM Models

In this Section, we investigate the relationship between the GR and the FM models, and also provide proper conditions such that both models become equivalent in the sense that one model can be embedded into the other and vice versa without increasing the dimension of the state-space. In fact, embedding the GR model into the FM model naturally preserves the dimension of the state-space since each local state in the GR model is the direct sum of d orthogonal components. On the other hand, embedding the FM model into the GR model in general cannot be accomplished without increasing the dimension of the state-space due to the fact that the state-space must be decomposed into d subspaces which may or may not overlap (see, e.g. [Gał01, MA98, Kac93] and the references therein). We shall show that under certain assumptions, embedding the FM model into the GR model can preserve the dimension of the state-space and hence, both models are equivalent.

3.5.1 Embedding Givone-Roesser into Fornasini-Marchesini

For simplicity, let us first consider the 2D GR model which is given by

$$\Sigma^{GR} = \begin{cases} \begin{bmatrix} x_1(n_1+1,n_2) \\ x_2(n_1,n_2+1) \end{bmatrix} &= \begin{bmatrix} A_{11}^{GR} & A_{12}^{GR} \\ A_{21}^{GR} & A_{22}^{GR} \end{bmatrix} \begin{bmatrix} x_1(n_1,n_2) \\ x_2(n_1,n_2) \end{bmatrix} + \begin{bmatrix} B_1^{GR} \\ B_2^{GR} \end{bmatrix} u(n_1,n_2) \\ y(n_1,n_2) &= \begin{bmatrix} C_1^{GR} & C_2^{GR} \end{bmatrix} \begin{bmatrix} x_1(n_1,n_2) \\ x_2(n_1,n_2) \end{bmatrix} + D^{GR} u(n_1,n_2). \end{cases} \tag{3.71}$$

We can write the state equation in (3.71) as

$$\begin{aligned} \begin{bmatrix} x_1(n_1,n_2) \\ x_2(n_1,n_2) \end{bmatrix} &= \begin{bmatrix} A_{11}^{GR} & A_{12}^{GR} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(n_1-1,n_2) \\ x_2(n_1-1,n_2) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A_{21}^{GR} & A_{22}^{GR} \end{bmatrix} \begin{bmatrix} x_1(n_1,n_2-1) \\ x_2(n_1,n_2-1) \end{bmatrix} \\ &\quad + \begin{bmatrix} B_1^{GR} \\ 0 \end{bmatrix} u(n_1-1,n_2) + \begin{bmatrix} 0 \\ B_2^{GR} \end{bmatrix} u(n_1,n_2-1), \end{aligned} \tag{3.72}$$

or equivalently,

$$\begin{aligned} \begin{bmatrix} x_1(n_1,n_2) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2(n_1,n_2) \end{bmatrix} &= \begin{bmatrix} A_{11}^{GR} & A_{12}^{GR} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} x_1(n_1-1,n_2) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2(n_1-1,n_2) \end{bmatrix} \right\} \\ &\quad + \begin{bmatrix} 0 & 0 \\ A_{21}^{GR} & A_{22}^{GR} \end{bmatrix} \left\{ \begin{bmatrix} x_1(n_1,n_2-1) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2(n_1,n_2-1) \end{bmatrix} \right\} \\ &\quad + \begin{bmatrix} B_1^{GR} \\ 0 \end{bmatrix} u(n_1-1,n_2) + \begin{bmatrix} 0 \\ B_2^{GR} \end{bmatrix} u(n_1,n_2-1). \end{aligned} \tag{3.73}$$

By setting $x(n_1,n_2) \triangleq \begin{bmatrix} x_1(n_1,n_2) \\ x_2(n_1,n_2) \end{bmatrix} = \begin{bmatrix} x_1(n_1,n_2) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2(n_1,n_2) \end{bmatrix}$, and

$$A_1^{FM} = \begin{bmatrix} A_{11}^{GR} & A_{12}^{GR} \\ 0 & 0 \end{bmatrix}, \quad A_2^{FM} = \begin{bmatrix} 0 & 0 \\ A_{21}^{GR} & A_{22}^{GR} \end{bmatrix}, \quad B_1^{FM} = \begin{bmatrix} B_1^{GR} \\ 0 \end{bmatrix}, \quad B_2^{FM} = \begin{bmatrix} 0 \\ B_2^{GR} \end{bmatrix},$$

$$C^{FM} = C^{GR} = \begin{bmatrix} C_1^{GR} & C_2^{GR} \end{bmatrix}, \quad D^{FM} = D^{GR},$$

we get

$$\begin{aligned} x(n_1,n_2) &= A_1^{FM}x(n_1-1,n_2) + A_2^{FM}x(n_1,n_2-1) + B_1^{FM}u(n_1-1,n_2) + B_2^{FM}u(n_1,n_2-1) \\ y(n_1,n_2) &= C^{FM}x(n_1,n_2) + D^{FM}u(n_1,n_2) \end{aligned} \tag{3.74}$$

which is exactly the 2D FM model.

It should be noted that the system equation written in the form of (3.73) is much more convenient to generalize to the d-D case than the original one in (3.71). Mathematically, there are two linear operators involved in the transformation from the state equation (3.71) into (3.73), namely

- the orthogonal projection $P_k : \mathcal{H} = \bigoplus_{i=1}^d \mathcal{H}_i \mapsto \mathcal{H}_k$ with image equal to $\mathcal{H}_k$,
- the inclusion map $\iota_k : \mathcal{H}_k \mapsto \mathcal{H} = \bigoplus_{i=1}^d \mathcal{H}_i$.

For instance, suppose

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \mathcal{H} \mapsto \mathcal{H}, \text{ where } \mathcal{H} = \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}.$$

Then,

$$P_1A = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} : \mathcal{H} \mapsto \mathcal{H}_1, \qquad \text{and} \qquad \iota_1P_1A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} : \mathcal{H} \mapsto \mathcal{H},$$

$$\text{likewise,} \quad P_2A = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} : \mathcal{H} \mapsto \mathcal{H}_2, \qquad \text{and} \qquad \iota_2P_2A = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} : \mathcal{H} \mapsto \mathcal{H}.$$

By using these operators, we can write (3.73) in a more compact form as

$$\begin{aligned} \iota_1x_1(n_1,n_2) + \iota_2x_2(n_1,n_2) &= \iota_1P_1A^{GR}\{\iota_1x_1(n_1-1,n_2) + \iota_2x_2(n_1-1,n_2)\} \\ &\quad + \iota_2P_2A^{GR}\{\iota_1x_1(n_1,n_2-1) + \iota_2x_2(n_1,n_2-1)\} \\ &\quad + \iota_1P_1B^{GR}u(n_1-1,n_2) + \iota_2P_2B^{GR}u(n_1,n_2-1) \end{aligned} \tag{3.75}$$

Now let $x(\cdot) = \sum_{k=1}^2 \iota_kx_k(\cdot)$, and $A_k^{FM} = \iota_kP_kA^{GR}, B_k^{FM} = \iota_kP_kB^{GR}$, $k = 1,2$, and hence we obtain the same result as in (3.74).

Now we are ready to generalize this idea to the d-D case. Recall that the operator colligation of the d-D GR model is given by

$$U^{GR} \triangleq \begin{bmatrix} A^{GR} & B^{GR} \\ C^{GR} & D^{GR} \end{bmatrix} = \begin{bmatrix} A^{GR}_{1,1} & \cdots & A^{GR}_{1,d} & B^{GR}_1 \\ \vdots & \ddots & \vdots & \vdots \\ A^{GR}_{d,1} & \cdots & A^{GR}_{d,d} & B^{GR}_d \\ C^{GR}_1 & \cdots & C^{GR}_d & D^{GR} \end{bmatrix} : \begin{bmatrix} \bigoplus_{i=1}^d \mathcal{H}_i \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \bigoplus_{i=1}^d \mathcal{H}_i \\ \mathcal{Y} \end{bmatrix}, \tag{3.76}$$

and the associated d-D GR system is

$$\begin{bmatrix} x_1(\mathbf{n}+\mathbf{e}_1) \\ \vdots \\ x_d(\mathbf{n}+\mathbf{e}_d) \end{bmatrix} = \begin{bmatrix} A^{GR}_{1,1} & \cdots & A^{GR}_{1,d} \\ \vdots & \ddots & \vdots \\ A^{GR}_{d,1} & \cdots & A^{GR}_{d,d} \end{bmatrix} \begin{bmatrix} x_1(\mathbf{n}) \\ \vdots \\ x_d(\mathbf{n}) \end{bmatrix} + \begin{bmatrix} B^{GR}_1 \\ \vdots \\ B^{GR}_d \end{bmatrix} u(\mathbf{n}) \tag{3.77}$$

$$y(\mathbf{n}) = \begin{bmatrix} C^{GR}_1 & \cdots & C^{GR}_d \end{bmatrix} \begin{bmatrix} x_1(\mathbf{n}) \\ \vdots \\ x_d(\mathbf{n}) \end{bmatrix} + D^{GR} u(\mathbf{n}). \tag{3.78}$$

Let P_k and ι_k, respectively be the orthogonal projection and the inclusion map as defined above. Thus we can rewrite the state equation (3.77) as

$$\sum_{k=1}^{d} \iota_k x_k(\mathbf{n}) = \sum_{k=1}^{d} \iota_k P_k A^{GR} \left\{ \sum_{j=1}^{d} \iota_j x_j(\mathbf{n}-\mathbf{e}_k) \right\} + \sum_{k=1}^{d} \iota_k P_k B^{GR} u(\mathbf{n}-\mathbf{e}_k) \tag{3.79}$$

Let us now set

$$\begin{aligned} A^{FM}_k = \iota_k P_k A^{GR}, \quad B^{FM}_k &= \iota_k P_k B^{GR} \text{ for } k = 1, \ldots, d \\ C^{FM} = C^{GR}, \quad D^{FM} &= D^{GR}, \end{aligned} \tag{3.80}$$

and define $x(\cdot) = \sum_{k=1}^{d} \iota_k x_k(\cdot)$. Then the state equation (3.77) and the output equation (3.78) of the d-D GR model (3.77) can be rewritten in the d-D FM formalism as

$$\begin{aligned} x(\mathbf{n}) &= \sum_{k=1}^{d} A^{FM}_k x(\mathbf{n}-\mathbf{e}_k) + \sum_{k=1}^{d} B^{FM}_k u(\mathbf{n}-\mathbf{e}_k) \\ y(\mathbf{n}) &= C^{FM} x(\mathbf{n}) + D^{FM} u(\mathbf{n}). \end{aligned}$$

3.5.2 Embedding Fornasini-Marchesini into Givone-Roesser

The d-D FM system is described by

$$x(\mathbf{n}) = \sum_{k=1}^{d} A_k^{FM} x(\mathbf{n}-\mathbf{e}_k) + \sum_{k=1}^{d} B_k^{FM} u(\mathbf{n}-\mathbf{e}_k) \tag{3.81}$$

$$y(\mathbf{n}) = C^{FM} x(\mathbf{n}) + D^{FM} u(\mathbf{n}) \tag{3.82}$$

together with the associated connecting operator

$$U^{FM} \triangleq \begin{bmatrix} A^{FM} & B^{FM} \\ C^{FM} & D^{FM} \end{bmatrix} = \begin{bmatrix} A_1^{FM} & B_1^{FM} \\ \vdots & \vdots \\ A_d^{FM} & B_d^{FM} \\ C^{FM} & D^{FM} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \bigoplus_1^d \mathcal{H} \\ \mathcal{Y} \end{bmatrix}. \tag{3.83}$$

It is of interest to write the state equation (3.81) as

$$x(\mathbf{n}) = \sum_{k=1}^{d} \left[A_k^{FM} x(\mathbf{n}-\mathbf{e}_k) + B_k^{FM} u(\mathbf{n}-\mathbf{e}_k) \right] \triangleq \sum_{k=1}^{d} x_k(\mathbf{n}). \tag{3.84}$$

In order to embed this model into the d-D GR formalism, we need to construct the Hilbert spaces $\mathcal{H}_k$ for $k = 1, \ldots, d$, such that the direct sum (not necessarily orthogonal) $\dot{+}_{k=1}^{d} \mathcal{H}_k = \mathcal{H}$. To do so, let us assume that

$$\operatorname{im} \begin{bmatrix} A_j^{FM} & B_j^{FM} \end{bmatrix} \cap \operatorname{im} \begin{bmatrix} A_k^{FM} & B_k^{FM} \end{bmatrix} = \{0\} \quad k \neq j, \tag{3.85}$$

and define $\mathcal{H}_k$ so that $\operatorname{im} \begin{bmatrix} A_k^{FM} & B_k^{FM} \end{bmatrix} \subset \mathcal{H}_k$. Then set

$$A_{i,j}^{GR} = P_i A_i^{FM}\big|_{\mathcal{H}_j} : \mathcal{H}_j \mapsto \mathcal{H}_i, \qquad B_i^{GR} = P_i B_i^{FM} : \mathcal{U} \mapsto \mathcal{H}_i,$$
$$C_j^{GR} = C^{FM}\big|_{\mathcal{H}_j} : \mathcal{H}_j \mapsto \mathcal{Y}, \qquad D^{GR} = D^{FM} : \mathcal{U} \mapsto \mathcal{Y}.$$

Thus, for each $k = 1, \ldots, d$,

$$\begin{aligned} x_k(\mathbf{n}) = P_k x(\mathbf{n}) &= P_k \sum_{\ell=1}^{d} \left[A_\ell^{FM} x(\mathbf{n}-\mathbf{e}_\ell) + B_\ell^{FM} u(\mathbf{n}-\mathbf{e}_\ell) \right] \\ &= P_k A_k^{FM} x(\mathbf{n}-\mathbf{e}_k) + P_k B_k^{FM} u(\mathbf{n}-\mathbf{e}_k) \\ &= \sum_{j=1}^{d} P_k A_k^{FM}\big|_{\mathcal{H}_j} x_j(\mathbf{n}-\mathbf{e}_k) + P_k B_k^{FM} u(\mathbf{n}-\mathbf{e}_k) \end{aligned}$$

$$= \sum_{j=1}^{d} A_{k,j}^{GR} x_j(\mathbf{n} - \mathbf{e}_k) + B_k^{GR} u(\mathbf{n} - \mathbf{e}_k)$$

which is equivalent to

$$x_k(\mathbf{n} + \mathbf{e}_k) = \sum_{j=1}^{d} A_{k,j}^{GR} x_j(\mathbf{n}) + B_k^{GR} u(\mathbf{n}). \tag{3.86}$$

Likewise, for the output equation (3.82), we have

$$\begin{aligned} y(\mathbf{n}) &= C^{FM} x(\mathbf{n}) + D^{FM} u(\mathbf{n}) \\ &= \sum_{j=1}^{d} C^{FM}\big|_{\mathcal{H}_j} x_j(\mathbf{n}) + D^{FM} u(\mathbf{n}) \\ &= \sum_{j=1}^{d} C_j^{GR} x_j(\mathbf{n}) + D^{GR} u(\mathbf{n}). \end{aligned} \tag{3.87}$$

Hence, (3.86) together with (3.87) form the d-D GR model as required.

For instance, let us consider the 2D FM model of the form

$$x(n_1, n_2) = x_1(n_1, n_2) + x_2(n_1, n_2) \tag{3.88}$$

$$y(n_1, n_2) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} x(n_1, n_2) + Du(n_1, n_2) \tag{3.89}$$

where

$$\begin{aligned} x_1(n_1, n_2) &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} x(n_1 - 1, n_2) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(n_1 - 1, n_2) \in \operatorname{im} \begin{bmatrix} A_1^{FM} & B_1^{FM} \end{bmatrix} \subset \mathcal{H}_1 \\ x_2(n_1, n_2) &= \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} x(n_1, n_2 - 1) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(n_1, n_2 - 1) \in \operatorname{im} \begin{bmatrix} A_2^{FM} & B_2^{FM} \end{bmatrix} \subset \mathcal{H}_2. \end{aligned}$$

Thus,

$$\begin{aligned} P_1 x(n_1, n_2) &= P_1 \left[x_1(n_1, n_2) + x_2(n_1, n_2) \right] = P_1 x_1(n_1, n_2) = x_1(n_1, n_2) \\ &= P_1 \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} x(n_1 - 1, n_2) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(n_1 - 1, n_2) \right) \\ &= \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} x(n_1 - 1, n_2) + B_1 u(n_1 - 1, n_2) \\ &= \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}\Big|_{\mathcal{H}_1} x_1(n_1 - 1, n_2) + \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}\Big|_{\mathcal{H}_2} x_2(n_1 - 1, n_2) + B_1 u(n_1 - 1, n_2) \\ &= A_{11} x_1(n_1 - 1, n_2) + A_{12} x_2(n_1 - 1, n_2) + B_1 u(n_1 - 1, n_2). \end{aligned}$$

Likewise,

$$\begin{aligned} P_2 x(n_1, n_2) &= x_2(n_1, n_2) \\ &= A_{21} x_1(n_1, n_2 - 1) + A_{22} x_2(n_1, n_2 - 1) + B_2 u(n_1, n_2 - 1). \end{aligned}$$

And the output $y(n_1, n_2)$ is

$$\begin{aligned} y(n_1, n_2) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}\Big|_{\mathcal{H}_1} x_1(n_1, n_2) + \begin{bmatrix} C_1 & C_2 \end{bmatrix}\Big|_{\mathcal{H}_2} x_2(n_1, n_2) + Du(n_1, n_2) \\ &= C_1 x_1(n_1, n_2) + C_2 x_2(n_1, n_2) + Du(n_1, n_2). \end{aligned}$$

Remark 7. Ball-Sadosky-Vinnikov [BSV] considered the GR and the FM models in the conservative case, and hence the conditions (3.85) are automatically achieved since the conditions such that the FM is conservative (see page 46) are stronger than the conditions (3.85). ▲

Before we end this section, let us consider Example 4 on page 44 for the moment. Recall that a 2D FM realization for the given transfer function is

$$x(h+1, k+1) = x(h, k+1) + 2x(h+1, k) + 2u(h, k+1) + u(h+1, k), \quad y(h, k) = x(h, k),$$

i.e., $A_1 = 1, A_2 = 2, B_1 = 2$, and $B_2 = 1$. Thus,

$$\begin{aligned} \operatorname{im} \begin{bmatrix} A_1 & B_1 \end{bmatrix} &= \operatorname{im} \begin{bmatrix} 1 & 2 \end{bmatrix} = \{x_1 \in \mathcal{H}_1 \mid x + 2u = x_1, \forall x \in \mathcal{H}, u \in \mathcal{U}\} \\ \operatorname{im} \begin{bmatrix} A_2 & B_2 \end{bmatrix} &= \operatorname{im} \begin{bmatrix} 2 & 1 \end{bmatrix} = \{x_2 \in \mathcal{H}_2 \mid 2x + u = x_2, \forall x \in \mathcal{H}, u \in \mathcal{U}\} \end{aligned}$$

In particular, for $x = 1, u = 1$, we have $x_1 = x_2 = 3$ which contradicts the condition (3.85) and this explains why the dimension of the state-space cannot be preserved when embedding this system into the GR model.

3.6 Conclusion

The purpose of this Chapter is to discuss chronologically the development of the state-space models (the Givone-Roesser (GR) and the Fornasini-Marchesini (FM) formalisms) including the advantage and disadvantage of each model. We generalize the original 2D systems to the d-D systems where $d \geq 2$, and establish the certain conditions to identify the models with each other. It is obvious that the GR model can be naturally embedded into the FM model. On the other hand, embedding the FM model into the GR model in general can not be accomplished without increasing the dimension of the state-space.

The time domain analysis and design such as stability, controllability, reachability, observability, similarity, observer design, output feedback, deadbeat control and H^∞ control in the state-space coordinate has been done by various researchers and is beyond the scope of this dissertation. For further references, readers should refer to, e.g. [Bos85, DX02, Kac85, LA92, Tza86, Zer00] and references therein.

Chapter 4

H^∞ Control, Model Matching and Interpolation Theory

In the case of classical 1D linear systems, the H^∞ control problems can be solved via either state-space analysis in the time domain, or interpolation theory in the frequency domain. The question considered here is whether or not these two approaches can be extended to the case of d-D systems and the answer is "Yes". The H^∞ control and filtering problems for 2D linear systems have already been solved (not completely satisfactorily) via an extended bounded real lemma for 2D systems in [DX02, DXZ01]. However, to the best of our knowledge, the H^∞ control problems in the frequency domain setting have been attacked for the first time in [BM, BM02] for the output feedback d-D linear systems ($d \geq 2$). This Chapter is an expanded version of the two papers mentioned above. The main goal here is to examine the connection between H^∞ control, model matching, and multivariable Nevanlinna-Pick interpolation problems for multidimensional d-D linear systems.

4.1 Introduction

Feedback stabilization and optimal control problems for the case of classical linear systems have been much studied over the past several decades. More recently, such problems for the case of multidimensional or d-D linear systems ($d > 1$) have been drawing the attention of researchers—see e.g., [Bos85, DX99, DXZ01, Lin00, SS92, Sul94, Tza86] and [Zer00], and the references therein. While most of these authors are motivated by applications to physical situations having d-D system models, Helton [Hel01] has pointed out a connection with adaptive control for a classical 1D system. After D. Givone-R. Roesser [GR72], E. Fornasini-G. Marchesini [FM78] and some other researchers proposed various types of multidimensional linear models in the seventies, most mathematicians and system engineers have been focusing on the development and extension of the existence theories for classical linear systems to the case of d-D systems. New theories and notions, all of which are more generalized and complicated than those in the classical case, have been introduced to describe the behavior of d-D systems.

After the H^∞ control theory was first introduced in the control community in the 1980's, it has been continually studied and progressively developed by mathematicians and system engineers, and effectively applied to various applications, although it is difficult to fully understand due to its intricate mathematical structure. Based on the fact that the H^∞ control theory involves the classical design methodology and the state-space analysis, it was said to be the first successful theory to diminish the gap between the classical control design and state-space theory. In addition, due to the remarkable structure of the H^∞ control design, the number of publications in the H^∞ control theory and its applications has been increasing considerably.

It is well known that for linear time-invariant 1D systems there are mainly two approaches to solve the H^∞ *control problems*: frequency-domain/interpolation theory, and time-domain/state-space analysis. In the early days of the H^∞ control, the frequency-domain/interpolation theory approach combined with implementation in terms of state-space coordinates had prominence. In this approach (see[Fra87]), one goes through coprime factorization to get the Q-parameter; with Q as the new design parameter rather than the controller K, one has a *model matching problem.* Let F be the performance function, which is affine in Q. Then, with the performance function F as the design parameter rather than Q, one has an *interpolation problem* for F. One then solves an interpolation problem to get F, and then backsolves for Q and finally for K, a desired controller. A criterion for internal stability can be expressed directly in terms of F: K is *internally stabilizing* for the closed loop system whenever F is *stable* and satisfies the appropriate *interpolation conditions.* Incorporation of a tolerance level on the performance function then leads to an *interpolation problem of Nevanlinna-Pick type.*

After the appearance of the seminal paper of Doyle et al. [DGKF89], however, the time-domain/state-space formulation has been the dominant approach. More recent refinements (see e.g., [PAJ91]) use the bounded real lemma as a tool for deriving the relevant coupled algebraic Riccati equations, which in turn can be expressed in an elegant form as a convex optimization problem in the form of a system of the so-called *Linear Matrix Inequalities or* LMI*s* for which available software exists to give a solution (see e.g., [BGFB94, GN00] and references therein).

Various authors have now extended both of these approaches to the case of d-D systems. The work of Du-Xie and Du-Xie-Zhang [DX99, DX02, DXZ01] uses a bounded real lemma for 2D systems to derive various systems of LMIs, the solutions of which lead to solutions of various 2D robust control and filtering problems; however, it is known that the 2D bounded real lemma gives only a sufficient (and not necessary) condition for a system to be bounded real, so one can expect the solutions based on this approach in general to be conservative, although they may well be satisfactory for some special examples.

The frequency-domain approach to the H^∞ control problem as described above is much more complicated for the d-D case for a number of reasons. First of all, the reduction to the model matching form is not obvious since the notion of coprime factorization splits in several independent ways in the d-D case (see [YG79]); a fundamental difference in the multivariable case is that factor coprime irreducible polynomials can have common zeros. Secondly, the multivariable analogue of Nevanlinna-Pick interpolation is much more complicated. By using the various notions of coprime in the d-D case, the matrix fraction description (MFD) approach for d-D linear systems and its connection with the properties of d-D polynomial and rational matrices were investigated by Z. Lin [Lin88]. He proved that, for the d-D case, the rational matrix function $P(\mathbf{z})$, where $\mathbf{z} = (z_1, \dots, z_d)$, does not always admit a minor right coprime decomposition. From this fact, he was able to produce a counterexample to illustrate that the determinant test for internal stability of 2D systems due to Humes-Jury [HJ77] may not be extended to the d-D case when $P(\mathbf{z})$ does not admit a minor right coprime decomposition. Therefore, he introduced the notion of *generating polynomials* (later renamed as *reduced minors*) and applied it to the stability test for d-D systems. The notion of reduced minors was introduced in connection with the feedback stabilization problem for d-D systems in [Lin98], [Lin99], and [Lin00]. However, in those papers, Lin studied the (output) feedback stabilization problem, which is the special case of the standard H^∞ control framework (see [Fra87]), and obtained an analogue of the famous Youla parametrization of the set of all stabilizing controllers. In his work, Lin did not take the next step of seeking to find a stabilizing controller which optimizes some performance function, i.e. the H^∞ control problem.

This Chapter uses the results of Lin to establish the connection between feedback stabilization and interpolation conditions for d-D linear systems for the case where the plant P admits a double coprime factorization (see Definition 21) in the so-called *1-block case*. When one goes on to demand performance in addition to internal stability as a design goal, there results an d-D matrix Nevanlinna-Pick interpolation problem. We apply recent work on Nevanlinna-Pick interpolation on the polydisk (see [Agl87, AM, AM02, BB, BT98]) to obtain a solution of the problem for the 2D case–the work in [BB] actually applies to more general types of domains than $\mathbb{D}^d$ but this does not concern us here. The same analysis applies in the d-D case ($d > 2$), but leads to solutions which are contractive in a norm (the "Schur-Agler norm") somewhat stronger than the H^∞ norm. We also present a solution based on the polydisk Commutant Lifting Theorem from [BLTT99]. It remains to determine how to streamline or short circuit the various steps in the solution procedure; this is discussed in the final section, where solution procedure is summarized. This connection between d-D matrix Nevanlinna-Pick interpolation and feedback stabilization with performance goal for d-D plants has previously been pointed out by Helton [Hel01] for the scalar case. Given that the sufficient conditions of Lin for the existence of a double coprime factorization of the original plant P are satisfied, our analysis, unlike the work of Du-Xie, is necessary and sufficient. When $d > 2$, our conditions, necessary and sufficient for performance in the Schur-Agler norm, are still sufficient (for performance with respect to the usual H^∞-norm).

This Chapter is organized as follows: Notation and some basic facts that will be used throughout this Chapter, the output feedback stabilization problem together with the set of solutions parametrized by the Youla parameter, say Q, and the connection between the closed loop transfer function and the model matching formalism are all established in Section 4.2. In Section 4.3, the interpolation theory for the d-D linear systems is developed and provides the interpolation conditions such that the model matching and the interpolation problems are equivalent. Then, Section 4.4 devotes to the Nevanlinna-Pick interpolation problem on the polydisk which establishes a close connection with the H^∞ control problem, followed by the application of the so-called *commutant lifting theorem* to the model matching problem in Section 4.5. The solution procedure for solving the H^∞ control problem is summarized in the last Section.

4.2 Preliminaries

In the following, we shall let $\mathbb{R}$ denote the field of real numbers; $\mathbb{R}[\mathbf{z}] = \mathbb{R}[z_1, \ldots, z_d]$ the polynomial ring over $\mathbb{R}$ in d-indeterminants $(z_1, \ldots, z_d)$, all of which are complex variables; $\mathbb{R}(\mathbf{z}) = \mathbb{R}(z_1, \ldots, z_d)$, the field of rational functions which is equal to the quotient field of $\mathbb{R}[\mathbf{z}]$; $\mathbb{R}_s(\mathbf{z}) \subseteq \mathbb{R}(\mathbf{z})$ the subset of rational functions in $\mathbb{R}(\mathbf{z})$ having no poles in the closed unit polydisk, defined as

$$\overline{\mathbb{D}^d} = \left\{ (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1| \leq 1, \ldots, |z_d| \leq 1 \right\}.$$

$\mathbb{R}^{m \times l}(\mathbf{z})$ the set of $m \times l$ matrices with entries in $\mathbb{R}(\mathbf{z})$ (i.e., entries are rational functions); $\mathbb{R}_s^{m \times l}(\mathbf{z})$ the set of $m \times l$ matrices with entries in $\mathbb{R}_s(\mathbf{z})$ (i.e., entries are stable real rational functions). The d-D polynomial is said to be stable if it has no zeros in $\overline{\mathbb{D}^d}$.

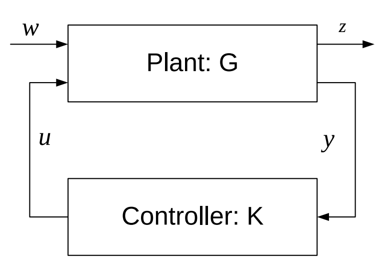

Figure 4.1: The standard H^∞ control framework

In the standard H^∞ control context (see Fig.4.1), the problem is to design a controller K

which minimizes the largest energy error signal z over all disturbances w of L^2-norm at most 1, subject to the additional constraint that K stabilizes the system:

$$\min_{K \text{stabilizing}} \max_{\| w \|_2 \leq 1} \| z \|_2 \tag{4.1}$$

where the L^2-norm of any signal $x(t)$ is regarded as the measure of energy of a vector-valued signal and defined by

$$\| x \|_2^2 = \int_0^\infty \| x(t) \|^2 \, dt \tag{4.2}$$

Loosely speaking, the goal of the H^∞ control problem is to find a stabilizing controller K so as to minimize the H^∞ norm of the desired performance function, say F. In other words, one needs to construct a controller K so that the closed loop system is *internally stable* (in the precise sense given in Definition 18 below) with L^2-induced operator-norm equal to at most a given tolerance level $\gamma > 0$.

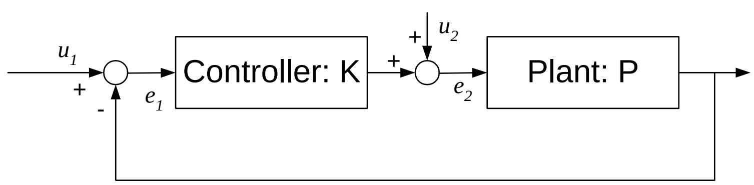

Figure 4.2: The output feedback system

Now consider the (output) feedback system depicted in Fig.4.2, where $P(\mathbf{z})$ and $K(\mathbf{z})$ denote, respectively the plant and controller in the multidimensional setting. Then the closed loop transfer matrix function from the input signals, u, to the error signals, e, is given by

$$\mathcal{H}_{eu} = \begin{bmatrix} (I + PK)^{-1} & -P\,(I + KP)^{-1} \\ K\,(I + PK)^{-1} & (I + KP)^{-1} \end{bmatrix} \tag{4.3}$$

The precise notion of internal stability for an output feedback system which we shall use here is given in the following definition (see [Fra87, Chapter 4] or [Vid85, Chapter 5]).

Definition 18. A given plant $P \in \mathbb{R}^{m \times l}(\mathbf{z})$ is said to be (output) feedback stabilizable if there exists a controller K such that the closed loop transfer matrix function $\mathcal{H}_{eu}$ in (4.3) is internally stable; i.e., each entry of $\mathcal{H}_{eu}$ has no poles in $\overline{\mathbb{D}^d}$.

It is well known that in the classical 1D linear system, the plant $P(z)$ always admits the so-called *double coprime factorization,* DCF and hence, one can construct a set of stabilizing linear controllers via the famous Youla parameter, say Q. The goal here is to get an analogous result as in the classical 1D linear system; however, the notion of coprimeness in several variables is not unique. For a simple example, let us consider the case when $f_1(z_1, z_2) = z_1 - 1$ and $f_2(z_1, z_2) = z_2 - 2$. Clearly, f_1 and f_2 have no common factor and hence they are factor coprime; however, the zero set of f_1 is the set such that $f_1 = 0$, i.e. an algebraic curve defined by

$$\mathcal{Z}(f_1) = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 - 1 = 0\}.$$

Likewise, the zero set of f_2 is an algebraic curve defined by

$$\mathcal{Z}(f_2) = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 - 2 = 0\}.$$

These two curves intersect at the point $(1, 2) \in \mathbb{C}^2$ and hence, f_1 and f_2 are not zero coprime.

Morf et al. [MLK77] studied the polynomial matrices and coprimeness for the 2D linear system, and observed the distinction between 1D and 2D primitive factorization. They proposed the *primitive factorization theorem* for two variables and observed that it applied well to 2D systems. However, this is not the case when $d > 2$. Youla and Gnani [YG79] showed by the counterexample that the primitive factorization theorem proposed by Morf et al. does not generalize to the case of three or more variables. They also provided several definitions of the *coprimeness* for polynomial matrices in several variables which all collapse to the classical notion in the one-variable case. In fact, there are at least three of them commonly used in the system theory, namely factor coprime, minor coprime and zero coprime.

Definition 19 ([YG79, Lin88]). Let $A \in \mathbb{R}^{l\times l}[\mathbf{z}]$, $B \in \mathbb{R}^{m\times l}[\mathbf{z}]$, and $F = \begin{bmatrix} A \\ B \end{bmatrix} \in \mathbb{R}^{(m+l)\times l}[\mathbf{z}]$. Then A and B are said to be:

1. *zero right coprime* (ZRC) if the $l \times l$ minors of $F(\mathbf{z})$ have no common zero in $\overline{\mathbb{D}^d} \subset \mathbb{C}^d$,
2. *minor right coprime* (MRC) if the above minors are relatively prime[1]; i.e., these minors are factor coprime,
3. *factor right coprime* (FRC) if, in any polynomial decomposition $F(\mathbf{z}) = F_1(\mathbf{z})F_2(\mathbf{z})$, the $l \times l$ matrix $F_2(\mathbf{z})$ is a unimodular matrix[2].

In a dual manner, $A_1 \in \mathbb{R}^{m\times m}[\mathbf{z}]$, and $B_1 \in \mathbb{R}^{m\times l}[\mathbf{z}]$ are zero left coprime (ZLC) *etc.*, if $A_1^\top$ and $B_1^\top$ are ZRC, *etc.*, where $A^\top$ denotes the transposed matrix of A.

Theorem 4.1 ([YG79]). *For $d = 1$, the three notions of coprimeness (zero, minor, and factor coprimes) are equivalent; For $d = 2$, minor and factor coprimes are equivalent, and for $d \geq 3$, all*

[1] One or more polynomials are said to be relatively prime provided that their greatest common polynomial divisor (g.c.d) is a nonzero constant.

[2] A square matrix $F_2(\mathbf{z})$ is said to be *unimodular* if it is elementary (i.e., $\det F_2(\mathbf{z}) = k \in \mathbb{R} \setminus \{0\}$)

of them in general are distinct. Always, zero coprime implies minor coprime and minor coprime implies factor coprime.

For analysis purposes, it is of interest to consider the strongest notion of coprime, namely the notion of *zero coprime*, and we also focus on the coprimeness over the ring $\mathbb{R}_s(\mathbf{z})$ rather than coprimeness over the polynomial ring $\mathbb{R}[\mathbf{z}]$. Thus, the definition of zero coprime in this case can be stated as follows:

Definition 20. Let $A \in \mathbb{R}_s^{l\times l}(\mathbf{z})$, $B \in \mathbb{R}_s^{m\times l}(\mathbf{z})$, and $F = \begin{bmatrix} A \\ B \end{bmatrix} \in \mathbb{R}_s^{(m+l)\times l}(\mathbf{z})$. Then A and B are said to be *zero right coprime over* $\mathbb{R}_s(\mathbf{z})$ if the $l \times l$ minors of F have no common zero in $\overline{\mathbb{D}^d}$.

The following Proposition establishes the connection between the zero right coprime and the well-known Bézout equation for multivariable case.

Proposition 4.2. *Let $A \in \mathbb{R}_s^{l\times l}(\mathbf{z})$ and $B \in \mathbb{R}_s^{m\times l}(\mathbf{z})$. Then A and B are* ZRC *if and only if there exists a matrix $\begin{bmatrix} X & Y \end{bmatrix}$ over $\mathbb{R}_s(\mathbf{z})$ which solves the Bézout equation*

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \mathbf{I}_l \tag{4.4}$$

Proof. See e.g. [Zer00] for the polynomial case. ■

Suppose we assume that two stable rational functions are zero coprime, then by Proposition 4.2 we have the Bézout identity and this leads to the notion of double coprime factorization (DCF). Thus, from this point on, we shall use *coprime* instead of *zero coprime*, unless otherwise specified.

Definition 21 ([Lin00]). Let $P \in \mathbb{R}^{m\times l}(\mathbf{z})$ be a proper real rational matrix d-D system. Then P is said to have a double coprime factorization (DCF) if

1. there exist $D_l \in \mathbb{R}_s^{m\times m}(\mathbf{z})$, $D_r \in \mathbb{R}_s^{l\times l}(\mathbf{z})$, and N_r, $N_l \in \mathbb{R}_s^{m\times l}(\mathbf{z})$;
2. there exist $X_l \in \mathbb{R}_s^{l\times l}(\mathbf{z})$, $X_r \in \mathbb{R}_s^{m\times m}(\mathbf{z})$, and Y_r, $Y_l \in \mathbb{R}_s^{l\times m}(\mathbf{z})$;
3. D_l, D_r, X_l, and X_r are all nonsingular;
4. $P = N_r D_r^{-1} = D_l^{-1} N_l$ and the following Bézout identity holds:

$$\begin{bmatrix} X_l & Y_l \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -Y_r \\ N_r & X_r \end{bmatrix} = I_{(m+l)\times(m+l)} \tag{4.5}$$

To analyze the stability issue, Lin (see [Lin88]) introduced the notion of *generating polynomials* and applied this notion to the stability test. He showed that, for the multivariable case, the rational matrix function $P(\mathbf{z})$ does not always admit a minor right coprime decomposition and from this fact, he constructed a counterexample to illustrate that the determinant test for internal stability of 2D systems due to Humes-Jury [HJ77] may not be extended to the d-D case ($d > 2$) when $P(\mathbf{z})$ does not admit a minor right coprime decomposition.

Since we are dealing with polynomial or rational matrix functions of several variables, the ordering of the submatrices and minors of a matrix are needed here. Let $F \in \mathbb{R}^{(m+l)\times l}[\mathbf{z}]$, and consider all submatrices $F_k \in \mathbb{R}^{l\times l}[\mathbf{z}], k = 1, \ldots, \beta$ where $\beta \triangleq \binom{m+l}{l}$. If submatrix F_k is formed by selecting rows $1 \le k_1 < \cdots < k_l \le m + l$, we associate F_k with an l-tuple $(k_1, \ldots, k_l)$. This forms a one-to-one correspondence between all the submatrices F_k of F and the collection of all strictly increasing l-tuples $(k_1, \ldots, k_l)$, where $1 \le k_1 < \cdots < k_l \le m+l$. Now by enumerating the above l-tuples $(k_1, \ldots, k_l)$ in the lexicographic order, the submatrices F_k are ordered accordingly. We shall assume this ordering throughout this Chapter. Now let a_k be the $l \times l$ minors of $F(\mathbf{z})$ and always be ordered in the same way as F_k, i.e., $a_k = \det F_k$, for $k = 1, \ldots, \beta$.

Definition 22. Let $F = \begin{bmatrix} A^\top & B^\top \end{bmatrix}^\top \in \mathbb{R}^{(m+l)\times l}[\mathbf{z}]$ be of normal full rank[3], and let $a_1(\mathbf{z}), \ldots, a_\beta(\mathbf{z})$ be the $l\times l$ minors of the matrix $F(\mathbf{z})$ as described above. Extracting the greatest common divisor (g.c.d.) $g(\mathbf{z})$ of $a_1(\mathbf{z}), \ldots, a_\beta(\mathbf{z})$ gives, $a_j(\mathbf{z}) = g(\mathbf{z})b_j(\mathbf{z})$, for $j = 1, \ldots, \beta$. Then $b_1(\mathbf{z}), \ldots, b_\beta(\mathbf{z})$ are called the **generating polynomials**, (later renamed as **reduced minors**) of $F(\mathbf{z})$.

Remark 8. It should be noted that the $l \times l$ minor $a_1(\mathbf{z})$ corresponds to $\det A(\mathbf{z})$ due to the ordering described above. If A and B in Definition 22 are MRC (i.e., all minors have no common factors), then $g(\mathbf{z}) = 1$, and hence one can take all minors $a_1(\mathbf{z}), \ldots, a_\beta(\mathbf{z})$ to be the generating polynomials of $F(\mathbf{z})$. ▲

Let us consider the following example borrowed from [Lin88].

Example 5. Suppose F is given by

$$F(\mathbf{z}) = \begin{bmatrix} (z_2+2)(z_3+2.5) & -(z_1+3)(z_3+2.5) \\ -(z_3+0.5)(z_2+2)(z_3+4.5) & (z_3+0.5)^2(z_1+3)(z_3+4.5) \\ (z_3+0.5)(z_3-0.5) & 0 \\ 0 & (z_3+0.5)(z_3-0.5) \end{bmatrix}$$

[3] An $m \times l$ matrix $A(\mathbf{z})$ is of normal full rank if there exists an $r \times r$ minor of $A(\mathbf{z})$ that is not identically zero, where $r = \min\{m, l\}$

Thus, we have

$$\begin{aligned}
a_1(\mathbf{z}) &= \det F_1 = (z_2+2)(z_3+2.5)(z_3+4.5)(z_1+3)(z_3+0.5)(z_3-0.5)\\
a_2(\mathbf{z}) &= \det F_2 = (z_1+3)(z_3+2.5)(z_3+0.5)(z_3-0.5)\\
a_3(\mathbf{z}) &= \det F_3 = (z_2+2)(z_3+2.5)(z_3+0.5)(z_3-0.5)\\
a_4(\mathbf{z}) &= \det F_4 = -(z_3+0.5)^2(z_1+3)(z_3+4.5)(z_3+0.5)(z_3-0.5)\\
a_5(\mathbf{z}) &= \det F_5 = -(z_3+0.5)(z_2+2)(z_3+4.5)(z_3+0.5)(z_3-0.5)\\
a_6(\mathbf{z}) &= \det F_6 = (z_3+0.5)(z_3-0.5)(z_3+0.5)(z_3-0.5)
\end{aligned}$$

Obviously, the common factor of all the minors a_k's is $g(\mathbf{z}) = (z_3+0.5)(z_3-0.5)$, and hence, the reduced minors are

$$\begin{aligned}
b_1(\mathbf{z}) &= (z_2+2)(z_3+2.5)(z_3+4.5)(z_1+3)\\
b_2(\mathbf{z}) &= (z_1+3)(z_3+2.5)\\
b_3(\mathbf{z}) &= (z_2+2)(z_3+2.5)\\
b_4(\mathbf{z}) &= -(z_3+0.5)^2(z_1+3)(z_3+4.5)\\
b_5(\mathbf{z}) &= -(z_3+0.5)(z_2+2)(z_3+4.5)\\
b_6(\mathbf{z}) &= (z_3+0.5)(z_3-0.5). \qquad \diamondsuit
\end{aligned}$$

Remark 9. Since $F(\mathbf{z})$ is of normal full rank and the order of all its minors $a_k(\mathbf{z})$'s is fixed, the reduced minors of $F(\mathbf{z})$ are essentially unique up to the multiplication by a nonzero constant. ▲

Now we are ready to state a necessary and sufficient condition to verify whether or not a given system is internally stable via the notion of reduced minor.

Proposition 4.3. *A d-D discrete-time system $P \in \mathbb{R}^{m\times l}(\mathbf{z})$ represented by right matrix fraction decomposition (or right* MFD*, for short) as $P = N_r D_r^{-1}$ is internally stable if and only if $b_1 \neq 0$ in the polydisk, $\overline{\mathbb{D}^d}$, where b_j are the reduced minors of $F = \begin{bmatrix} D^\top & N^\top \end{bmatrix}^\top$.*

Application of this Proposition to the stability test is given in the following example.

Example 6. Let $P(\mathbf{z})$ be a rational matrix function in three complex variables $\mathbf{z} = (z_1, z_2, z_3)$ given by

$$P(\mathbf{z}) = \begin{bmatrix} \frac{z_3^2+z_3+0.25}{(z_2+2)(z_3+2.5)} & \frac{1}{(z_2+2)(z_3+4.5)} \\ \frac{z_3+0.5}{(z_1+3)(z_3+2.5)} & \frac{1}{(z_1+3)(z_3+4.5)} \end{bmatrix}.$$

Clearly, $P(\mathbf{z}) \in \mathbb{R}_s^{3\times 3}(\mathbf{z})$, since $P(\mathbf{z})$ has no poles in $\overline{\mathbb{D}^3}$. Decompose $P(\mathbf{z})$ into a right MFD as

$P = N_r D_r^{-1}$, where

$$N_r(\mathbf{z}) = \begin{bmatrix} (z_3+0.5)(z_3-0.5) & 0 \\ 0 & (z_3+0.5)(z_3-0.5) \end{bmatrix},$$
$$D_r(\mathbf{z}) = \begin{bmatrix} (z_2+2)(z_3+2.5) & -(z_1+3)(z_3+2.5) \\ -(z_3+0.5)(z_2+2)(z_3+4.5) & (z_3+0.5)^2(z_1+3)(z_3+4.5) \end{bmatrix}$$

It can be checked that

$$\det D_r(\mathbf{z}) = (z_2+2)(z_3+2.5)(z_3+4.5)(z_1+3)(z_3+0.5)(z_3-0.5)$$

which has zeros in the unstable region $\overline{\mathbb{D}^3}$.

However, it is clear from Example 5 that the reduced minor $b_1(\mathbf{z})$ has no zeros in the closed polydisk $\overline{\mathbb{D}^3}$. Therefore, by Proposition 4.3 we conclude that the 3D system $P(\mathbf{z})$ is internally stable, which agrees with the fact that $P(\mathbf{z})$ has no poles in $\overline{\mathbb{D}^3}$.

Now let us define $F = \begin{bmatrix} D_r^\top & N_r^\top \end{bmatrix}^\top$. Then, from Example 5 we have seen that all minors of F have common factor, i.e. $a_k(\mathbf{z}) = g(\mathbf{z})b_k(\mathbf{z}), k = 1, \ldots, \beta$, where $g(\mathbf{z}) = (z_3+0.5)(z_3-0.5)$, and hence, N_r and D_r are not MRC (see Remark 8); however, Lin was able to show that they are FRC (see [Lin88]). $\diamondsuit$

We observe that the zero set of a polynomial factor in $\det D_r(\mathbf{z})$ does not necessarily appear as part of the polar variety of the system $P(\mathbf{z})$. This example shows that the determinant test for structural stability of 2D systems due to Humes-Jury may not be extended to the general d-D case when $P(\mathbf{z})$ does not admit a MRC MFD.

Suppose now that a d-D discrete system $P = N_r D_r^{-1} \in \mathbb{R}^{m\times l}(\mathbf{z})$ is not internally stable (i.e., $b_1(\mathbf{z})$ has a zero in $\overline{\mathbb{D}^d}$). Then one needs to find a controller K so that the closed loop system is internally stable. However, not all P's are feedback stabilizable even in the scalar case (unlike in the 1D case). The next theorem provides a necessary and sufficient condition for such a P to be stabilizable.

Proposition 4.4 ([Lin98]). *Let $P = N_r D_r^{-1} \in \mathbb{R}^{m\times l}(\mathbf{z})$ represent a d-D system which is a proper rational matrix function, and let $b_1, \ldots, b_\beta$ be the reduced minors of $F = \begin{bmatrix} D^\top & N^\top \end{bmatrix}^\top$, with $\beta = \binom{m+l}{l}$. Then P is feedback stabilizable if and only if the reduced minors b_j of F ($j = 1, \ldots, \beta$) have no common zeros in $\overline{\mathbb{D}^d}$.*

The analogue of Proposition 4.4 for the standard H^∞ control problem in full generality (i.e., the d-D version of the so-called *standard* problem of H^∞ control in [Fra87]) does not seem to be known.

Definition 23 ([Lin98]). A rational function $\dfrac{n(\mathbf{z})}{d(\mathbf{z})}$ with $n, d \in \mathbb{R}[\mathbf{z}]$ is said to be *causal* if

$d(\mathbf{0}) = d(0,\ldots,0) \neq 0$. It is called *strictly causal* if in addition $n(\mathbf{0}) = 0$. A rational matrix function $P \in \mathbb{R}^{m\times l}(\mathbf{z})$ is said to be *causal* if all its entries are causal. It is called *strictly causal* if all its entries are strictly causal.

Proposition 4.5 ([Lin98]). *If $P \in \mathbb{R}^{m\times l}(\mathbf{z})$ is causal (strictly causal), there exists a right* MFD *$P = N_r D_r^{-1}$ such that $\det D_r(\mathbf{0}) \neq 0$ (in addition, $N_r(\mathbf{0}) = 0_{m\times l}$). On the other hand, if $P = N_r D_r^{-1} \in \mathbb{R}^{m\times l}(\mathbf{z})$, and $\det D_r(\mathbf{0}) \neq 0$, then P is causal. If in addition $N_r(\mathbf{0}) = 0_{m\times l}$, then P is strictly causal.*

Suppose that the plant P is feedback stabilizable; i.e., P satisfies the condition given in Proposition 4.4. Then the following theorem provides a sufficient condition so that P admits the double coprime factorization, (DCF).

Proposition 4.6 ([Lin00]). *Let $P \in \mathbb{R}^{m\times l}(\mathbf{z})$ represent a causal feedback stabilizable* MIMO *d-D system. Let $P = N_r D_r^{-1}$ be a right* MFD *of P (not necessarily coprime), and $F = \begin{bmatrix} D^\top & N^\top \end{bmatrix}^\top \in \mathbb{R}^{(m+l)\times l}[\mathbf{z}]$. If there exists a unimodular matrix $U \in \mathbb{R}^{(m+l)\times(m+l)}[\mathbf{z}]$ such that some single reduced minor of the polynomial matrix $F_1 = UF$ is devoid of any zeros in the closed unit polydisk, $\overline{\mathbb{D}^d}$, then P has a* DCF *satisfying the Bézout Identity* (4.5).

Corollary 4.7. *Suppose P admits a* DCF. *Then the set of all stabilizing controllers is given by*

$$\begin{aligned} K &= (X_l - QN_l)^{-1}(Y_l + QD_l) && \textit{where } \det(X_l - QN_l) \neq 0 \qquad (4.6)\\ &= (D_rQ + Y_r)(-N_rQ + X_r)^{-1} && \textit{where } \det(-N_rQ + X_r) \neq 0 \qquad (4.7)\\ &\textit{where } Q \in \mathbb{R}_s^{l\times m}(\mathbf{z}) \end{aligned}$$

Proof. For the complete proof, we refer to e.g., [FFGK98, Chapter VII.5]. ■

According to this parametrization of controllers K, the following lemma establishes the connection between the closed loop transfer function and the model matching formulation via the Youla parameter Q.

Corollary 4.8. *Consider the transfer matrix function in* (4.3). *If P is given as in Proposition 4.6 with the set of all feedback stabilizing controllers given by* (4.6) *or* (4.7)*, then*

$$\begin{aligned} \mathcal{H}_{eu} &= \begin{bmatrix} (I+PK)^{-1} & -P(I+KP)^{-1} \\ K(I+PK)^{-1} & (I+KP)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} X_rD_l - N_rQD_l & -N_rX_l + N_rQN_l \\ Y_rD_l + D_rQD_l & D_rX_l - D_rQN_l \end{bmatrix} \end{aligned} \qquad (4.8)$$

Proof.

$$\begin{aligned}
(I+PK)^{-1} &= \left[I + D_l^{-1} N_l \left(D_r Q + Y_r\right)\left(-N_r Q + X_r\right)^{-1}\right]^{-1} \\
&= \left(-N_r Q + X_r\right)\left[\underbrace{N_l Y_r + D_l X_r}_{=I} + \underbrace{\left(N_l D_r - D_l N_r\right)}_{=0} Q\right]^{-1} D_l \\
&= X_r D_l - N_r Q D_l \\
(I+KP)^{-1} &= \left[I + \left(X_l - Q N_l\right)^{-1}\left(Y_l + Q D_l\right) N_r D_r^{-1}\right]^{-1} \\
&= D_r \left[\underbrace{X_l D_r + Y_l N_r}_{=I} + Q \underbrace{\left(D_l N_r - N_l D_r\right)}_{=0}\right]^{-1} \left(X_l - Q N_l\right) \\
&= \left(D_r X_l - D_r Q N_l\right) \\
-P\left(I+KP\right)^{-1} &= -N_r\left(X_l - Q N_l\right) \quad = \left(-N_r X_l + N_r Q N_l\right) \\
K\left(I+PK\right)^{-1} &= \left(D_r Q + Y_r\right) D_l \quad = \left(Y_r D_l + D_r Q D_l\right)
\end{aligned}$$

■

Obviously, each entry in (4.8) is in the form $T_1 - T_2 Q T_3$, i.e., in the model matching form. For example, by letting $T_1 = X_r D_l$, $T_2 = N_r$, and $T_3 = D_l$, the first entry of the closed loop transfer matrix function known as the *sensitivity function:* $S \triangleq X_r D_l - N_r Q D_l$ can be rewritten as $T_1 - T_2 Q T_3$. However, the well-posedness condition (the requirement that the determinants in (4.6) and (4.7) not vanish identically) imposes the condition that $S = X_r D_l - N_r Q D_l = (X_r - N_r Q) D_l$ have determinant not vanishing identically. Similarly, the well-posedness condition forces the (1,2) and (2,2) blocks of (4.8) to have determinants which do not vanish identically. Note also that $X_l, X_r, Y_l, Y_r, N_l, N_r, D_l$ and D_r are all stable by construction. Therefore, without loss of generality, in the next Section, we could assume that all T_i for $i = 1, 2, 3$ are stable.

Remark 10. In fact, if we let $T_1 = \begin{bmatrix} X_r D_l & -N_r X_l \\ Y_r D_l & D_r X_l \end{bmatrix}, T_2 = \begin{bmatrix} -N_r \\ D_r \end{bmatrix}$ and $T_3 = \begin{bmatrix} -D_l & N_l \end{bmatrix}$, then the closed loop transfer matrix function $\mathcal{H}_{eu}$ in (4.8) is in the model matching formulation. Due to the fact that D_r and D_l are assumed to be nonsingular (also square), T_2 and T_3 in this case are nonsquare. Consequently, this problem is a so-called *four-block problem.* In the single-variable case, one may apply spectral factorization to reduce a four-block problem to a one-block Nehari problem (see e.g. [Fra87, Chapter 8]); however it is known that outer spectral factorization is in general not possible in the multivariable setting (see e.g. [Rud69]). ▲

4.3 Equivalence of Model Matching and Interpolation

In the previous Section, we provide a sufficient condition proven by Lin [Lin00] for the existence of a DCF of the original plant $P(\mathbf{z})$, and hence we obtain a set of stabilizing controllers. In addition, at the end of the previous Section, we also show that by using the Youla parameter, Q, one could rewrite the closed loop transfer matrix function $\mathcal{H}_{eu}$ in the model matching form (4.8). Now let F be the performance function, which is affine in Q. Then the goal of this section is to construct the interpolation conditions for F, which is a design parameter rather than Q. Once we solve an interpolation problem to get F, we can backsolve for Q and finally for K, a desired controller. A criterion for internal stability can be expressed directly in terms of F: A controller K is internally stabilizing for the closed loop system if and only if the performance function F is stable and satisfies the appropriate interpolation conditions.

Let us consider the model matching problem in general stated as follows: given stable rational matrix functions T_1, T_2, and T_3 of compatible sizes, find the stable Q so as to achieve

$$\min_{Q} \| T_1 - T_2 Q T_3 \| \tag{4.9}$$

where the norm is the supremum norm over $\mathbb{D}^d$.

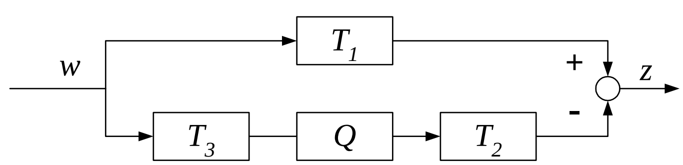

Figure 4.3: The standard model matching framework

Here T_1, T_2 and T_3 are all stable rational matrix function in $\mathbf{z} = (z_1, \ldots, z_d)$ of the appropriate sizes, say, $T_1 \in \mathbb{R}_s^{l\times m}(\mathbf{z})$, $T_2 \in \mathbb{R}_s^{l\times l}(\mathbf{z})$, and $T_3 \in \mathbb{R}_s^{m\times m}(\mathbf{z})$. We shall focus on the so-called *1-block case* (see [Fra87]), i.e., we shall assume that T_2 and T_3 are invertible in $\mathbb{R}^{l\times l}(\mathbf{z})$ and $\mathbb{R}^{m\times m}(\mathbf{z})$, respectively with inverses T_2^{-1} and T_3^{-1} (not necessarily stable) existing and uniformly bounded on the distinguished boundary $\mathbb{T}^d$ of the polydisk.
The performance function F is given by

$$F = T_1 - T_2 Q T_3, \text{ where } Q \in \mathbb{R}^{l\times m}(\mathbf{z}) \tag{4.10}$$

Since T_1, T_2, T_3 are all stable, if $Q \in \mathbb{R}_s^{l\times m}(\mathbf{z})$ (stable rational matrix function), then F is also stable. Conversely, if $F \in \mathbb{R}_s^{l\times m}(\mathbf{z})$, then one can backsolve for Q,

$$Q = T_2^{-1}(T_1 - F)T_3^{-1}. \tag{4.11}$$

Since all quantities on the right hand side of the above expression are bounded on the distinguished boundary $\mathbb{T}^d$, it follows that Q is bounded on $\mathbb{T}^d$; by the maximum modulus principle, it then follows that Q is stable once it is guaranteed that Q is holomorphic on $\mathbb{D}^d$. Since T_2^{-1} and T_3^{-1} may or may not be stable, holomorphicity of F on $\mathbb{D}^d$ does not guarantee holomorphicity of Q on $\mathbb{D}^d$ in general, unless some additional interpolation conditions are imposed on F (see Theorem 4.10). Thus we see that stability for the closed loop system is equivalent to stability of the performance function F together with holomorphicity of the rational matrix function Q given by (4.11). In case the Model Matching Problem arises from the sensitivity minimization problem for an output feedback configuration as sketched in the previous Section, then $l = m$ and we must also impose the well-posedness condition that $\det F$ not vanish identically.

In this Section, for convenience, we shall drop the requirement that Q and F be real and rational; these constraints can always be reincorporated at a later stage. With these relaxations, from the discussion above we see that the stability question, formulated with the performance function F taken as the free parameter, reduces to: *characterize those* $(l \times m)$*-matrix valued functions* F *(with* $\det F$ *not identically equal to 0) for which*

1. *F is holomorphic and uniformly bounded on the polydisk $\mathbb{D}^d$, and*

2. *the function Q given by* (4.11) *is holomorphic on $\mathbb{D}^d$.*

Theorem 4.9. *Suppose that we are given an irreducible polynomial $g(\mathbf{z})$ in $\mathbf{z} = (z_1, \ldots, z_d)$ and that ℓ is a given positive integer. Then a necessary and sufficient condition for a scalar-valued holomorphic function f on the polydisk $\mathbb{D}^d$ to have the form*

$$f(\mathbf{z}) = g(\mathbf{z})^\ell \varphi(\mathbf{z}); \quad \mathbf{z} \in \mathbb{D}^d \tag{4.12}$$

for some scalar-valued function φ holomorphic on $\mathbb{D}^d$ is that f satisfies the interpolation conditions

$$\left. \frac{\partial^{|\mathbf{j}|} f}{\partial \mathbf{z}^{\mathbf{j}}} \right|_{\mathcal{Z}(g)} = 0 \quad \textit{for } |\mathbf{j}| = 0, 1, \ldots, \ell - 1. \tag{4.13}$$

on a generic subset of $\mathcal{Z}(g)$.

Proof. Suppose that f is a scalar-valued holomorphic function on $\mathbb{D}^d$ with a representation of the form $f(\mathbf{z}) = g(\mathbf{z})^\ell \varphi(\mathbf{z})$ for some scalar-valued function φ holomorphic on $\mathbb{D}^d$. Then $f|_{\mathcal{Z}(g)} = 0$. Also, all partial derivatives of f with respect to all variables $z_1, \ldots, z_d$ of order m are equal to zero along $\mathcal{Z}(g)$ for $m = 1, \ldots, \ell - 1$ since each such derivative necessarily contains a factor of $g(\mathbf{z})$. Hence the interpolation conditions (4.13) hold.

Conversely, assume now that f is holomorphic in $\mathbb{D}^d$ and satisfies the interpolation conditions (4.13). Let $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \subset \mathbb{D}^d$ be a neighborhood around a point $\mathbf{z}^0 \in \mathbb{D}^d$ for small $\boldsymbol{\delta} > 0$.

Since f is holomorphic in $\mathbb{D}^d$, for any $\mathbf{z} \in U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$, f admits a multivariable power series representation:

$$f(\mathbf{z}) = f(z_1, \ldots, z_d) = \sum_{|\mathbf{j}|=0}^{\infty} C_{\mathbf{j}}\left(\mathbf{z} - \mathbf{z}^0\right)^{\mathbf{j}} \tag{4.14}$$

where $C_{\mathbf{j}} = C_{\mathbf{j}}(\mathbf{z}^0) = \dfrac{1}{\mathbf{j}!} \left.\dfrac{\partial^{|\mathbf{j}|} f}{\partial \mathbf{z}^{\mathbf{j}}}\right|_{\mathbf{z}=\mathbf{z}^0}$, and $\mathbf{j}! = j_1! \cdots j_d!$.

As we shall vary the point $\mathbf{z}^0$ in this representation, we shall make the dependence of $C_{\mathbf{j}}$ on $C_{\mathbf{j}}(\mathbf{z}^0)$ explicit. Since any partial derivative of a holomorphic function is again holomorphic, $C_{\mathbf{j}}$ is also a holomorphic function since f is.

Recall that for any $z_k \in \mathbb{C}$, $\grave{z}_k = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_d)$. By using this notation, the equation (4.14) may be rewritten as

$$f(\mathbf{z}) = f(\grave{z}_k, z_k) = \sum_{\substack{\grave{j}_k \in \mathbb{N}^{d-1} \\ j_k \in \mathbb{N}}} C_{\grave{j}_k, j_k}(\grave{z}_k^0, z_k^0)(\grave{z}_k - \grave{z}_k^0)^{\grave{j}_k}(z_k - z_k^0)^{j_k}. \tag{4.15}$$

We assume now that $\mathbf{z}^0 = \left(\grave{z}_k^0, z_k^0\right) \in \mathcal{Z}(g) \cap U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$ is a smooth point of $\mathcal{Z}(g)$; i.e., $\mathbf{z}^0 \in \mathcal{Z}(g)$ and $\dfrac{\partial g}{\partial z_j}(\mathbf{z}^0) \neq 0$ for at least one $j = 1, \ldots, d$. Since g is an irreducible polynomial by assumption, the zero variety $\mathcal{Z}(g)$ is also an irreducible subvariety (see Lemma 2.8). Without loss of generality we assume that $j = k$. Then the Implicit Function Theorem 2.5 implies that there exists a holomorphic function h defined on $U(\grave{z}_k^0, \grave{\delta}_k)$ with $h(\grave{z}_k^0) = z_k^0$ so that $g(\grave{z}_k, z_k) = 0$ for a $\grave{z}_k \in U(\grave{z}_k^0, \grave{\delta}_k)$ and z_k in a sufficiently small neighborhood of z_k^0 if and only if $z_k = h(\grave{z}_k)$. This in turn implies that the Weierstrass polynomial for g at $\mathbf{z}^0$ is of degree 1 (see Remark 2 on page 13), and hence g is *irreducible at* $\mathbf{z}^0$ in the sense that g cannot be factored as $g(\mathbf{z}) = g_1(\mathbf{z})g_2(\mathbf{z})$ on any neighborhood $U(\mathbf{z}^0, \boldsymbol{\delta}')$ of $\mathbf{z}^0$ with both g_1 and g_2 holomorphic on $U(\mathbf{z}^0, \boldsymbol{\delta}')$ and with $g_1(\mathbf{z}^0) = g_2(\mathbf{z}^0) = 0$ (see Definition 4 on page 13).

Define $\tilde{g}(\mathbf{z}) = \tilde{g}(\grave{z}_k, z_k) = z_k - h(\grave{z}_k)$ for $\mathbf{z} \in U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$. Then $\mathcal{Z}(\tilde{g}) = \left\{\mathbf{z} \in U\left(\mathbf{z}^0, \boldsymbol{\delta}\right) \middle|\, z_k = h(\grave{z}_k)\right\}$ $= \mathcal{Z}(g) \cap U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$. Since, as was mentioned above, g is irreducible at $\mathbf{z}^0$, by Lemma 2.7 it follows that $\tilde{g}(\mathbf{z}) = g(\mathbf{z})\psi(\mathbf{z})$, where ψ is holomorphic on some neighborhood of $\mathbf{z}^0$, without loss of generality still denoted as $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$.

Since f satisfies the interpolation conditions (4.13), $C_{\grave{0}_k, j_k}(\grave{z}_k^0, z_k^0) = 0$ for $j_k = 0, 1, \ldots, \ell - 1$ for each $\mathbf{z}^0 = (\grave{z}_k^0, z_k^0) \in \mathcal{Z}(g)$. Hence, if we set $(\grave{z}_k^0, z_k^0) = (\grave{z}_k, h(\grave{z}_k))$, then $(\grave{z}_k - \grave{z}_k^0)^{\grave{j}_k} = (\grave{z}_k - \grave{z}_k)^{\grave{j}_k} = 0$ for $0 \neq \grave{j}_k \in \mathbb{N}^{d-1}$ and (4.15) collapses to

$$\begin{aligned} f(\mathbf{z}) = f(\grave{z}_k, z_k) &= (z_k - h(\grave{z}_k))^{\ell} \sum_{j_k=0}^{\infty} C_{\grave{0}_k, j_k+\ell}\left(\grave{z}_k, h(\grave{z}_k)\right)\left(z_k - h(\grave{z}_k)\right)^{j_k} \\ &= \tilde{g}(\mathbf{z})^{\ell}\Phi(\mathbf{z}) \end{aligned} \tag{4.16}$$

where, $\Phi(\mathbf{z}) = \sum_{j_k=0}^{\infty} C_{\grave{0}_k, j_k+\ell}\left(\grave{z}_k, h(\grave{z}_k)\right)\left(z_k - h(\grave{z}_k)\right)^{j_k}$.

Then, $f(\mathbf{z}) = g(\mathbf{z})^\ell \varphi(\mathbf{z})$ on $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$, where $\varphi(\mathbf{z}) = \psi(\mathbf{z})^\ell \Phi(\mathbf{z})$ is holomorphic on $U\left(\mathbf{z}^0, \boldsymbol{\delta}\right)$ since ψ and Φ are. We conclude that $\varphi(\mathbf{z}) = f(\mathbf{z})/g(\mathbf{z})^\ell$, initially defined only on $\mathbb{D}^d \setminus \mathcal{Z}(g)$, has analytic continuation to the generic set of smooth points of $\mathcal{Z}(g)$.

Since g is irreducible, the singular points of $\mathcal{Z}(g)$, consisting of the intersection of the zero sets of $g, \dfrac{\partial g}{\partial z_1}, \cdots, \dfrac{\partial g}{\partial z_d}$, is an algebraic variety of co-dimension at least 2. This set together with the nongeneric set of smooth exceptional points of $\mathcal{Z}(g)$ where the interpolation conditions fail is still contained in an algebraic variety of co-dimension at least 2. By applying the Riemann Extension Theorem 2.9 to φ, we conclude that φ has analytic continuation to all of $\mathbb{D}^d$ and the Theorem follows. ■

Remark 11. In case $g(\mathbf{z}) = z_1$, the interpolation conditions (4.13) can be collapsed to

$$\left.\frac{\partial^j f}{\partial z_1^j}\right|_{\mathcal{Z}(g)} = 0 \text{ for } j = 0, 1, \ldots, \ell - 1. \tag{4.17}$$

Indeed, the vanishing of partial derivatives involving the other variables $z_2, \ldots, z_d$ along $\mathcal{Z}(g) = \{\mathbf{z} \in \mathbb{D}^d \colon z_1 = 0\}$ is automatic from the vanishing of f along $\mathcal{Z}(g)$. More generally, one could do a change of coordinates $\mathbf{z} = (z_1, \ldots, z_d) \mapsto \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ in such a way that $\lambda_1(z) = g(z)$. Then, with respect to these new local coordinates, the interpolation conditions (4.13) can be reduced to

$$\left.\frac{\partial^j f}{\partial \lambda_1^j}\right|_{\boldsymbol{\lambda} \colon \lambda_1 = 0} = 0 \text{ for } j = 0, 1, \ldots, \ell - 1.$$

This is how the criterion for (4.12) is expressed in [DMV00]. In the context here such a change of coordinates is not useful for engineering purposes as it would destroy the rationality of the functions in the interpolation data set. ▲

We now explain the type of interpolation problem to which the model matching problem can be converted in the 1-block case. For $u = 1, \ldots, \eta$, assume that we are given distinct irreducible (scalar) polynomials q_u with zero variety $\mathcal{Z}(q_u)$ having nontrivial intersection with $\mathbb{D}^d$, meromorphic matrix functions G_u and $\widetilde{G}_u$ (of compatible sizes for the interpolation conditions to follow to make sense) with polar divisor not including $\mathcal{Z}(q_u)$, and positive integers k_u. For $v = 1, \ldots, \mu$ assume that similarly we are given distinct irreducible polynomials s_v together with meromorphic matrix functions H_v and $\widetilde{H}_v$ (of compatible sizes) with polar divisor not including $\mathcal{Z}(s_v)$, and positive integers ℓ_v. For each pair of indices (u, v) for which $q_u = s_v =: h_{u,v}$, assume that we are given an additional matrix function R_{uv} meromorphic on a neighborhood of each point of $\mathcal{Z}(h_{u,v})$. The whole aggregate

$$\mathcal{D} = \{q_u, G_u, \widetilde{G}_u, k_u; s_v, H_v, \widetilde{H}_v, \ell_v; R_{uv}\} \tag{4.18}$$

we call a *1-block interpolation data set.*

We say that an $l \times m$ matrix-valued function F holomorphic on $\mathbb{D}^d$ satisfies the interpolation conditions associated with $\mathcal{D}$ (denoted by $F \in \mathcal{I}(\mathcal{D})$) if

$$\left\{ \frac{\partial^{|\mathbf{i}|}}{\partial \mathbf{z}^{\mathbf{i}}} G_u(\mathbf{z}) F(\mathbf{z}) \right\}\Bigg|_{\mathcal{Z}(q_u)} = \left\{ \frac{\partial^{|\mathbf{i}|}}{\partial \mathbf{z}^{\mathbf{i}}} \widetilde{G}_u(\mathbf{z}) \right\}\Bigg|_{\mathcal{Z}(q_u)} \quad \text{generically on } \mathcal{Z}(q_u),$$

$$\text{for } u = 1, \ldots, \eta \text{ and } |\mathbf{i}| = 0, 1, \ldots, k_u - 1, \tag{4.19}$$

$$\left\{ \frac{\partial^{|\mathbf{j}|}}{\partial \mathbf{z}^{\mathbf{j}}} F(\mathbf{z}) H_v(\mathbf{z}) \right\}\Bigg|_{\mathcal{Z}(s_v)} = \left\{ \frac{\partial^{|\mathbf{j}|}}{\partial \mathbf{z}^{\mathbf{j}}} \widetilde{H}_v(\mathbf{z}) \right\}\Bigg|_{\mathcal{Z}(s_v)} \quad \text{generically on } \mathcal{Z}(s_v),$$

$$\text{for } v = 1, \ldots, \mu \text{ and } |\mathbf{j}| = 0, 1, \ldots, \ell_v - 1, \text{ and} \tag{4.20}$$

$$\left\{ \frac{\partial^{|\mathbf{l}|}}{\partial \mathbf{z}^{\mathbf{l}}} G_u(\mathbf{z}) F(\mathbf{z}) H_v(\mathbf{z}) \right\}\Bigg|_{\mathcal{Z}(h_{u,v})} = \left\{ \frac{\partial^{|\mathbf{l}|}}{\partial \mathbf{z}^{\mathbf{l}}} R_{u,v}(\mathbf{z}) \right\}\Bigg|_{\mathcal{Z}(h_{u,v})} \quad \text{generically on } \mathcal{Z}(h_{u,v})$$

$$\text{for all pairs of indices } (u, v) \text{ with } q_u = s_v \text{ and for } |\mathbf{l}| = 0, 1, \ldots, k_u + \ell_v - 1 \tag{4.21}$$

Given T_1, T_2 and T_3 (of respective sizes $l \times m$, $l \times l$ and $m \times m$, say) as in the 1-block case of the model matching problem, we associate an interpolation data set $\mathcal{D}$ as follows. Write the $l \times l$ rational matrix valued function $T_2^{-1}(\mathbf{z})$ as $T_2^{-1}(\mathbf{z}) = \left[\frac{p_{ij}}{q_{ij}}(\mathbf{z}) \right]_{i,j=1,\ldots,l}$, and consider the set of unstable entries of T_2^{-1}, say $\left\{ \frac{p_{i_a,j_a}}{q_{i_a,j_a}}(\mathbf{z}) \right\}$ for $a = 1, \ldots, \alpha$. Let $q(\mathbf{z})$ be the least common multiple (or, l.c.m.) of $\{q_{i_1,j_1}(\mathbf{z}), \ldots, q_{i_\alpha,j_\alpha}(\mathbf{z})\}$. Also write the $m \times m$ rational matrix valued function $T_3^{-1}(\mathbf{z})$ as $T_3^{-1}(\mathbf{z}) = \left[\frac{r_{ij}}{s_{ij}}(\mathbf{z}) \right]_{i,j=1,\ldots,m}$, and consider the set of unstable entries of T_3^{-1}, say $\left\{ \frac{r_{i_b,j_b}}{s_{i_b,j_b}}(\mathbf{z}) \right\}$ for $b = 1, \ldots, \beta$. Let $s(\mathbf{z})$ be the l.c.m. of $\{s_{i_1,j_1}(\mathbf{z}), \ldots, s_{i_\beta,j_\beta}(\mathbf{z})\}$.

Suppose now that $q(\mathbf{z})$ and $s(\mathbf{z})$, respectively, can be factored into irreducible polynomials, say $q(\mathbf{z}) = q_1^{k_1}(\mathbf{z}) \cdots q_\eta^{k_\eta}(\mathbf{z})$, where $k_u > 0$ for $u = 1, \ldots, \eta$, and $s(\mathbf{z}) = s_1^{\ell_1}(\mathbf{z}) \cdots s_\mu^{\ell_\mu}(\mathbf{z})$, where $\ell_v > 0$ for $v = 1, \ldots, \mu$,. Then for each $u \in \{1, \ldots, \eta\}$, $T_2^{-1}(\mathbf{z}) = \frac{G_u}{q_u^{k_u}}(\mathbf{z})$, where $G_u(\mathbf{z})$ is a meromorphic matrix function in $\mathbb{D}^d$ with polar divisor not including $\mathcal{Z}(q_u)$, and $q_u^{k_u}$ is an unstable irreducible polynomial with multiplicity k_u. In addition we set $\widetilde{G}_u(\mathbf{z}) = G_u(\mathbf{z}) T_1(\mathbf{z})$, so $\widetilde{G}_u(\mathbf{z})$ is also meromorphic with polar divisor not including $\mathcal{Z}(q_u)$. Analogously, for each $v \in \{1, \ldots, \mu\}$, $T_3^{-1}(\mathbf{z}) = \frac{H_v}{s_v^{\ell_v}}(\mathbf{z})$, where $H_v(\mathbf{z})$ is a meromorphic matrix function on $\mathbb{D}^d$ with polar divisor not including $\mathcal{Z}(s_v)$. Set $\widetilde{H}_v(\mathbf{z}) = T_1(\mathbf{z}) H_v(\mathbf{z})$, so $\widetilde{H}_v(\mathbf{z})$ is meromorphic with polar divisor not including $\mathcal{Z}(s_v)$. In addition, if q and s have some common factors, say $q_u = s_v$ for some pair of indices u and v, set $h_{u,v} = q_u = s_v$ and $R_{u,v}(\mathbf{z}) = G_u(\mathbf{z}) T_1(\mathbf{z}) H_v(\mathbf{z})$, so $R_{u,v}$ is meromorphic with polar divisor not including $\mathcal{Z}(h_{u,v})$. In this way we have formed an interpolation data set $\mathcal{D}$ as in (4.18). When $\mathcal{D}$ is formed in this way from T_1, T_2, T_3, let us write

$\mathcal{D} = \mathcal{D}_{T_1,T_2,T_3}$.

Now we are ready to state the main theorem, which gives the connection between the model matching and interpolation problems.

Theorem 4.10. *Let T_1, T_2, T_3 be the data set for a 1-block model matching problem, and let $\mathcal{D}_{T_1,T_2,T_3}$ be the associated interpolation data set as delineated in the previous paragraph. Then a necessary and sufficient condition for a given function F holomorphic on $\mathbb{D}^d$ to have the model matching form $F = T_1 - T_2QT_3$ for a stable Q is that F satisfy the interpolation conditions* (4.19), (4.20) *and* (4.21) *associated with the data set $\mathcal{D}_{T_1,T_2,T_3}$ (i.e., $F \in \mathcal{I}(\mathcal{D}_{T_1,T_2,T_3})$).*

Proof. Assume that F has the model matching form; i.e.,

$$F = T_1 - T_2QT_3 \;\text{ for a stable } Q \tag{4.22}$$

Suppose now that the index u is chosen so that q_u is not equal to any s_v. Then we can write (4.22) as

$$\begin{aligned} Q(\mathbf{z})T_3(\mathbf{z}) &= T_2^{-1}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z}) \\ &= \frac{G_u}{q_u^{k_u}}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z}) \end{aligned} \tag{4.23}$$

and hence $G_u(\mathbf{z})\,(T_1 - F)\,(\mathbf{z}) = q_u^{k_u}(\mathbf{z})Q(\mathbf{z})T_3(\mathbf{z})$. Application of a localized version of Theorem 4.9 entrywise leads to the interpolation conditions (4.19) holding on $\mathcal{Z}(g_u) \setminus (\mathcal{Z}(g_u) \cap \mathcal{P}(G_u))$, a generic subset of $\mathcal{Z}(g_u)$.

Similarly, suppose that the index v is such that s_v is not equal to any q_u. Then

$$\begin{aligned} T_2(\mathbf{z})Q(\mathbf{z}) &= (T_1 - F)\,(\mathbf{z})T_3^{-1}(\mathbf{z}) \\ &= (T_1 - F)\,(\mathbf{z})\frac{H_v}{s_v^{\ell_v}}(\mathbf{z}) \end{aligned} \tag{4.24}$$

and hence $(T_1 - F)\,(\mathbf{z})H_v(\mathbf{z}) = s_v^{\ell_v}(\mathbf{z})T_2(\mathbf{z})Q(\mathbf{z})$. Again, application of Theorem 4.9 entrywise leads to the interpolation conditions (4.20) holding on $\mathcal{Z}(s_v) \setminus (\mathcal{Z}(s_v) \cap \mathcal{P}(H_v))$.

Suppose finally that the indices (u, v) are such that $q_u = s_v =: h_{uv}$. Then

$$\begin{aligned} Q(\mathbf{z}) &= T_2^{-1}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})T_3^{-1}(\mathbf{z}) \\ &= \frac{G_u}{q_u^{k_u}}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})\frac{H_v}{s_v^{\ell_v}}(\mathbf{z}) \\ &= \frac{1}{h_{u,v}^{k_u+\ell_v}}G_u(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})H_v(\mathbf{z}) \end{aligned} \tag{4.25}$$

and hence $G_u(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})H_v(\mathbf{z}) = h_{u,v}^{k_u+\ell_v}(\mathbf{z})Q(\mathbf{z})$. Apply Theorem 4.9 entrywise to obtain

(4.21) holding on $\mathcal{Z}(h_{u,v}) \setminus (\mathcal{Z}(h_{u,v}) \cap (\mathcal{P}(G_u) \cup \mathcal{P}(H_v)))$.

Conversely, suppose that F satisfies the interpolation conditions (4.19), (4.20), and (4.21). Then Theorem 4.9 implies that:

1. For each u such that q_u is not equal to any s_v, by (4.19) we have

$$\begin{aligned}\widetilde{G}_u(\mathbf{z}) - G_u(\mathbf{z})F(\mathbf{z}) &= G_u(\mathbf{z})\,(T_1 - F)\,(\mathbf{z}) \\ &= q_u^{k_u} Q_u(\mathbf{z}) \quad \text{for some } Q_u \text{ holomorphic on } \mathcal{Z}(q_u) \setminus (\mathcal{Z}(q_u) \cap \mathcal{P}(G_u)).\end{aligned}$$

   Note that $T_2^{-1}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z}) = Q_u(\mathbf{z})$ since $T_2^{-1} = \dfrac{G_u}{q_u^{k_u}}$.
   Hence $T_2^{-1}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})T_3^{-1}(\mathbf{z}) = Q_u(\mathbf{z})T_3^{-1}(\mathbf{z})$ is holomorphic on $\mathcal{Z}(q_u) \setminus (\mathcal{Z}(q_u) \cap (\mathcal{P}(G_u) \cup \mathcal{P}(T_3^{-1})))$. This last set is a generic subset of $\mathcal{Z}(q_u)$ since $\mathcal{P}(T_3^{-1})$ does not include $\mathcal{Z}(q_u)$ since we are in the case where q_u is not equal to any s_v.

2. For each v such that s_v is not equal to any q_u, then (4.20) implies that

$$\begin{aligned}\widetilde{H}_v(\mathbf{z}) - F(\mathbf{z})H_v(\mathbf{z}) &= (T_1 - F)\,(\mathbf{z})H_v(\mathbf{z}) \\ &= s_v^{\ell_v} Q_v(\mathbf{z}) \quad \text{for some } Q_v \text{ holomorphic on } \mathcal{Z}(s_v) \setminus (\mathcal{Z}(s_v) \cap \mathcal{P}(H_v)).\end{aligned}$$

   Note that $(T_1 - F)\,(\mathbf{z})T_3^{-1}(\mathbf{z}) = Q_v(\mathbf{z})$ since $T_3^{-1} = \dfrac{H_v}{s_v^{\ell_v}}$.
   Hence $T_2^{-1}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})T_3^{-1}(\mathbf{z}) = T_2^{-1}(\mathbf{z})Q_v(\mathbf{z})$ is holomorphic on $\mathcal{Z}(s_v) \setminus (\mathcal{Z}(s_v) \cap (\mathcal{P}(H_v) \cup \mathcal{P}(T_2^{-1})))$. This last set is still generic in $\mathcal{Z}(s_v)$ since we are here in the case where s_v is not equal to any q_u.

3. For each pair (u, v) such that $q_u = s_v =: h_{u,v}$, then (4.21) implies that

$$\begin{aligned}R_{u,v}(\mathbf{z}) - G_u(\mathbf{z})F(\mathbf{z})H_v(\mathbf{z}) &= G_u(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})H_v(\mathbf{z}) \\ &= h_{u,v}^{k_u+\ell_v} Q_{u,v}(\mathbf{z}) \quad \text{for some } Q_{u,v} \text{ holomorphic on} \\ &\qquad \mathcal{Z}(h_{u,v}) \setminus (\mathcal{Z}(h_{u,v}) \cap (\mathcal{P}(G_u) \cup \mathcal{P}(H_v))).\end{aligned}$$

   Hence $T_2^{-1}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})T_3^{-1}(\mathbf{z}) = Q_{u,v}(\mathbf{z})$.

4. For any points $\mathbf{z}^0$ not in any $\mathcal{Z}(q_u)$ nor in any $\mathcal{Z}(s_v)$ for $u = 1, \dots, \eta$, and for $v = 1, \dots, \mu$, $T_2^{-1}(\mathbf{z}), T_3^{-1}(\mathbf{z})$, and $(F - T_1)(\mathbf{z})$ are holomorphic at $\mathbf{z}^0$. This implies that $T_2^{-1}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})T_3^{-1}(\mathbf{z})$ is holomorphic at any such $\mathbf{z}^0$.

We conclude that $Q := T_2^{-1}(\mathbf{z})\,(T_1 - F)\,(\mathbf{z})T_3^{-1}(\mathbf{z})$ is holomorphic on a set of the form $\mathbb{D}^d \setminus A$ where the exceptional set A is contained in an algebraic variety of co-dimension at least 2. Another application of the Riemann Extension Theorem 2.9 gives us that Q is analytic on all of $\mathbb{D}^d$, and hence $F = T_1 - T_2QT_3$ has the desired model-matching form. This completes the proof of Theorem 4.10. ■

Remark 12. Theorem 4.10 is a multivariable analogue of Theorem 16.9.3 in [BGR90]. To be consistent with the terminology given in [BGR90], the equations in (4.19), (4.20), and (4.21) are called respectively the left-, right- and two-sided interpolation conditions for a tangential interpolation problem. The proof for the single-variable case relies heavily in the end on the existence of a local Smith-McMillan form for rational matrix functions; as the Smith-McMillan form is unavailable in the multivariable setting, our proof here relies exclusively on making use of the notion of zero coprime. ▲

Example 7. Suppose $P(z_1, z_2, z_3)$ is given by

$$P = \begin{bmatrix} 1 & 0 \\ \frac{z_2}{z_1+0.5} & \frac{z_3+2}{z_1+0.5} \end{bmatrix}.$$

Then Lin (see [Lin99]) has shown that there exists a set of stabilizing controllers K satisfying the Bézout identity (4.5), where

$$D_l = \begin{bmatrix} 1 & 0 \\ 0 & z_1+0.5 \end{bmatrix}, \quad D_r = \begin{bmatrix} z_3+2 & 0 \\ -z_2 & z_1+0.5 \end{bmatrix}, \quad N_l = \begin{bmatrix} 1 & 0 \\ z_2 & z_3+2 \end{bmatrix}, \quad N_r = \begin{bmatrix} z_3+2 & 0 \\ 0 & z_3+2 \end{bmatrix}$$

$$X_l = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_r = \begin{bmatrix} -(z_3+2) & 0 \\ z_2 & -1 \end{bmatrix}, \quad Y_l = \frac{1}{z_3+2}\begin{bmatrix} z_3+3 & 0 \\ -z_2 & z_1+1.5 \end{bmatrix}$$

and $$Y_r = \frac{1}{z_3+2}\begin{bmatrix} (z_3+3)(z_3+2) & 0 \\ -z_2(z_3+z_1+3.5) & z_1+1.5 \end{bmatrix}$$

Suppose we are interested in the sensitivity function $S = (I+PK)^{-1}$. By Corollary 4.8, the model matching problem associated with performance function equal to S is given by

$$S = X_rD_l - N_rQD_l = T_1 - T_2QT_3 \tag{4.26}$$

where

$$T_1 = X_rD_l = \begin{bmatrix} -(z_3+2) & 0 \\ z_2 & -(z_1+0.5) \end{bmatrix},$$

$$T_2 = N_r = \begin{bmatrix} z_3+2 & 0 \\ 0 & z_3+2 \end{bmatrix} \Rightarrow \quad T_2^{-1} = \begin{bmatrix} \frac{1}{z_3+2} & 0 \\ 0 & \frac{1}{z_3+2} \end{bmatrix},$$

$$T_3 = D_l = \begin{bmatrix} 1 & 0 \\ 0 & z_1+0.5 \end{bmatrix} \Rightarrow \quad T_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{z_1+0.5} \end{bmatrix}.$$

Since only T_3^{-1} contains an unstable polar variety with polar variety $\{\mathbf{z} : z_1 = -0.5\}$ disjoint from $\mathbb{T}^3$, we need to impose the right interpolation condition (4.20) on S. To be precise, we can rewrite (4.26) as $(T_1 - S)\,T_3^{-1} = T_2 Q$, or

$$(T_1 - S)\begin{bmatrix}(z_1+0.5) & 0\\ 0 & 1\end{bmatrix} = (z_1+0.5)T_2Q. \tag{4.27}$$

Then the interpolation condition (4.20) implies that

$$S|_{\mathcal{Z}(f)}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \underline{0}, \text{ where } f = z_1 + 0.5. \tag{4.28}$$

In particular, $S = \begin{bmatrix}z_1+0.5 & 0\\ 0 & z_1+0.5\end{bmatrix}$ satisfies the above condition. Note also that $\det S \neq 0$ so the well-posedness condition (4.6) or (4.7) is satisfied. One can next backsolve for Q as

$$\begin{aligned} Q &= T_2^{-1}(T_1 - S)T_3^{-1}\\ &= \frac{1}{z_3+2}\begin{bmatrix}-(z_1+z_3+2.5) & 0\\ z_2 & -2\end{bmatrix} \end{aligned} \tag{4.29}$$

Obviously, Q is holomorphic on $\mathbb{D}^3$ as required. Then one can solve back for a compensator K via $K = P^{-1}(S^{-1} - I)$, i.e.

$$K = \tfrac{z_1-0.5}{z_1+0.5}\begin{bmatrix}-1 & 0\\ \frac{z_2}{(z_3+2)} & -\frac{(z_1+0.5)}{(z_3+2)}\end{bmatrix}. \qquad \diamondsuit$$

Example 8. Let $T_3(z)^{-1} = \begin{bmatrix}\frac{1}{z_1^2} & \frac{1}{z_2+2}\\ \frac{z_3}{z_1z_2-0.5} & \frac{1}{z_1}\end{bmatrix}$ Obviously, the set of unstable entries of T_3^{-1} is $\left\{\frac{1}{z_1^2}, \frac{z_3}{z_1z_2-0.5}, \frac{1}{z_1}\right\}$ with polar variety having empty intersection with $\mathbb{T}^3$. Let $q(\mathbf{z})$ = l.c.m. of $\{z_1^2,\, z_1z_2 - 0.5,\, z_1\} = z_1^2\,(z_1z_2 - 0.5)$. Set $q_1 = z_1$ with multiplicity 2, and $q_2 = z_1z_2 - 0.5$. Then, the corresponding G_1 and G_2, respectively are given by

$$\begin{aligned} G_1(\mathbf{z}) &= \begin{bmatrix}1 & \frac{z_1^2}{z_2+2}\\ \frac{z_1^2z_3}{z_1z_2-0.5} & z_1\end{bmatrix},\\ \text{and}\quad G_2(\mathbf{z}) &= \begin{bmatrix}\frac{z_1z_2-0.5}{z_1^2} & \frac{z_1z_2-0.5}{z_2+2}\\ z_3 & \frac{z_1z_2-0.5}{z_1}\end{bmatrix}. \end{aligned}$$

Consider first when $q_1 = z_1$, then $G_1(\mathbf{z})$ can be written as $G_1(\mathbf{z}) = z_1^2\tilde{Q}(\mathbf{z})$ and the zero variety $\mathcal{Z}(q_1) = \left\{ (z_1,\, z_2,\, z_3) \in \mathbb{D}^3 \middle|\, z_1 = 0 \right\}$. Then the interpolation conditions are:

$$\left.(T_1 - F)(\mathbf{z})\, G_1(\mathbf{z})\right|_{\mathcal{Z}(q_1)} = \left.(T_1 - F)(\mathbf{z})\right|_{\mathcal{Z}(q_1)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \underline{0} \tag{4.30}$$

$$\begin{aligned} \left.\frac{\partial}{\partial z_1}\{(T_1 - F)(\mathbf{z})\, G_1(\mathbf{z})\}\right|_{\mathcal{Z}(q_1)} &= \frac{\partial}{\partial z_1}\left.(T_1 - F)(\mathbf{z})\right|_{\mathcal{Z}(q_1)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \left.(T_1 - F)(\mathbf{z})\right|_{\mathcal{Z}(q_1)} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \underline{0} \end{aligned} \tag{4.31}$$

When $q_2 = z_1 z_2 - 0.5$, then $G_2(\mathbf{z})$ can be written as $G_2(\mathbf{z}) = (z_1 z_2 - 0.5)\tilde{Q}(\mathbf{z})$ and the zero variety $\mathcal{Z}(q_2) = \left\{ (z_1,\, z_2,\, z_3) \in \mathbb{D}^3 \middle|\, z_1 z_2 = 0.5 \right\}$. Then the interpolation condition is:

$$\left.(T_1 - F)(\mathbf{z})\, G_2(\mathbf{z})\right|_{\mathcal{Z}(q_2)} = \left.(T_1 - F)(\mathbf{z})\right|_{\mathcal{Z}(q_2)} \begin{bmatrix} 0 & 0 \\ z_3 & 0 \end{bmatrix} = \underline{0} \tag{4.32}$$

Hence, the content of Theorem 4.10 for this example is: $F \in \mathbb{R}_s^{l \times m}(\mathbf{z})$ *satisfies the interpolation conditions* (4.30)–(4.32) *if and only if* $Q = T_2^{-1}(T_1 - F)T_3^{-1}$ *is holomorphic on* $\mathbb{D}^d$. ◊

Remark 13. If one loosens the 1-block assumption on (T_1, T_2, T_3), the model matching form for F is equivalent to interpolation conditions for F on subvarieties of other co-dimensions, including the possibility of interpolation conditions at isolated points, or, at the opposite extreme, interpolation conditions on the whole of $\mathbb{D}^d$. For the single-variable case ($d = 1$), there are only the two possibilities of co-dimension equal to 1 or to 0, i.e. interpolation at isolated points or interpolation along the whole unit disk—see [BR92, BR94] for a thorough treatment. ▲

Remark 14. We now consider the special case where $f(\mathbf{z}) = g(\mathbf{z})^{\ell}\varphi(\mathbf{z})$ where $\ell = 1$. Then the interpolation conditions (4.19) – (4.21) simplify to

$$\left.G_u(\mathbf{z})\big(T_1(\mathbf{z}) - F(\mathbf{z})\big)\right|_{\mathcal{Z}(q_u)} = 0 \text{ for } u = 1, \ldots, \eta, \tag{4.33}$$

$$\left.\big(T_1(\mathbf{z}) - F(\mathbf{z})\big)H_v(\mathbf{z})\right|_{\mathcal{Z}(s_v)} = 0 \text{ for } v = 1, \ldots, \mu, \text{ and} \tag{4.34}$$

$$\left.\frac{\partial^{\ell}}{\partial z_i^{\ell}}\left[G_u(\mathbf{z})\big(T_1(\mathbf{z}) - F(\mathbf{z})\big)H_v(\mathbf{z})\right]\right|_{\mathcal{Z}(h_{u,v})} = 0 \text{ for } i = 1, \ldots, d; \ell = 0, 1, \text{ and}$$

$$\text{for all pairs of indices } (u, v) \text{ with } q_u = s_v \tag{4.35}$$

Note that all these formulations of interpolation conditions depend heavily on a particular choice of coordinates for the various varieties $\mathcal{Z}(q_u)$ and $\mathcal{Z}(s_v)$. It is of interest to note that conditions (4.33) and (4.34) can be expressed in a more coordinate-free form by using the Poincaré residue map (see [GH78, page 147]). Indeed, in connection with (4.33) e.g., application of the Poincaré residue map to the d-form

$$T_2(\mathbf{z})^{-1}(T_1(\mathbf{z}) - F(\mathbf{z}))\, dz_1 \wedge \cdots \wedge dz_d = \frac{G_u(\mathbf{z})}{q_u(\mathbf{z})}(T_1(\mathbf{z}) - F(\mathbf{z}))\, dz_1 \wedge \cdots \wedge dz_d$$

yields the $(d-1)$-form on $\mathcal{Z}(q_u)$

$$(-1)^{i-1} G_u(\mathbf{z})(T_1(\mathbf{z}) - F(\mathbf{z})) \frac{dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_d}{\partial q_u / \partial z_i}\Bigg|_{\mathcal{Z}(q_u)}$$

(where the notation $\widehat{dz_i}$ indicates that the i-th term is omitted) for any i such that $\dfrac{\partial q_u}{\partial z_i} \neq 0$. Thus the interpolation condition (4.33) on F can be expressed as the vanishing of the Poincaré residue (a $d-1$ form) of the d-form $T_2(\mathbf{z})^{-1}(T_1(\mathbf{z}) - F(\mathbf{z}))\, dz_1 \wedge \cdots \wedge dz_d$ along the variety $\mathcal{Z}(q_u)$. ▲

4.4 The Nevanlinna-Pick Interpolation Problem on the Polydisk

We consider the following d-D version of the bitangential Nevanlinna-Pick interpolation problem, (or, NPIP) (for the 1D version, see e.g. [BGR90, Dym89]): *given an interpolation data set (or, ID set) $\mathcal{D}$ as in* (4.18)*, find an $l \times m$ matrix-valued function F holomorphic on $\mathbb{D}^d$ satisfying the interpolation conditions* (4.19), (4.20), (4.21) *($F \in \mathcal{I}(\mathcal{D})$) for which in addition*

$$\sup_{\mathbf{z} \in \mathbb{D}^d} \|F(\mathbf{z})\| \leq 1, \tag{4.36}$$

i.e., find $F \in \mathcal{I}(\mathcal{D}) \cap \mathcal{S}_d(\mathbb{C}^m, \mathbb{C}^l)$.

For $d > 2$, it turns out that the norm constraint (4.36) is not so convenient to work with, and hence we shall relax the norm constraint (4.36) to

$$\sup\{\|F(T_1, \ldots, T_d\|\} \leq 1, \tag{4.37}$$

where the supremum is over any d-tuple of commuting strict contractions $(T_1, \ldots, T_d)$ on some Hilbert space $\mathcal{H}$ (see also Section 2.3), and work with the Schur-Agler class rather than seeking a function in the Schur class. For a discussion on the Schur class and the Schur-Agler class, readers should refer to Section 2.3.

The bitangential NPIP for the class $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l)$ can be stated as follows: *given an ID set $\mathcal{D}$ as in* (4.18)*, find an $l \times m$ matrix–valued function F holomorphic on $\mathbb{D}^d$ satisfying the interpolation conditions* (4.19)*,* (4.20)*,* (4.21) *which in addition satisfies* (4.37), i.e., we seek $F \in \mathcal{I}(\mathcal{D}) \cap \mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l)$.

We have seen from the Section 2.3 and the remarks above that the Schur-Agler-modified bitangential NPIP ($\mathcal{I}(\mathcal{D})$ with (4.37)) is exactly the same as the d-D bitangential NPIP given above ($\mathcal{I}(\mathcal{D})$ with (4.36)) in case $d = 1, 2$; while, for $d > 2$, a necessary and sufficient condition for solving the Schur-Agler variant gives only a sufficient condition for solving the original version. We shall next discuss results concerning the Schur-Agler version of the bitangential NPIP.

For the statement of the next result we need one more piece of terminology. For Ω any set and P a function defined on $\Omega \times \Omega$ with value $P(\omega', \omega)$ at $(\omega', \omega) \in \Omega \times \Omega$ equal to an operator from the Hilbert space $\mathcal{K}_\omega$ to the Hilbert space $\mathcal{K}_{\omega'}$ (i.e., $P : \Omega \times \Omega \to \mathcal{L}(\mathcal{K}_\omega, \mathcal{K}_{\omega'})$), we say that P is a *positive kernel* if for any choice of N points, say $\omega_1, \dots, \omega_N \in \Omega$, and of N vectors $x_1, \dots, x_N$ with $x_i \in \mathcal{K}_{\omega_i}$ for $i = 1, \dots, N$ (where N is any finite number)

$$\sum_{i,j=1}^{N} \langle P(\omega_i, \omega_j) x_j, x_i \rangle_{\mathcal{K}_{\omega_i}} \geq 0.$$

Such objects are closely connected with the theory of reproducing kernel Hilbert spaces (see e.g. [Dym89]). It is well known that an equivalent condition for a given $P(\omega', \omega)$ as above to be a positive kernel is that there be an auxiliary Hilbert space $\mathcal{H}$ and an operator-valued function $\omega \to T(\omega)$ on Ω, where the value $T(\omega)$ at $\omega \in \Omega$ is an operator from $\mathcal{H}$ into $\mathcal{K}_\omega$, such that we have the factorization

$$P(\omega', \omega) = T(\omega')T(\omega)^*.$$

More concretely, one can view an operator-valued function $P(\cdot, \cdot)$ as above as an infinite block-matrix, with rows and columns indexed by the arbitrary (possibly countably infinite or uncountably infinite) set Ω.

For example, let $M = \left[M^{i,j}\right]_{i,j=1,\dots,N}$ be a finite block-operator matrix with matrix entries equal to operators from $\mathcal{K}_\omega$ to $\mathcal{K}_{\omega'}$. We say that M is a *positive-semidefinite matrix* if the following condition holds:

$$\sum_{i,j=1}^{N} \langle M^{i,j} x_j, x_i \rangle_{\mathcal{K}_{\omega_i}} \geq 0 \quad \text{for all } x_1, \dots, x_N \text{ with } x_i \in \mathcal{K}_{\omega_i} \ \text{ for all } N = 1, 2, \dots .$$

If we set $\Omega = \{1, 2, \dots, N\}$ and set $P(\omega_i, \omega_j) = M^{i,j}$, then we see that P is a positive kernel exactly when M is a positive-semidefinite matrix. Thus the condition that P be a positive kernel can then be viewed as an infinite analogue of a positive-semidefinite matrix.

In case no q_u is also an s_v (so the third set of interpolation conditions (4.21) is vacuous) and the multiplicities k_u and ℓ_v are all equal to 1, we have the following solution of the Schur-Agler variant of the bitangential NPIP from [BB] (see also [BT98] for the case of interpolation along finitely many points).

Theorem 4.11. *Suppose we are given an* ID *set* (4.18) *such that, for all* $u = 1, \dots, \eta$ *and*

$v = 1, \dots, \mu$, $q_u \neq s_v$ *(so the interpolation condition* (4.21) *is vacuous),* $k_u = 1$ *and* $\ell_v = 1$*. Then there exists a matrix-valued function holomorphic on* $\mathbb{D}^d$ *satisfying the interpolation conditions* (4.19) *and* (4.20) *together with the norm constraint* (4.36) *if and only if there exists d positive kernels* $P_1, \dots, P_d$*, where*

$$P_j(\omega', \omega)\colon \Omega \times \Omega \to \begin{cases} \mathbb{C}^{l\times l}, & \text{if } \omega' \in \mathcal{Z}(q_{u'}), \omega \in \mathcal{Z}(q_u) \text{ for some } u', u; \\ \mathbb{C}^{l\times m}, & \text{if } \omega' \in \mathcal{Z}(q_{u'}), \omega \in \mathcal{Z}(s_v) \text{ for some } u', v; \\ \mathbb{C}^{m\times m}, & \text{if } \omega' \in \mathcal{Z}(s_{v'}), \omega \in \mathcal{Z}(s_v) \text{ for some } v', v; \end{cases} \tag{4.38}$$

satisfying the equation

$$\sum_{k=1}^{d} \left[M_k(\omega')^* P_k(\omega', \omega) M_k(\omega) - N_k(\omega')^* P_k(\omega', \omega) N_k(\omega) \right] = X(\omega')^* X(\omega) - Y(\omega')^* Y(\omega) \tag{4.39}$$

for all $\omega', \omega \in \Omega$*, where*

$$\Omega := \left(\bigcup_{u=1}^{\eta} \mathcal{Z}(q_u) \right) \bigcup \left(\bigcup_{v=1}^{\mu} \mathcal{Z}(s_v) \right)$$

and where

$$M_k(\omega) = I_l, \qquad N_k(\omega) = \overline{\omega_k} I_l, \qquad X(\omega) = G_u(\omega)^*, \qquad Y(\omega) = \widetilde{G}_u(\omega)^* \tag{4.40}$$

in case $\omega \in \mathcal{Z}(q_u)$ *for some* $u = 1, \dots, \eta$*,*

$$M_k(\omega) = \omega_k I_m, \qquad N_k(\omega) = I_m, \qquad X(\omega) = \widetilde{H}_v(\omega), \qquad Y(\omega) = H_v(\omega) \tag{4.41}$$

in case $\omega \in \mathcal{Z}(s_v)$ *for some* $v = 1, \dots, \mu$*.*

Remark 15. Were it the case that the set Ω in Theorem 4.11 were finite, then the problem of solving (4.39) for d positive-semidefinite matrices $P_1, \dots, P_d$ would be a particular instance of a semidefinite programming subject to a Linear Matrix Inequality (LMI) constraint for which much research and software is now well developed. A thorough survey of semidefinite programming and its applications can be found in, e.g. [BGFB94, GN00] and the references therein. It should be noted also that there are several commercial and non-commercial software packages which allow users to represent LMI problems with the higher-level language and to interface with MATLAB program, for instance, LMILAB [GN93], LMITOOL [GDN95], SDPSOL [WB96], and SDPHA [PSB97]. There does not appear to be much experience developed with infinite LMIs such as (4.39). ▲

Remark 16. A basic result in the theory of holomorphic functions of several complex variables is the following special case of the work of H. Cartan on the sheaf cohomology on Stein domains (see [Car67] or [GR65, page 245], [Hör73, Chapter 7] or [HL84, Theorem 4.11.1] for more modern treatments): *if $\mathcal{Z}$ is an analytic variety in a domain of holomorphy Ω and if f is a (complex-valued) holomorphic function on V, then there is a holomorphic function g on Ω so that $g|_V = f$.* A finer result is the holomorphic extension theorem in the book of Henkin and Leiterer, where there is also given some norm control (see [HL84, Theorem 4.11.1]). Uniqueness issues related to this problem with preservation of the norm ($\sup_{\mathbf{z}\in\Omega}|g(z)| = \sup_{\mathbf{z}\in\mathcal{Z}}|f(z)|$) are explored in [AM03]. The paper of Cotlar-Sadosky [CS94] considered interpolation along a variety with control of a *Bounded Mean Oscillation (*BMO*)* norm. ▲

The scalar case of the interpolation problem with solution sought in Schur-Agler class and with the interpolation nodal varieties all taken to have dimension zero, is simply: *given interpolation nodes $\mathbf{z}^1, \dots, \mathbf{z}^n \in \mathbb{D}^d$ and interpolation values $w_1, \dots, w_n \in \mathbb{C}$, find a scalar function $F \in \mathcal{SA}_d$* satisfying the interpolation conditions

$$F(\mathbf{z}^k) = w_k \text{ for } k = 1, \dots, n. \tag{4.42}$$

The original result of Agler [Agl87] on this problem is stated in Theorem 2.20 (see page 27). This result was extended to the matrix-valued setting (with the interpolation nodal varieties still assumed to be zero-dimensional and without consideration of two-sided interpolation conditions) in [BT98, AM]. A contour integral formulation which incorporated higher-order interpolation conditions but still at isolated points was solved in [ABB00].

To end this Section, we provide two simple numerical examples to demonstrate that one cannot expect to find the solution of the full infinite LMI by approximating with solutions of finite sub-LMIs. Here we use the LMILAB package in MATLAB to perform the experiments; the source code is given in Appendix B.

The interpolation data $\mathbf{z}^k$ provided in the following examples are assumed to be nodes (or points in $\mathbb{D}^2$ with zero dimension) on the curve of zero variety in the Bidisk, and w_k be the interpolation values in $\mathbb{C}$. Suppose for simplicity that $n = 3$ and the 3-point ID set is given by

$$\mathcal{D}_3^2 = \{(\mathbf{z}^k, w_k) \in (\mathbb{D}^2 \times \mathbb{C}) : k = 1, 2, 3\}.$$

First, we have to verify whether the data set $\mathcal{D}_3^2$ is feasible; i.e., we are seeking the matrices $P_1 = \left[P_1^{k,\ell}\right]_{k,\ell=1}^3$ and $P_2 = \left[P_2^{k,\ell}\right]_{k,\ell=1}^3$ to satisfy the Agler's condition:

$$1 - w_k\overline{w}_\ell = \sum_{j=1}^{2}\left(1 - z_j^k\overline{z_j^\ell}\right)P_j^{k,\ell} \quad \text{for } k, \ell = 1, 2, 3. \tag{4.43}$$

See also the equation (2.38) on page 27. If such P_1 and P_2 exist and are positive-semidefinite, we shall call $\mathcal{D}_3^2$ the *feasible 3-point* ID *set.*

Suppose that the 3-point ID set is feasible. Then we consider only the first two data points for a moment and recompute for the matrices $\tilde{P}_1 = \left[\tilde{P}_1^{k,\ell}\right]_{k,\ell=1}^2$, and $\tilde{P}_2 = \left[\tilde{P}_2^{k,\ell}\right]_{k,\ell=1}^2$ satisfying the Agler's condition (4.43) for $k = 1, 2$.

One can check that such $\tilde{P}_1$ and $\tilde{P}_2$ do exist and are positive-semidefinite since they are constructed from the feasible ID set $\mathcal{D}_3^2$. Also, it turns out in general that $\tilde{P}_1$ and $\tilde{P}_2$ generated by LMItoolbox are different from the 2×2 block-matrices of the top-left corner of P_1 and P_2. Finally, we add the last data points and compute P_1' and P_2' satisfying the following:

1. both must satisfy the Agler's condition (4.43),
2. both must be positive-semidefinite, and
3. $P_1' = \begin{bmatrix} \tilde{P}_1 & * \\ * & * \end{bmatrix}$, and $P_2' = \begin{bmatrix} \tilde{P}_2 & * \\ * & * \end{bmatrix}$.

However, it is obvious from the following examples that the extension matrices P_1' and P_2' may or may not exist in the sense that the solutions of LMI are infeasible even though P_1 and P_2 are feasible solutions.

Example 9. The 3-point interpolation data set $\mathcal{D}_3^2$ on the Bidisk is given by:

k	$\mathbf{z}^k = (\alpha^k, \beta^k)$	w_k
1	$(0.2i, 0.15)$	0.2
2	$(-0.8 - 0.1i, 0.5)$	$0.1i$
3	$(0.2 + 0.5i, -0.8i)$	$0.25 + 0.25i$

Table 4.1: Interpolation data

Using MATLAB's LMItoolbox, two positive-semidefinite matrices P_1 and P_2 satisfying the Agler's condition (2.38) for 3 points are given by:

$$P_1 = \begin{bmatrix} 0.4754 & 0.6472 - 0.0756i & 0.3183 + 0.0443i \\ 0.6472 + 0.0756i & 1.5348 & 0.4031 + 0.1453i \\ 0.3183 - 0.0443i & 0.4031 - 0.1453i & 0.5100 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.5152 & 0.3544 - 0.0070i & 0.6496 + 0.1009i \\ 0.3544 + 0.0070i & 0.6038 & 0.3888 + 0.1079i \\ 0.6496 - 0.1009i & 0.3888 - 0.1079i & 1.4247 \end{bmatrix}$$

Since P_1 and P_2 are positive-definite, the given ID set $\mathcal{D}_3^2$ is feasible. Now we drop the last data points (i.e., the 3-rd row of Table 4.1) and using LMItoolbox to recompute the 2×2 matrices

$\tilde{P}_1$ and $\tilde{P}_2$:

$$\tilde{P}_1 = \begin{bmatrix} 0.6286 & 0.7769 - 0.1082i \\ 0.7769 + 0.1082i & 1.7615 \end{bmatrix}$$
$$\tilde{P}_2 = \begin{bmatrix} 0.3647 & 0.2057 + 0.0066i \\ 0.2057 - 0.0066i & 0.4980 \end{bmatrix}$$

One can verify that $\tilde{P}_1$ and $\tilde{P}_2$ are positive-definite and satisfy the Agler's condition (2.38) for 2 points. In fact, these two matrices are constructed from the feasible ID set, and hence they must be positive-definite. What we want to do next is to add the last data point $(\mathbf{z}^3, w_3)$ and recompute the 3×3 matrices satisfying the conditions 1–3 given above. In this particular ID set, we obtain:

$$P_1' = \begin{bmatrix} 0.6286 & 0.7769 - 0.1082i & 0.4253 + 0.0573i \\ 0.7769 + 0.1082i & 1.7615 & 0.4893 + 0.1777i \\ 0.4253 - 0.0573i & 0.4893 - 0.1777i & 0.5444 \end{bmatrix}$$
$$P_2' = \begin{bmatrix} 0.3647 & 0.2057 + 0.0066i & 0.5551 + 0.0821i \\ 0.2057 - 0.0066i & 0.4980 & 0.2905 + 0.0621i \\ 0.5551 - 0.0821i & 0.2905 - 0.0621i & 1.3570 \end{bmatrix}$$

The extension matrices P_1' and P_2' are positive-definite and the 2×2 top-left corner block-matrix of P_i' is exactly $\tilde{P}_i$ for $i = 1, 2$. $\diamondsuit$

In the previous example, one can obtain the extension matrices P_1' and P_2' as desired. However, this is not always the case. The next example illustrates the case when P_1' and P_2' do not exist (i.e., the solutions of LMI are infeasible).

Example 10. The 3-point interpolation data set $\mathcal{D}_3^2$ on the Bidisk is given by:

k	$\mathbf{z}^k = (\alpha^k, \beta^k)$	w_k
1	$(0.15 - 0.5i, 0.21)$	$0.2i$
2	$(0.2i, -0.05i)$	-0.19
3	$(0.9 + 0.1i, -0.65)$	$-0.85 + 0.25i$

Table 4.2: Interpolation data

Using MATLAB's LMIToolbox, two positive-semidefinite matrices P_1 and P_2 satisfying the

Agler's condition (2.38) for 3 points are given by:

$$P_1 = \begin{bmatrix} 1.1806 & 0.8466 - 0.0319i & 0.8789 - 0.2706i \\ 0.8466 + 0.0319i & 0.8895 & 0.8076 + 0.0912i \\ 0.8789 + 0.2706i & 0.8076 - 0.0912i & 1.1084 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.1058 & 0.0673 + 0.0484i & 0.0232 + 0.0103i \\ 0.0673 - 0.0484i & 0.1102 & 0.0309 + 0.0075i \\ 0.0232 - 0.0103i & 0.0309 - 0.0075i & 0.0268 \end{bmatrix}$$

Since both P_1 and P_2 are positive-definite, the given interpolation data set is feasible. Now we drop the last data point $(\mathbf{z}^3, w_3)$ and recompute $\tilde{P}_1, \tilde{P}_2$ as before:

$$\tilde{P}_1 = \begin{bmatrix} 0.9411 & 0.7319 + 0.0077i \\ 0.7319 - 0.0071i & 0.7184 \end{bmatrix}$$

$$\tilde{P}_2 = \begin{bmatrix} 0.2880 & 0.1950 + 0.0097i \\ 0.1950 - 0.0097i & 0.2749 \end{bmatrix}$$

However, when the last data point $(\mathbf{z}^3, w_3)$ is added back to the ID set and recompute the extension 3×3 matrices satisfying the conditions 1–3, the result yields the infeasible solutions. Thus, in this case, the extension matrices P_1', P_2' do not exist. $\diamondsuit$

From these experiments, we observe that even though the 3-point ID set is feasible, if we choose inappropriate 2×2 matrices $\tilde{P}_i$, the extension matrices P_i' may or may not be feasible. This fact is fundamentally different from the Schur algorithm for one variable case in which one can construct an $n \times n$ positive-semidefinite matrix from the extension of the $(n-1) \times (n-1)$ positive-semidefinite matrix associated with dropping one ID point.

4.5 An Operator-theoretic Formulation of the Model Matching Problem

This Section presents an operator-theoretic formulation of the Model Matching Problem which lends itself to a solution via the recent polydisk Commutant Lifting Theorem in [BLTT99]. For a thorough introduction to the Commutant Lifting approach for the classical case, we refer the reader to [FF90, FFGK98].

Assume that we have arrived at the Model Matching Problem: *given stable T_1, T_2 and T_3, find stable Q (all matrix functions of compatible sizes) so that $F = T_1 - T_2 Q T_3$ satisfies the norm constraint* (4.36); here in general we say that the $(m \times n)$-matrix valued function X is *stable* if each matrix entry of X is bounded and holomorphic on $\mathbb{D}^d$, i.e. $X \in H^\infty(\mathbb{D}^d, \mathbb{C}^{m \times n})$.

In this Section, we assume that T_3 is invertible with T_3^{-1} also stable. In this case the new Q parameter $Q' := QT_3$ sweeps all holomorphic functions on the polydisk $\mathbb{D}^d$ while Q does, and we may use Q' as our new Q parameter. When this is done, the problem reduces to the case where $T_3(z) = I$ and our model-matching form is simply $F = T_1 - T_2Q$. Thus, for the rest of this Section, we assume that $T_3 = I$. To be specific, we assume that T_1 and T_2 have sizes $l \times m$ and $l \times p$ respectively. The size of the Q-parameter then is $p \times m$.

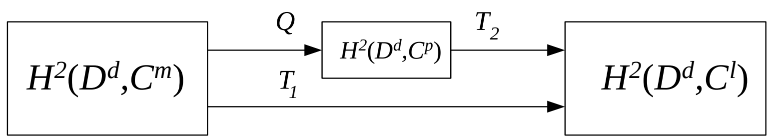

Figure 4.4: The model matching set up in the operator theoretic approach.

The next result is formulated in the operator-theoretic approach. Let $H^2(\mathbb{D}^d, \mathbb{C}^m)$ be the Hardy space of $\mathbb{C}^m$-valued holomorphic functions $f(\mathbf{z}) = \sum_{|\mathbf{j}|=0}^{\infty} C_{\mathbf{j}}\mathbf{z}^{\mathbf{j}}$ with square-summable coefficients ($C_{\mathbf{j}} \in \mathbb{C}^m$ with $\sum_{|\mathbf{j}|=0}^{\infty} \|C_{\mathbf{j}}\|^2 < \infty$). For G any bounded holomorphic matrix function of size $l \times m$, we denote by $M_G \colon H^2(\mathbb{D}^d, \mathbb{C}^m) \to H^2(\mathbb{D}^d, \mathbb{C}^l)$ the multiplication operator given by

$$M_G \colon f(\mathbf{z}) \to G(\mathbf{z})f(\mathbf{z}) \in H^2(\mathbb{D}^d, \mathbb{C}^l).$$

Thus, for any $f \in H^2(\mathbb{D}^d, \mathbb{C}^m)$, we have

$$\begin{aligned} M_{T_1} &: f(\mathbf{z}) \mapsto T_1(\mathbf{z})f(\mathbf{z}) \in H^2(\mathbb{D}^d, \mathbb{C}^l) = T_1 H^2(\mathbb{D}^d, \mathbb{C}^m), \\ M_{T_2} &: Q(\mathbf{z})f(\mathbf{z}) \mapsto T_2(\mathbf{z})Q(\mathbf{z})f(\mathbf{z}) \in H^2(\mathbb{D}^d, \mathbb{C}^l) = T_2 H^2(\mathbb{D}^d, \mathbb{C}^p). \end{aligned}$$

The only assumption which we impose on T_2 is that $M_{T_2}H^2(\mathbb{D}^d, \mathbb{C}^p)$ is a closed subspace of $H^2(\mathbb{D}^d, \mathbb{C}^l)$; this is the case for example with $T_2 \in \mathbb{R}_s^{l\times l}(\mathbf{z})$ with inverse T_2^{-1} existing and uniformly bounded on the distinguished boundary $\mathbb{T}^d$. Then we define $\mathcal{M} \subset H^2(\mathbb{D}^d, \mathbb{C}^l)$ to be the orthogonal complement of $T_2H^2(\mathbb{D}^d, \mathbb{C}^p)$ in $H^2(\mathbb{D}^d, \mathbb{C}^l)$, i.e.

$$\mathcal{M} = H^2(\mathbb{D}^d, \mathbb{C}^l) \ominus T_2 H^2(\mathbb{D}^d, \mathbb{C}^p). \tag{4.44}$$

Since the matrix function $T_1 \in H^\infty(\mathbb{D}^d, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^l))$ is given, we then use T_1 to define an

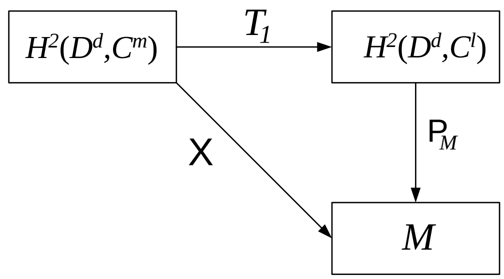

Figure 4.5: The lifting diagram for an operator X

operator X acting from $H^2(\mathbb{D}^d, \mathbb{C}^m)$ directly to the subspace $\mathcal{M}$ by:

$$X = P_{\mathcal{M}} M_{T_1} : H^2(\mathbb{D}^d, \mathbb{C}^m) \to \mathcal{M} \tag{4.45}$$

where $P_{\mathcal{M}}$ is the orthogonal projection from $H^2(\mathbb{D}^d, \mathbb{C}^l)$ onto $\mathcal{M}$ (see Figure 4.5).

The following Lemma provides a necessary and sufficient condition for a matrix-valued holomorphic function to have a Model Matching Form.

Lemma 4.12. *The $l \times m$ matrix-valued function F holomorphic on $\mathbb{D}^d$ has the model matching form $F = T_1 - T_2 Q$ for some bounded $p \times m$ matrix function Q holomorphic on $\mathbb{D}^d$ if and only if $P_{\mathcal{M}} M_F = X$ where X is given by* (4.45).

Proof. Suppose now that there exists such an F of the model matching form, i.e. $F = T_1 - T_2 Q$. Then, from (4.45), we have

$$P_{\mathcal{M}} M_F = P_{\mathcal{M}} M_{T_1 - T_2 Q} = P_{\mathcal{M}} M_{T_1} - P_{\mathcal{M}} M_{T_2 Q} = P_{\mathcal{M}} M_{T_1}$$

which is the definition of an operator X. Note that the last equality in the above expression comes from the fact that $\mathcal{M} \perp T_2 H^2(\mathbb{D}^d, \mathbb{C}^p)$.

Conversely, suppose that $P_{\mathcal{M}} M_F = X$. Then $P_{\mathcal{M}}(M_{T_1} - M_F) = 0$. This implies that

$$(M_{T_1} - M_F) : H^2(\mathbb{D}^d, \mathbb{C}^m) \mapsto T_2 H^2(\mathbb{D}^d, \mathbb{C}^p) \quad \text{since } \mathcal{M} = H^2(\mathbb{D}^d, \mathbb{C}^l) \ominus T_2 H^2(\mathbb{D}^d, \mathbb{C}^p).$$

Thus, one can solve for $Q = \begin{bmatrix} q_1 & \cdots & q_m \end{bmatrix} \in H^2(\mathbb{D}^d, \mathcal{L}(\mathbb{C}^m, \mathbb{C}^p))$ so that $T_1 - F = T_2 Q$; i.e., choose $q_j \in H^2(\mathbb{D}^d, \mathbb{C}^p)$ so that

$$(T_1 - F)e_j = T_2 q_j, \text{ where } \{e_1, \ldots, e_m\} \text{ is the standard basis in } \mathbb{C}^m.$$

Since multiplication by Q on the left necessarily maps $H^2(\mathbb{D}^d, \mathbb{C}^m)$ into $H^2(\mathbb{D}^d, \mathbb{C}^p)$, one can argue that in fact $Q \in H^\infty(\mathbb{D}^d, \mathbb{C}^{p \times m})$. Thus, one can deduce that $F = T_1 - T_2 Q$ with Q stable as required. ■

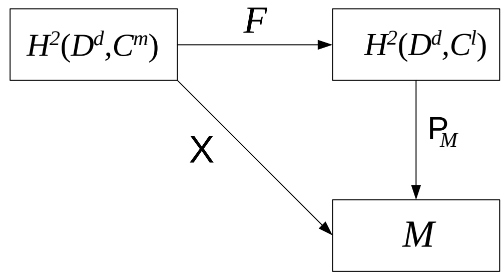

Figure 4.6: The lifting diagram for the Model Matching Problem

Suppose now that F has the Model Matching Form, i.e. $F = T_1 - T_2Q$. Then by Lemma 4.12, we define the operator $X_F : H^2(\mathbb{D}^d, \mathbb{C}^m) \mapsto \mathcal{M}$ so that $P_{\mathcal{M}} M_F = X_F$ where the subscript of X indicates its corresponding lifting operator. If $\tilde{F}$ also admits the Model Matching Form, say $\tilde{F} = T_1 - T_2\tilde{Q}$. Then for any function $f \in H^2(\mathbb{D}^d, \mathbb{C}^m)$,

$$\begin{aligned}
(X_F - X_{\tilde{F}})f &= P_{\mathcal{M}} M_F f - P_{\mathcal{M}} M_{\tilde{F}} f \\
&= P_{\mathcal{M}} \left(M_{T_1 - T_2 Q} - M_{T_1 - T_2\tilde{Q}} \right) f \\
&= P_{\mathcal{M}} M_{T_2(Q-\tilde{Q})} f \\
&= 0 \qquad (\text{since } M_{T_2(Q-\tilde{Q})} f \in T_2 H^2(\mathbb{D}^d, \mathbb{C}^p))
\end{aligned}$$

and hence $X_F = X_{\tilde{F}} := X$ is independent of the choice of Q. In other words, X depends only on the data T_1 and T_2.

Two more operators needed here are the multiplication by the coordinate function z_j on $H^2(\mathbb{D}^d, \mathbb{C}^m)$ denoted by M_{z_j}, and the so-called *model operator* $S_{\mathcal{M},j}$ on $\mathcal{M}$ which is given by:

$$S_{\mathcal{M},j} = P_{\mathcal{M}} M_{z_j}\big|_{\mathcal{M}} \qquad \text{for } j = 1, \ldots, d.$$

Finally, let us verify that such an operator X defined above intertwines the multiplication operator M_{z_j} on $H^2(\mathbb{D}^d, \mathbb{C}^m)$ with the operator $S_{\mathcal{M},j}$ on $\mathcal{M}$:

$$X\ M_{z_j}\big|_{H^2(\mathbb{D}^d, \mathbb{C}^m)} = S_{\mathcal{M},j} X \text{ for } j = 1, \ldots, d.$$

To see this, let $f \in H^2(\mathbb{D}^d, \mathbb{C}^m)$. Then,

$$\begin{aligned}
X M_{z_j} f &= P_{\mathcal{M}} M_{T_1} M_{z_j} f \quad = P_{\mathcal{M}} M_{z_j} M_{T_1} f \\
&= P_{\mathcal{M}} M_{z_j} \left(P_{\mathcal{M}} + P_{T_2 H^2(\mathbb{D}^d, \mathbb{C}^p)} \right) M_{T_1} f
\end{aligned}$$

$$
\begin{aligned}
&= P_{\mathcal{M}} M_{z_j} P_{\mathcal{M}} M_{T_1} f + P_{\mathcal{M}} M_{z_j} P_{T_2 H^2(\mathbb{D}^d,\mathbb{C}^p)} M_{T_1} f \\
&= P_{\mathcal{M}} M_{z_j} P_{\mathcal{M}} M_{T_1} f + P_{\mathcal{M}} P_{T_2 H^2(\mathbb{D}^d,\mathbb{C}^p)} M_{z_j} P_{T_2 H^2(\mathbb{D}^d,\mathbb{C}^p)} M_{T_1} f \\
&= P_{\mathcal{M}} M_{z_j} P_{\mathcal{M}} M_{T_1} f \quad = S_{\mathcal{M},j} X f
\end{aligned}
$$

(This idea originates from the seminal paper of [Sar67]).

We are thus in a position to apply the polydisk Commutant Lifting Theorem from [BLTT99]. One more piece of notation is required: for Y any operator on $\mathcal{M}$, we let Γ_Y denote the completely positive operator on $\mathcal{L}(\mathcal{M})$ given by

$$\Gamma_Y \colon X \to \Gamma_Y[X] := Y X Y^* \text{ for } X \in \mathcal{L}(\mathcal{M}).$$

The following result, an immediate application of Theorem 5.1 from [BLTT99], reduces the problem to solving a *Linear Operator Inequality (* LOI*)*.

Theorem 4.13. *Let T_1 and T_2 be bounded matrix functions holomorphic on $\mathbb{D}^d$ with $T_2 H^2(\mathbb{D}^d, \mathbb{C}^p)$ a closed subspace of $H^2(\mathbb{D}^d, \mathbb{C}^l)$ as above, and define the subspace $\mathcal{M}$ and the operator X on $\mathcal{M}$ as in* (4.44) *and* (4.45)*. Then there is a bounded matrix function Q holomorphic on $\mathbb{D}^d$ such that $F := T_1 - T_2 Q$ is in the Schur-Agler class $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l)$ (i.e. F satisfies* (4.37)*) if and only if there exists positive operators*

$$G_1 \geq 0, \dots, G_d \geq 0 \text{ on } \mathcal{M}$$

such that

$$I - XX^* = G_1 + \cdots + G_d \text{ and } \prod_{j:j\neq i} \left(I - \Gamma_{S_{\mathcal{M},j}}\right)[G_i] \geq 0 \text{ for } i = 1, \dots, d. \tag{4.46}$$

For example, let us consider the case when $d = 3$. Then the LOI (4.46) becomes: for $I - XX^* = G_1 + G_2 + G_3$ where $G_i \geq 0$ on $\mathcal{M}$, and

$$
\begin{aligned}
\text{for } i = 1, \quad & \prod_{j=2,3} \left(I - \Gamma_{S_{\mathcal{M},j}}\right)[G_1] \\
&= \left(I - \Gamma_{S_{\mathcal{M},3}}\right)\left[\left(I - \Gamma_{S_{\mathcal{M},2}}\right)[G_1]\right] \\
&= \left(I - \Gamma_{S_{\mathcal{M},2}}\right)[G_1] - \Gamma_{S_{\mathcal{M},3}}\left[\left(I - \Gamma_{S_{\mathcal{M},2}}\right)[G_1]\right] \\
&= G_1 - S_{\mathcal{M},2} G_1 S_{\mathcal{M},2}^* - S_{\mathcal{M},3}\left(G_1 - S_{\mathcal{M},2} G_1 S_{\mathcal{M},2}^*\right) S_{\mathcal{M},3}^* \\
&= G_1 - S_{\mathcal{M},2} G_1 S_{\mathcal{M},2}^* - S_{\mathcal{M},3} G_1 S_{\mathcal{M},3}^* - S_{\mathcal{M},3} S_{\mathcal{M},2} G_1 S_{\mathcal{M},2}^* S_{\mathcal{M},3}^* \geq 0, \\
\text{for } i = 2, \quad & \prod_{j=1,3} \left(I - \Gamma_{S_{\mathcal{M},j}}\right)[G_2] \\
&= G_2 - S_{\mathcal{M},1} G_2 S_{\mathcal{M},1}^* - S_{\mathcal{M},3} G_2 S_{\mathcal{M},3}^* - S_{\mathcal{M},3} S_{\mathcal{M},1} G_2 S_{\mathcal{M},1}^* S_{\mathcal{M},3}^* \geq 0, \\
\text{for } i = 3, \quad & \prod_{j=1,2} \left(I - \Gamma_{S_{\mathcal{M},j}}\right)[G_3]
\end{aligned}
$$

$$= G_3 - S_{\mathcal{M},1}G_3S^*_{\mathcal{M},1} - S_{\mathcal{M},2}G_3S^*_{\mathcal{M},2} - S_{\mathcal{M},2}S_{\mathcal{M},1}G_3S^*_{\mathcal{M},1}S^*_{\mathcal{M},2} \geq 0.$$

Remark 17. As was the case for the interpolation approach in Section 4.4, Theorem 4.13 gives necessary and sufficient condition for the existence of a solution of the Model Matching Problem in the smaller class $\mathcal{SA}_d(\mathbb{C}^m, \mathbb{C}^l)$. Thus, this is also necessary and sufficient for the existence of solutions in the physically desired class $\mathcal{S}_d(\mathbb{C}^m, \mathbb{C}^l)$ for the case $d = 1, 2$, but in general only sufficient for the case $d > 2$. ▲

Remark 18. In Theorem 4.13, we do not require T_2 to be square and invertible. Thus Theorem 4.13 includes interpolation along varieties of any co-dimension (see Remark 13). ▲

4.6 Solution of the d-D H^∞ control problem

We now return to the H^∞ control problem introduced in Section 4.2: *given a d-D plant P (see Fig. 4.2) design a stabilizing controller K for which the sensitivity function* $S = (I + PK)^{-1}$ *achieves* $\|S\| = \|(I + PK)^{-1}\| \leq 1$. Collecting the results of the previous sections, we now have the following solution procedure of the H^∞ control problem:

1. Construct a DCF (4.5) for P satisfying the Bézout identity. While the existence of such a DCF in general appears not to have been proved at the moment, it is conjectured that such a DCF always does exist; one scenario guaranteeing the existence of such a DCF is the set of conditions in Proposition (4.6).

2. Identify T_1, T_2, T_3 so that $(I + PK)^{-1} = T_1 - T_2QT_3$ where Q is the Youla parameter, as in (4.8). Assume that T_2 and T_3 are invertible in $\mathbb{R}^{l\times l}(\mathbf{z})$ and $\mathbb{R}^{m\times m}(\mathbf{z})$ respectively with inverse T_2^{-1} and T_3^{-1} existing and uniformly bounded on the distinguished boundary $\mathbb{T}^d$.

3. Form the ID set $\mathcal{D}_{T_1,T_2,T_3}$ from T_1, T_2, and T_3 as in Theorem 4.10.

4. Assume that the interpolation data set is such that no q_u is also a s_v as in the hypotheses of Theorem 4.11. Then a sufficient (and also necessary if $d \leq 2$) condition for the H^∞ control problem to have a (not necessarily well-posed) solution is that the Schur-Agler bitangential NPIP (4.19)–(4.20) with the norm constraint (4.37) and the ID set $\mathcal{D}_{T_1,T_2,T_3}$ have a solution, or equivalently, that the infinite LMI (4.39) have a positive solution $P(\omega', \omega)$.

   In this case, there are explicit realization formulas for solutions S of the Schur-Agler bitangential NPIP (see [BB, Problem 1.4 and Theorem 1.5]); choose any such S with meets in addition the well-posedness condition $\det S \neq 0$. Then $K = (D_rQ + Y_r)(-N_rQ + X_r)^{-1}$ with $Q = T_2^{-1}(T_1 - S)T_3^{-1}$ gives a stabilizing controller K which meets the H^∞ performance criterion $\|(I + PK)^{-1}\| \leq 1$.

Alternatively, in case T_3^{-1} is stable, at Step 3 one could apply Theorem 4.13 to arrive at a solution criterion for the existence of (not necessarily well-posed) solutions in terms of the LOI (4.46). The procedure described above can be viewed as an extension of the basic idea in [Hel01] from the SISO, stable case to a MIMO, possibly unstable setting.

Part II

Noncommutative Multidimensional Linear Systems

Chapter 5

Introduction to Noncommutative Linear Systems

In this Part, we introduce an input-state-output (i/s/o) linear system with evolution along a free semigroup. The corresponding transfer function for such a system is a formal power series in noncommuting indeterminants. Motivation for the study of such formal power series comes from the paper of C. Beck and J. Doyle [BD99] on the robust control for systems with structured uncertainty. In that paper, they used the linear fractional transformations (LFT's) as a tool for modeling the uncertain systems with structured perturbations on a nominal model. We denote by $\widetilde{\mathbf{\Delta}} = \operatorname{diag}\{\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_d\}$, $\boldsymbol{\delta}_i\boldsymbol{\delta}_j \neq \boldsymbol{\delta}_j\boldsymbol{\delta}_i$ unless $i = j$ the uncertainty operator, where for each i, $\boldsymbol{\delta}_i$ represents noise or small disturbance entering to the system in different location, and can be regarded as arbitrary time-varying operator on the square summable space ℓ^2, a real-valued parameter uncertainty, or a noncommuting indeterminant. If the system realization matrix of the nominal model is assumed to be known and is partitioned as $\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{\mathbf{A}} & \widetilde{\mathbf{B}} \\ \widetilde{\mathbf{C}} & \widetilde{\mathbf{D}} \end{bmatrix}$, then the input/output (i/o) map from an input sequence $\widetilde{u}$ to an output sequence $\widetilde{y}$ is given by the upper LFT representation as:

$$LFT_u(\widetilde{\mathbf{U}}, \widetilde{\mathbf{\Delta}}) = \widetilde{\mathbf{D}} + \widetilde{\mathbf{C}}\widetilde{\mathbf{\Delta}} \left(I - \widetilde{\mathbf{A}}\widetilde{\mathbf{\Delta}}\right)^{-1} \widetilde{\mathbf{B}}, \tag{5.1}$$

provided that the inverse of $\left(I - \widetilde{\mathbf{A}}\widetilde{\mathbf{\Delta}}\right)$ exists.

The authors in [BD99] also reviewed the realization theory and the Lyapunov stability theory for uncertain systems, proposed a necessary and sufficient condition for reducibility in terms of a coupling Lyapunov inequalities, and discussed the controllability and observability of an uncertain system realization; however, they did not provide the state-space interpretation for these objects. The main goal here is to formulate an i/s/o system, with evolution along elements of a free semigroup, with transfer function of the form (5.1) (with $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_d$ interpreted as formal noncommuting indeterminants), with tests for controllability and observability giving rise to the controllability/observability matrices appearing in [BD99], and with system minimality connected with minimality of the LFT representation (5.1). The LFT representation (5.1) for the transfer function of the system arises via the application of a noncommutative d-variable Z-transform to the system equations, just as in the classical case.

This Chapter is organized as follows: In Section 5.1, we first briefly review some fundamental facts from the noncommutative algebra, and also introduce some terminologies and notation that will be used throughout this Part of dissertation. We summarize the LFT framework commonly used in the robust control literature for structured perturbations on a nominal model for 1D discrete-time linear system, and establish the connection between the robust control theory and the multidimensional linear system theory in Section 5.2. Since the i/s/o linear system we are dealing with in this Part is the system with evolution along elements of a free semigroup, we then present the interpretation of the so-called "time-axis" of such a system via the notion of a *homogeneous tree with root* in Section 5.3. Finally, in Section 5.4 we demonstrate some examples of other types of systems close to the systems considered here which have appeared in the engineering literature. It is not too much to conceive of the possibility that the system models we describe here should also be interesting and applicable in engineering applications in addition to the connection with robust control mentioned above.

5.1 Noncommutative Algebra

In the mathematical system theory, it is well known that one way to explore the structure of the system we consider is by using an algebraic approach. The semigroup theory is one of the most powerful tools for studying the behavior of the finite state machine in the automata theory. Also, a formal power series plays an important role as a tool for studying the behavior of noncommutative systems in the frequency domain analysis. As an introduction, we present here a brief review on some properties of a free semigroup followed by the notion of formal power series and its applications. For a more detailed treatment of this subject, see e.g. [BR88].

5.1.1 Free Semigroup

Definition 24 (Semigroup). A nonempty set $\mathcal{E}$ with a binary operation $*$ is called a *semigroup* if the following properties are satisfied:

- $\mathcal{E}$ is closed under $*$, i.e. $x * y \in \mathcal{E}$ for each x and y in $\mathcal{E}$.
- $*$ is associative, i.e. for any x, y, and $z \in \mathcal{E}$, $x * (y * z) = (x * y) * z$.

Let $\mathcal{F}$ be a nonempty finite set of d generators (letters), say $\mathcal{F} = \{g_1, \ldots, g_d\}$. A *string* or *word* is any finite sequence of letters from the set $\mathcal{F}$. The *length* of a word w, denoted by $|w|$, is a number of letters contained in the word. For instance, a word $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$ has length $|w| = n$. Given a word w, we denote by $w^\top$ the word with the same letters but listed in reverse order, i.e. if $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$ then $w^\top = g_{i_1} g_{i_2} \cdots g_{i_n}$.

Let $\mathcal{F}_d$ be a set of all words of finite length generated by letters from the set $\mathcal{F}$. Then for any words $w_1 = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$ and $w_2 = g_{j_m} g_{j_{m-1}} \cdots g_{j_1} \in \mathcal{F}_d$ where $i_k, j_\ell \in \mathcal{I}_d := \{1, \ldots, d\}$, we define a closed binary operation $*$ on this set by

$$w_1 * w_2 = g_{i_n} g_{i_{n-1}} \cdots g_{i_1} g_{j_m} g_{j_{m-1}} \cdots g_{j_1} \in \mathcal{F}_d. \tag{5.2}$$

Such an operation is known as a *concatenation*. It is easy to verify that $*$ is an associative operation; i.e., for any $w_1, w_2, w_3 \in \mathcal{F}_d$, $(w_1 * w_2) * w_3 = w_1 * (w_2 * w_3)$. Therefore $\mathcal{F}_d$ is a free semigroup[1] generated by letters from the set $\mathcal{F}$ with respect to the concatenation. Note that the concatenation is not a commutative operation since, in general, $w_1 * w_2 \neq w_2 * w_1$ unless $w_1 = w_2$.

Suppose we now add the *null word*, denoted by λ, which is a word with no letters at all and satisfies the following properties:

- for any $w \in \mathcal{F}_d, \lambda * w = w * \lambda = w$,
- the length of the null word by definition is equal to 0 ($|\lambda| = 0$).

For convenience, let us still write $\mathcal{F}_d$ for $\mathcal{F}_d \cup \{\lambda\}$. Then $(\mathcal{F}_d, *)$ is a monoid or semigroup with identity. Also this is called the free monoid generated by the set $\mathcal{F}$.

Although $(\mathcal{F}_d, *)$ is a monoid, for convenience in analysis, we shall define the notion of inverse on words in $\mathcal{F}_d$ as follows: for any words $u, v \in \mathcal{F}_d$, if there exists $w \in \mathcal{F}_d$ such that $u = w * v$, we shall assign the expression $w = u * v^{-1}$ for w; otherwise, we shall say that $u * v^{-1}$ is undefined. The expression of $v^{-1} * u$ can be defined in the similar way. It should be noted here that since there are no "real" inverse elements in a monoid, the associativity property involving with the inverse operation may fail: in general it is not the case that $(u * v^{-1}) * w = u * (v^{-1} * w)$, for any u, v and $w \in \mathcal{F}_d$. For instance, let $u = g_1, v = g_1$ and $w = g_2$, then $(u * v^{-1}) * w = g_2$ but $u * (v^{-1} * w)$ is not defined. Thus $\mathcal{F}_d$ has the structure of a groupoid (multiplication by inverse of an element is defined only on a certain domain).

5.1.2 Formal Power Series

In the automata theory and formal languages literature, a *formal power series* is abstractly defined as a function $T : \mathcal{F}_d \mapsto K$ such that

$$T = \sum_{v \in \mathcal{F}_d} T_v \cdot v,$$

[1] A semigroup $(\mathcal{F}_d, *)$ is said to be *free* if it consists of words of finite length generated by the set $\mathcal{F}$.

where K is a semiring defined as follows:

Definition 25 (Semiring). A set K equipped with two operations: sum $+$ and product $\cdot$, is said to be a *semiring* if the following properties hold:

1. $(K, +)$ is a commutative monoid with identity element denoted by 0,
2. $(K, \cdot)$ is a monoid with identity element denoted by 1,
3. The product is distributive with respect to the sum,
4. For all $k \in K$, $0 \cdot k = k \cdot 0 = 0$.

The *support* of T is a subset of $\mathcal{F}_d$ defined by

$$\operatorname{supp}(T) = \{w \in \mathcal{F}_d \mid T_v \neq 0\}.$$

For our analysis purposes here, we consider in particular a formal power series of the form

$$T := T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v, \tag{5.3}$$

where T_v is called the *coefficient* of z^v in $T(z)$ and K in this case is either $\mathbb{C}^p$, or $\mathbb{C}^{p\times q}$ (or in coordinate-free notation $K = \mathcal{U}, \mathcal{Y}$ or $K = \mathcal{L}(\mathcal{U}, \mathcal{Y})$). A formal power series $T(z)$ is said to be *proper* if the coefficient of the null word vanishes (i.e., $T_\lambda = 0$). The set of all formal power series with coefficients in K is denoted by $K_{nc}[[z_1, \ldots, z_d]]$. A *polynomial* is a formal power series with finite support. The set of polynomials is denoted by $K_{nc}[z_1, \ldots, z_d]$.

Given two formal power series $T(z)$ and $T'(z)$ with compatible coefficient spaces K (e.g., both $T(z)$ and $T'(z)$ in $\mathbb{C}^{p\times q}_{nc}[[z_1, \ldots, z_d]]$ for addition, $T(z) \in \mathbb{C}^{p\times r}_{nc}[[z_1, \ldots, z_d]]$ and $T'(z) \in \mathbb{C}^{r\times q}_{nc}[[z_1, \ldots, z_d]]$ for multiplication), then their sum is given by

$$[T(z) + T'(z)]_v = T_v + T'_v,$$

and their product by

$$[T(z) \cdot T'(z)]_v = \sum_{ww'=v} T_w T'_{w'}.$$

Define $T^* = \sum_{k=0}^{\infty} T(z)^k$ and $T^+ = \sum_{k=1}^{\infty} T(z)^k$. Clearly, $T^* = I + T^+$, and $T^+ = T(z) \cdot T^* = T^* \cdot T(z)$. From these, we have

$$T^*(I - T(z)) = T^* - T^* \cdot T(z) = T^* - T^+ = I.$$

Thus, it follows that if K is a ring, then T^*, the *star* of a formal power series $T(z)$, is the inverse of $I - T(z)$ (i.e., $(I - T(z))^{-1} = \sum_{k=0}^{\infty} T(z)^k$). These three operations (the sum, the product, the star operation or the inverse) are called the *rational operations* and a formal power series $T(z)$ is called a *rational series* if it is an element of the smallest subset of $\mathbb{C}_{nc}[[z_1, \ldots, z_d]]$ which is closed under the rational operations. An element of $\mathbb{C}_{ns}^{p \times q}[[z_1, \ldots, z_d]]$ is said to be *rational* if its matrix entries are all rational. Basically, a formal power series is a rational series if it can be expressed as a finite number of sums, products and inversions of polynomials.

Definition 26 (Recognizable Series). A formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ (here $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional linear spaces) is said to be *recognizable* if there exist a finite-dimensional linear space $\mathcal{H}$ and operators $F_1, \ldots, F_d$ on $\mathcal{H}$, $G : \mathcal{U} \mapsto \mathcal{H}$, and $H : \mathcal{H} \mapsto \mathcal{Y}$ such that $T_v = HF^v G$ where $F^v := F_{i_n} F_{i_{n-1}} \cdots F_{i_1}$ if $v = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$.

The following is the fundamental theorem of rational series proposed by M. Schützenberger in 1961.

Theorem 5.1 ([Sch61]). *A formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v \in K_{nc}[[z_1, \ldots, z_d]]$ is rational if and only if it is recognizable.*

Given a formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v \in K_{nc}[[z_1, \ldots, z_d]]$, the *Hankel operator* H_T associated with $T(z)$ has a matrix representation with rows indexed by $v \in \mathcal{F}_d$ and columns indexed by $w \in \mathcal{F}_d$, and each entry of H_T is defined by

$$[H_T]_{v,w} = T_{vw}. \tag{5.4}$$

If $T(z) \in K_{nc}[[z_1, \ldots, z_d]]$ is recognizable, then $[H_T]_{v,w} = T_{vw} = HF^{vw}G$ for all $v, w \in \mathcal{F}_d$ (for some $F = (F_1, \ldots, F_d), H, G$).

Theorem 5.2 ([Fli74]). *A formal power series $T(z)$ is rational if and only if*

$$\operatorname{rank}(H_T) = n < \infty.$$

In this case, n is equal to the smallest possible size for the square matrices $F_1, \ldots, F_d$ so that T_v has a representation of the form $T_v = HF^v G$.

5.2 Linear Fractional Transformation

It is well known in the robust control literature that many control problems can be formulated in a linear fractional transformation (LFT) framework which provides a structural paradigm to analyze and design stabilizing linear controllers for the closed-loop system in an effective way. In fact, the terminology of LFT originates from the theory of one complex variable which is stated as follows: *An* LFT *is a mapping $f : \mathbb{C} \mapsto \mathbb{C}$ of the form*

$$f(s) = \frac{a + bs}{c + ds}$$

where a, b, c *and* $d \in \mathbb{C}$.

This definition can be generalized to the matrix case. Let M be a complex matrix which is partitioned as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(n_1+n_2)},$$

and let $\Delta_u \in \mathbb{C}^{n_1 \times m_1}$, and $\Delta_\ell \in \mathbb{C}^{n_2 \times m_2}$ be two other complex matrices. Then the mapping

$$LFT_u(M, \Delta_u) := D + C\Delta_u(I - A\Delta_u)^{-1}B$$

is called an *upper* LFT with respect to Δ_u whenever the inverse of $(I - A\Delta_u)$ exists. Likewise, the mapping

$$LFT_\ell(M, \Delta_\ell) := A + B\Delta_\ell(I - D\Delta_\ell)^{-1}C$$

is called a *lower* LFT with respect to Δ_ℓ whenever the inverse of $(I - D\Delta_\ell)$ exists.

We now show the connection between the LFT described above and the classical control system. Let us consider the following diagrams

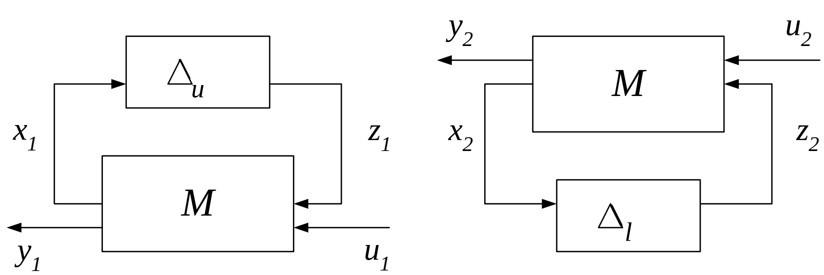

Figure 5.1: Block diagrams of the upper and the lower LFTs

The diagram on the left hand side of Figure 5.1 represents the set of equations:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = M \begin{bmatrix} z_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z_1 \\ u_1 \end{bmatrix},$$
$$z_1 = \Delta_u x_1.$$

It is easy to verify that the mapping from u_1 to y_1 is given by $D + C\Delta_u(I - A\Delta_u)^{-1}B$ which is exactly the $LFT_u(M, \Delta_u)$. The diagram on the right corresponds to the system equations:

$$\begin{bmatrix} y_2 \\ x_2 \end{bmatrix} = M \begin{bmatrix} u_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} u_2 \\ z_2 \end{bmatrix},$$
$$z_2 = \Delta_\ell x_2,$$

and the closed-loop system is given by $LFT_\ell(M, \Delta_\ell)$.

Thus, if M represents a system realization and Δ_u is replaced by a complex variable z, then the upper LFT is neither more nor less than a transfer function of the classical discrete-time linear system, i.e.

$$LFT_u(M, z) = D + zC(I - zA)^{-1}B.$$

For further discussion, see e.g. [ZDG96].

5.2.1 Classical Discrete-time Linear Time-invariant Systems

The main purpose of this Subsection is to present the system model for the classical discrete-time linear time-invariant systems in the operator theoretical approach, which will be used to illustrate the connection between the robust control and the noncommutative d-D linear system in Subsection 5.2.3.

For any Hilbert space $\mathcal{H}$, we denote by $\ell^2_+(\mathcal{H}) = \ell^2(\mathbb{Z}_+, \mathcal{H})$ the set of all $\mathcal{H}$-valued functions consisting of all infinite tuples of the form $f = \begin{bmatrix} f_0 & f_1 & f_2 & \dots \end{bmatrix}^\top$, for each $f_k \in \mathcal{H}$, and such that

$$\|f\|^2_{\ell^2_+(\mathcal{H})} \triangleq \sum_{k \in \mathbb{Z}_+} \|f_k\|^2_{\mathcal{H}} < \infty. \tag{5.5}$$

Let $\mathcal{S}$ be the forward shift operator acting on $\ell^2_+(\mathcal{H})$ defined by

$$\mathcal{S} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 & \cdots \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ f_0 \\ f_1 \\ \vdots \end{bmatrix}, \tag{5.6}$$

where the size of the identity operator I is determined from the context. Note that we shall write S rather than $\mathcal{S}$ if the identity operator I is replaced by 1.

Suppose now that the Hilbert spaces $\mathcal{H}, \mathcal{U}$ and $\mathcal{Y}$ are the state space, the input space and the output space, respectively, and U denotes the connecting operator. Then the system equation of the classical discrete-time linear time-invariant system can be described by (see Section 3.2 for further discussion):

$$\begin{bmatrix} x_{k+1} \\ y_k \end{bmatrix} = U \begin{bmatrix} x_k \\ u_k \end{bmatrix} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \quad x_0 = 0 \tag{5.7}$$

where $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}), B \in \mathcal{L}(\mathcal{U}, \mathcal{H}), C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. In particular, one may consider the case when $\mathcal{H} = \mathbb{C}^n, \mathcal{U} = \mathbb{C}^{n_u}$, and $\mathcal{Y} = \mathbb{C}^{n_y}$ for simplicity.

Under assumption that the state sequence $x := \{x_k\}_{k=0}^\infty$, the input sequence $u := \{u_k\}_{k=0}^\infty$, and the output sequence $y := \{y_k\}_{k=0}^{\infty}$ are all square summable, i.e. $x \in \ell_+^2(\mathcal{H}), u \in \ell_+^2(\mathcal{U})$, and $y \in \ell_+^2(\mathcal{Y})$, respectively, the system equation (5.7) can be expressed as an infinite block matrix

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & I \end{bmatrix} \mathbf{U} \begin{bmatrix} x \\ u \end{bmatrix} := \begin{bmatrix} \mathcal{S} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \tag{5.8}$$

where $\mathbf{U} : \begin{bmatrix} \ell_+^2(\mathbb{C}^n) \\ \ell_+^2(\mathbb{C}^{n_u}) \end{bmatrix} \mapsto \begin{bmatrix} \ell_+^2(\mathbb{C}^n) \\ \ell_+^2(\mathbb{C}^{n_y}) \end{bmatrix}$, and $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ are block-diagonal operators defined as follows:

Definition 27. For any Hilbert spaces $\mathcal{F}$ and $\mathcal{G}$, a bounded linear operator $\mathbf{Q} \in \mathcal{L}(\ell^2(\mathcal{F}), \ell^2(\mathcal{G}))$ is said to admit a *block-diagonal* structure if there exists an operator $Q \in \mathcal{L}(\mathcal{F}, \mathcal{G})$ such that, if $g = \mathbf{Q}f$ for any $f \in \mathcal{F}$ and $g \in \mathcal{G}$, then $g_k = Qf_k$ for all $k = 0, 1, \ldots$. The representation of such $\mathbf{Q}$ can be expressed as an infinite diagonal matrix: $\mathbf{Q} = \mathrm{diag}\{Q, Q, \ldots\}$.

5.2.2 Systems with Uncertainty

Suppose that the system in (5.7) is disturbed by some external perturbation sources or involved with system uncertainty—e.g., noises, small disturbances, unmodeled dynamics, non-dominant nonlinearities, or parameter variations—which we shall model as Δ, the uncertainty operator. Since each perturbation source is likely to enter the system in which we are interested at a different location, the uncertainty operator can be represented by a block diagonal matrix:

$$\Delta := \mathrm{diag}\left\{ \bigoplus_{j=1}^{r_1} \delta_1, \ldots, \bigoplus_{j=1}^{r_s} \delta_s, \Delta_1, \ldots, \Delta_f \right\} : \mathcal{H}_\Delta \mapsto \mathcal{H}_\Delta, \tag{5.9}$$

where $\mathcal{H}_\Delta$ is the space on which the uncertainty operator acts and has a direct sum structure as

$$\mathcal{H}_\Delta = \bigoplus_{i=1}^{s} \left[\bigoplus_{j=1}^{r_i} \mathcal{H}_{\delta_i} \right] \bigoplus \left[\bigoplus_{j=1}^{f} \mathcal{H}_{\Delta_j} \right]. \tag{5.10}$$

In the robust control literature, the nominal plant in general can be represented by

$$\Sigma_k = \begin{cases} x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k \\ z_k &= C_1 x_k + D_1 w_k + D_2 u_k\,; \quad x_0 = 0 \\ y_k &= C_2 x_k + D_3 w_k + D_4 u_k \end{cases} \tag{5.11}$$

together with the uncertainty portion: $w_k = \Delta z_k$.

To keep the exposition simple, we here assume that $\mathcal{H}_{\delta_i} = \mathbb{C}$ with $\delta_i \in \mathbb{C}$, and $\mathcal{H}_{\Delta_j} = \mathbb{C}^{n_j}$ with $\Delta_j \in \mathcal{L}(\mathbb{C}^{n_j})$ (i.e., Δ_j equal to any $n_j \times n_j$ matrix). Thus the perturbation space $\mathcal{H}_\Delta$ is

$\mathbb{C}^{n_\Delta}$ where $n_\Delta = \sum_{i=1}^{s} r_i + \sum_{j=1}^{f} n_j$.

If we also assume that all signals $x := \{x_k\}_{k=0}^{\infty}$, $u := \{u_k\}_{k=0}^{\infty}$, $y := \{y_k\}_{k=0}^{\infty}$, $z := \{z_k\}_{k=0}^{\infty}$, and $w := \{w_k\}_{k=0}^{\infty}$ are square summable, then the system equation (5.11) can be expressed as

$$\Sigma = \begin{cases} x &= \mathcal{S}\left[\mathbf{A}x + \mathbf{B}_1 w + \mathbf{B}_2 u\right] \\ z &= \mathbf{C}_1 x + \mathbf{D}_1 w + \mathbf{D}_2 u \\ y &= \mathbf{C}_2 x + \mathbf{D}_3 w + \mathbf{D}_4 u \end{cases}, \quad \text{and} \quad w = \mathbf{\Delta} z, \tag{5.12}$$

where $\mathbf{\Delta} = \text{diag}\{\Delta, \Delta, \dots\}$. The block diagram of the overall system (the nominal system plus disturbances, $\mathbf{\Delta}$) is depicted in Figure 5.2.

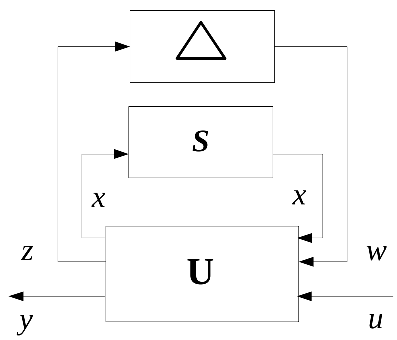

Figure 5.2: A block diagram of the system with disturbances, $\mathbf{\Delta}$

To condense notation, let us define an operator $\mathbf{U}$ as

$$\mathbf{U} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} : \begin{bmatrix} \ell_+^2(\mathbb{C}^n) \\ \ell_+^2(\mathbb{C}^{n_\Delta}) \\ \ell_+^2(\mathbb{C}^{n_u}) \end{bmatrix} \mapsto \begin{bmatrix} \ell_+^2(\mathbb{C}^n) \\ \ell_+^2(\mathbb{C}^{n_\Delta}) \\ \ell_+^2(\mathbb{C}^{n_y}) \end{bmatrix},$$

where $\mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ are partitioned as follows:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}, \text{ and } \mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{D}_2 \\ \mathbf{D}_3 & \mathbf{D}_4 \end{bmatrix}.$$

Thus, we have

$$\begin{bmatrix} x \\ z \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & I \end{bmatrix} \mathbf{U} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \quad \text{and} \quad w = \mathbf{\Delta} z, \tag{5.13}$$

If we consider now that $\begin{bmatrix} w \\ u \end{bmatrix}$ is the input and $\begin{bmatrix} z \\ y \end{bmatrix}$ is the output of the plant, and by assuming that the system is well-defined[2], then the input/output (i/o) mapping operator

$$G(\mathcal{S}) : \begin{bmatrix} \ell^2_+(\mathbb{C}^{n_\Delta}) \\ \ell^2_+(\mathbb{C}^{n_u}) \end{bmatrix} \mapsto \begin{bmatrix} \ell^2_+(\mathbb{C}^{n_\Delta}) \\ \ell^2_+(\mathbb{C}^{n_y}) \end{bmatrix},$$

is given by

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(\mathcal{S}) \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{where } G(\mathcal{S}) := \mathbf{D} + \mathbf{C}(I - \mathcal{S}\mathbf{A})^{-1}\mathcal{S}\mathbf{B}. \tag{5.14}$$

Clearly, this operator can also be viewed as an upper LFT, i.e. $G(\mathcal{S}) = LFT_u(\mathbf{U}, \mathcal{S})$. Since $w = \mathbf{\Delta} z$, one can write (5.14) as

$$\begin{bmatrix} w \\ y \end{bmatrix} = \begin{bmatrix} \mathbf{\Delta} & 0 \\ 0 & I \end{bmatrix} G(\mathcal{S}) \begin{bmatrix} w \\ u \end{bmatrix}.$$

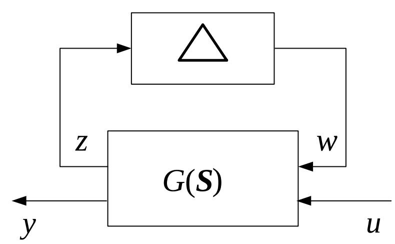

Figure 5.3: An equivalent diagram of the system in Figure 5.2 where $G(\mathcal{S}) = LFT_u(\mathbf{U}, \mathcal{S})$.

We then apply the upper LFT again with $G(\mathcal{S})$ acting on the uncertainty operator $\mathbf{\Delta}$ to get the closed loop system $T_{\mathbf{\Delta}}$ (i.e., the i/o map from u to y) when the perturbation $\mathbf{\Delta}$ is presented as shown in Figure 5.3. Thus, we have

$$T_{\mathbf{\Delta}}(\mathcal{S}) = LFT_u(G(\mathcal{S}), \mathbf{\Delta}) = LFT_u(LFT_u(\mathbf{U}, \mathcal{S}), \mathbf{\Delta}).$$

For analysis purposes, it is of interest to combine the shift operator $\mathcal{S}$ and the uncertainty operator $\mathbf{\Delta}$ together in the same block. To this end, let us consider the system equation (5.12) for the moment. Since $w = \mathbf{\Delta} z$, one can replace $z = \mathbf{C}_1 x + \mathbf{D}_1 w + \mathbf{D}_2 u$ by $w =$

[2]If $\mathbf{U} := \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ is contraction, then $G(\mathcal{S})$ is well-defined as

$$\lim_{r \uparrow 1} \mathbf{C}(I - r\mathcal{S}\mathbf{A})^{-1} x \quad \text{exists for each } x$$

(see [BLTT99] for an explanation).

$\mathbf{\Delta}\left[\mathbf{C}_1 x + \mathbf{D}_1 w + \mathbf{D}_2 u\right]$. Thus, we have the following

$$\Sigma = \begin{cases} \begin{bmatrix} x \\ w \end{bmatrix} &= \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathbf{\Delta} \end{bmatrix} \left(\begin{bmatrix} \mathbf{A} & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_2 \end{bmatrix} u \right) \\ y &= \begin{bmatrix} \mathbf{C}_2 & \mathbf{D}_3 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \mathbf{D}_4 u. \end{cases} \tag{5.15}$$

For notational convenience, let us introduce some operators as follows:

$$\widehat{\mathbf{\Delta}} := \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathbf{\Delta} \end{bmatrix} : \begin{bmatrix} \ell^2_+(\mathbb{C}^n) \\ \ell^2_+(\mathbb{C}^{n_\Delta}) \end{bmatrix} \mapsto \begin{bmatrix} \ell^2_+(\mathbb{C}^n) \\ \ell^2_+(\mathbb{C}^{n_\Delta}) \end{bmatrix},$$

$$\widehat{\mathbf{A}} := \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix}, \quad \widehat{\mathbf{B}} := \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_2 \end{bmatrix}, \quad \widehat{\mathbf{C}} := \begin{bmatrix} \mathbf{C}_2 & \mathbf{D}_3 \end{bmatrix}, \quad \widehat{\mathbf{D}} := \mathbf{D}_4.$$

Hence, (5.15) becomes

$$\begin{bmatrix} x \\ w \\ y \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{\Delta}} & 0 \\ 0 & I \end{bmatrix} \widehat{\mathbf{U}} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{5.16}$$

where

$$\widehat{\mathbf{U}} = \begin{bmatrix} \widehat{\mathbf{A}} & \widehat{\mathbf{B}} \\ \widehat{\mathbf{C}} & \widehat{\mathbf{D}} \end{bmatrix} : \begin{bmatrix} \ell^2_+(\mathbb{C}^n) \\ \ell^2_+(\mathbb{C}^{n_\Delta}) \\ \ell^2_+(\mathbb{C}^{n_u}) \end{bmatrix} \mapsto \begin{bmatrix} \ell^2_+(\mathbb{C}^n) \\ \ell^2_+(\mathbb{C}^{n_\Delta}) \\ \ell^2_+(\mathbb{C}^{n_y}) \end{bmatrix}.$$

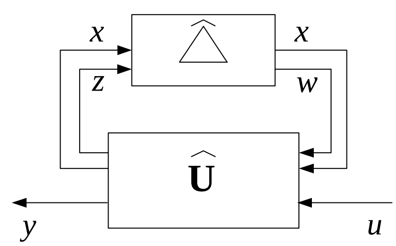

Figure 5.4: An equivalent diagram of the system in Figure 5.2 where $\widehat{\mathbf{\Delta}} = \operatorname{diag}\{\mathcal{S}, \mathbf{\Delta}\}$

The system equation in the form of (5.16) yields the i/o map (from the input u to the output y), which can be expressed directly in terms of an upper LFT with constant coefficients acting on the uncertainty operator $\mathbf{\Delta}$ augmented by the shift operator $\mathcal{S}$:

$$y = LFT_u(\widehat{\mathbf{U}}, \widehat{\mathbf{\Delta}}) \cdot u = \left[\widehat{\mathbf{D}} + \widehat{\mathbf{C}} \left(I - \widehat{\mathbf{\Delta}} \widehat{\mathbf{A}} \right)^{-1} \widehat{\mathbf{\Delta}} \widehat{\mathbf{B}} \right] \cdot u. \tag{5.17}$$

By the definition of the upper LFT and the application of the inversion lemma, one can easily verify that

$$LFT_u(\widehat{\mathbf{U}}, \widehat{\boldsymbol{\Delta}}) = LFT_u(LFT_u(\mathbf{U}, \mathcal{S}), \boldsymbol{\Delta}) = T_{\boldsymbol{\Delta}}(\mathcal{S}). \tag{5.18}$$

Before we move on to the next Section, let us point out that the space $\ell^2_+(\mathbb{C}^p)$ is isomorphic to p copies of the space $\ell^2_+(\mathbb{C})$, i.e.

$$\ell^2_+(\mathbb{C}^p) \cong \bigoplus_{i=1}^{p} \ell^2_+(\mathbb{C}).$$

The application of this fact is illustrated in the following. Let us first consider the system equation (5.15) for a moment. By using the rows and columns permutation appropriately, one can verify that the system described as in (5.15) is equivalent to

$$\widetilde{\Sigma} = \begin{cases} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} &= \begin{bmatrix} \bigoplus_{i=1}^n S & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix} \left(\begin{bmatrix} \widetilde{A} & \widetilde{B}_1 \\ \widetilde{C}_1 & \widetilde{D}_1 \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} + \begin{bmatrix} \widetilde{B}_2 \\ \widetilde{D}_2 \end{bmatrix} \widetilde{u} \right) \\ \quad \widetilde{y} &= \begin{bmatrix} \widetilde{C}_2 & \widetilde{D}_3 \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} + \widetilde{D}_4 \widetilde{u}, \end{cases} \tag{5.19}$$

where

1. $\widetilde{f} = \begin{bmatrix} \widetilde{f}_1 \\ \vdots \\ \widetilde{f}_p \end{bmatrix}$, each $\widetilde{f}_j \in \ell^2_+(\mathbb{C})$ if $f = \begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix}$, each $f_j \in \mathbb{C}$,

2. the uncertainty operator $\widetilde{\Delta}$ is given by

$$\widetilde{\Delta} = \operatorname{diag}\left\{ \bigoplus_{j=1}^{r_1} \boldsymbol{\delta}_1, \ldots, \bigoplus_{j=1}^{r_s} \boldsymbol{\delta}_s, \widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_f \right\} : \mathcal{H}_{\widetilde{\Delta}} \mapsto \mathcal{H}_{\widetilde{\Delta}}, \tag{5.20}$$

where $\boldsymbol{\delta}_j = \operatorname{diag}\{\delta_j, \delta_j, \ldots\} = \delta_j I_{\ell^2}$, $\widetilde{\Delta}_k = [\boldsymbol{\delta}^k_{i,j}]_{i,j=1}^{n_k} = [\delta^k_{i,j} I_{\ell^2}]_{i,j=1}^{n_k}$, and $\mathcal{H}_{\widetilde{\Delta}} = \bigoplus_{i=1}^{n_\Delta} \ell^2_+(\mathbb{C})$,

3. the operator $\widetilde{A} = [\mathbf{a}_{i,j}]_{i,j=1}^n$, where $\mathbf{a}_{i,j} := \operatorname{diag}\{a_{i,j}, a_{i,j}, \ldots\} = a_{i,j} I_{\ell^2}$ if $A = [a_{i,j}]_{i,j=1}^n$.

Note that the other operators in (5.19) are also defined in the similar way as $\widetilde{A}$.

Let us now define the following operators:

$$\widetilde{\boldsymbol{\Delta}} := \begin{bmatrix} \bigoplus_{i=1}^n S & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix} : \begin{bmatrix} \bigoplus_{i=1}^n \ell^2_+(\mathbb{C}) \\ \bigoplus_{i=1}^{n_\Delta} \ell^2_+(\mathbb{C}) \end{bmatrix} \mapsto \begin{bmatrix} \bigoplus_{i=1}^n \ell^2_+(\mathbb{C}) \\ \bigoplus_{i=1}^{n_\Delta} \ell^2_+(\mathbb{C}) \end{bmatrix},$$

$$\widetilde{\mathbf{A}} := \begin{bmatrix} \widetilde{A} & \widetilde{B}_1 \\ \widetilde{C}_1 & \widetilde{D}_1 \end{bmatrix}, \quad \widetilde{\mathbf{B}} := \begin{bmatrix} \widetilde{B}_2 \\ \widetilde{D}_2 \end{bmatrix}, \quad \widetilde{\mathbf{C}} := \begin{bmatrix} \widetilde{C}_2 & \widetilde{D}_3 \end{bmatrix}, \quad \widetilde{\mathbf{D}} := \widetilde{D}_4,$$

and hence the system $\widetilde{\Sigma}$ in (5.19) becomes

$$\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \\ \widetilde{y} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{\Delta}} & 0 \\ 0 & I \end{bmatrix} \widetilde{\mathbf{U}} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \\ \widetilde{u} \end{bmatrix}, \tag{5.21}$$

where

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{\mathbf{A}} & \widetilde{\mathbf{B}} \\ \widetilde{\mathbf{C}} & \widetilde{\mathbf{D}} \end{bmatrix} : \begin{bmatrix} \bigoplus_{i=1}^{n} \ell_+^2(\mathbb{C}) \\ \bigoplus_{i=1}^{n_\Delta} \ell_+^2(\mathbb{C}) \\ \bigoplus_{i=1}^{n_u} \ell_+^2(\mathbb{C}) \end{bmatrix} \mapsto \begin{bmatrix} \bigoplus_{i=1}^{n} \ell_+^2(\mathbb{C}) \\ \bigoplus_{i=1}^{n_\Delta} \ell_+^2(\mathbb{C}) \\ \bigoplus_{i=1}^{n_y} \ell_+^2(\mathbb{C}) \end{bmatrix}.$$

The above discussion is illustrated in the following example.

Example 11. Let $x_k = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_k \in \mathbb{C}^2$, $w_k = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}_k \in \mathbb{C}^3$, $u_k \in \mathbb{C}$, and $\Delta = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{bmatrix}$. To save the space, let us also assume that $\widehat{\mathbf{A}} = 0$. Then, from (5.15) we have

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathbf{\Delta} \end{bmatrix} \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_2 \end{bmatrix} u \quad \text{where } x \in \ell_+^2(\mathbb{C}^2), w \in \ell_+^2(\mathbb{C}^3), u \in \ell_+^2(\mathbb{C})$$

$$\begin{bmatrix} x_{k=0} \\ x_{k=1} \\ x_{k=2} \\ \vdots \\ w_{k=0} \\ w_{k=1} \\ w_{k=2} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0_{2\times2} & 0_{2\times2} & \cdots & 0_{2\times3} & 0_{2\times3} & \cdots \\ I_{2\times2} & 0_{2\times2} & \cdots & 0_{2\times3} & 0_{2\times3} & \cdots \\ 0_{2\times2} & I_{2\times2} & \cdots & 0_{2\times3} & 0_{2\times3} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0_{3\times3} & 0_{3\times3} & \cdots & \Delta & 0_{3\times3} & \cdots \\ 0_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & \Delta & \cdots \\ 0_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & 0_{3\times3} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \begin{bmatrix} b_{2,1} \\ b_{2,2} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \cdots \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} b_{2,1} \\ b_{2,2} \end{bmatrix} & \cdots \\ \vdots & \vdots & \ddots \\ \begin{bmatrix} d_{2,1} \\ d_{2,2} \\ d_{2,3} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \cdots \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} d_{2,1} \\ d_{2,2} \\ d_{2,3} \end{bmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_{k=0} \\ u_{k=1} \\ u_{k=2} \\ \vdots \end{bmatrix}$$

$$
= \begin{bmatrix}
\begin{bmatrix}0\\0\end{bmatrix} & \begin{bmatrix}0\\0\end{bmatrix} & \cdots \\
\begin{bmatrix}b_{2,1}\\b_{2,2}\end{bmatrix} & \begin{bmatrix}0\\0\end{bmatrix} & \cdots \\
\begin{bmatrix}0\\0\end{bmatrix} & \begin{bmatrix}b_{2,1}\\b_{2,2}\end{bmatrix} & \cdots \\
\vdots & \vdots & \ddots \\
\begin{bmatrix}\delta_1 d_{2,1}\\ \delta_1 d_{2,2}\\ \delta_2 d_{2,3}\end{bmatrix} & \begin{bmatrix}0\\0\\0\end{bmatrix} & \cdots \\
\begin{bmatrix}0\\0\\0\end{bmatrix} & \begin{bmatrix}\delta_1 d_{2,1}\\ \delta_1 d_{2,2}\\ \delta_2 d_{2,3}\end{bmatrix} & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}u_{k=0}\\u_{k=1}\\u_{k=2}\\ \vdots\end{bmatrix},
$$

which is equivalent to

$$
\begin{bmatrix}
\begin{pmatrix}\widetilde{x}_1\\ \widetilde{x}_2\end{pmatrix}\\
\begin{pmatrix}\widetilde{w}_1\\ \widetilde{w}_2\\ \widetilde{w}_3\end{pmatrix}
\end{bmatrix}
=
\begin{bmatrix}
\begin{bmatrix}0 & 0 & 0 & \cdots\\ b_{2,1} & 0 & 0 & \cdots\\ 0 & b_{2,1} & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}\\
\begin{bmatrix}0 & 0 & 0 & \cdots\\ b_{2,2} & 0 & 0 & \cdots\\ 0 & b_{2,2} & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}\\
\begin{bmatrix}\delta_1 d_{2,1} & 0 & 0 & \cdots\\ 0 & \delta_1 d_{2,1} & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}\\
\begin{bmatrix}\delta_1 d_{2,2} & 0 & 0 & \cdots\\ 0 & \delta_1 d_{2,2} & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}\\
\begin{bmatrix}\delta_2 d_{2,3} & 0 & 0 & \cdots\\ 0 & \delta_2 d_{2,3} & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}
\end{bmatrix}
\widetilde{u}
=
\begin{bmatrix}
\begin{bmatrix}S & 0\\ 0 & S\end{bmatrix} & 0\\
0 & \begin{bmatrix}\boldsymbol{\delta}_1 & 0 & 0\\ 0 & \boldsymbol{\delta}_1 & 0\\ 0 & 0 & \boldsymbol{\delta}_2\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}b_{2,1} & 0 & \cdots\\ 0 & b_{2,1} & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix}\\
\begin{bmatrix}b_{2,2} & 0 & \cdots\\ 0 & b_{2,2} & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix}\\
\begin{bmatrix}d_{2,1} & 0 & \cdots\\ 0 & d_{2,1} & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix}\\
\begin{bmatrix}d_{2,2} & 0 & \cdots\\ 0 & d_{2,2} & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix}\\
\begin{bmatrix}d_{2,3} & 0 & \cdots\\ 0 & d_{2,3} & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix}
\end{bmatrix}
\widetilde{u}
$$

$$
\begin{bmatrix}\widetilde{x}\\ \widetilde{w}\end{bmatrix}
=
\begin{bmatrix}\bigoplus_{i=1}^{2} S & 0\\ 0 & \widetilde{\boldsymbol{\Delta}}\end{bmatrix}
\begin{bmatrix}\widetilde{\mathbf{B}}_2\\ \widetilde{\mathbf{D}}_2\end{bmatrix}
\widetilde{u}, \quad \text{where } \widetilde{x} \in \bigoplus_{i=1}^{2} \ell_+^2(\mathbb{C}), \widetilde{w} \in \bigoplus_{i=1}^{3} \ell_+^2(\mathbb{C}), \widetilde{u} \in \ell_+^2(\mathbb{C}). \qquad \diamondsuit
$$

5.2.3 The Connection with Multidimensional Linear Systems

The point here of working with i/o map in the time domain rather than the transfer function in the frequency domain as has been done in the earlier μ-synthesis literature is that we can easily introduce time-varying disturbances in this formalism. Recall that the system $\widetilde{\Sigma}$ is described by

$$\widetilde{\Sigma} = \begin{cases} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} &= \begin{bmatrix} \bigoplus_{i=1}^{n} S & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix} \left(\widetilde{\mathbf{A}} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} + \widetilde{\mathbf{B}}\widetilde{u} \right) \\ \widetilde{y} &= \widetilde{\mathbf{C}} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} + \widetilde{\mathbf{D}}\widetilde{u}. \end{cases} \tag{5.22}$$

Formally, if we replace $\delta_j I_{\ell^2}$ and $\delta_{i,j}^k I_{\ell^2}$, respectively with general operators on ℓ^2, say $\boldsymbol{\delta}_j \in \mathcal{L}(\ell^2)$ and $\boldsymbol{\delta}_{i,j}^k \in \mathcal{L}(\ell^2)$ (these operators are regarded as time-varying uncertainty operators), and form the uncertainty operator $\widetilde{\Delta}$ as in (5.20), then with this more general meaning for $\widetilde{\Delta}$, all the formulas derived previously go through in the same way; i.e., one can check that the i/o map from $\widetilde{u} \in \bigoplus_{i=1}^{n_u} \ell_+^2(\mathbb{C})$ to $\widetilde{y} \in \bigoplus_{i=1}^{n_y} \ell_+^2(\mathbb{C})$ is given by the upper LFT

$$\widetilde{y} = LFT_u(\widetilde{\mathbf{U}}, \widetilde{\boldsymbol{\Delta}}) \cdot \widetilde{u} = \left[\widetilde{\mathbf{D}} + \widetilde{\mathbf{C}} \left(I - \widetilde{\boldsymbol{\Delta}}\widetilde{\mathbf{A}} \right)^{-1} \widetilde{\boldsymbol{\Delta}}\widetilde{\mathbf{B}} \right] \cdot \widetilde{u}. \tag{5.23}$$

In this case, $LFT_u(\widetilde{\mathbf{U}}, \widetilde{\boldsymbol{\Delta}})$ represents the i/o operator for a time-varying linear system, where the entries of $\widetilde{\Delta}$ no longer commute with the forward shift operator S on ℓ_+^2. Here the nominal system ((5.22) with $\widetilde{\Delta} = 0$) is time-invariant, but the system perturbed by the time-varying disturbance $\widetilde{\Delta}$ is time-varying. The role of formal power series in analyzing linear time-invariant plants having time-varying structured uncertainties was introduced in the work of Beck, D'Andrea, Doyle and Glover [Bec01, BD99, BD97, ZDG96] in a more formal, but less precise way.

From the above arguments, we now define

$$\delta := (S, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_s, \boldsymbol{\delta}_{1,1}^1, \boldsymbol{\delta}_{1,2}^1, \ldots, \boldsymbol{\delta}_{n_1,n_1}^1, \ldots, \boldsymbol{\delta}_{1,1}^f, \boldsymbol{\delta}_{1,2}^f, \ldots, \boldsymbol{\delta}_{n_f,n_f}^f),$$

and recall that $\boldsymbol{\delta}_i$ and $\boldsymbol{\delta}_{i,j}^k$ are general operators in ℓ^2 representing noises or small perturbation parameters entering to the system in different locations. From the mathematical point of view these operators including the shift operator in general can be regarded as noncommuting indeterminants.

Thus, one may replace δ by $z = (z_1, z_2, \ldots, z_d)$ the noncommutative variables where $d = 1 + s + \sum_{i=1}^{f} n_i^2$ and $z_i z_j \neq z_j z_i$ unless $i = j$, and hence the i/o map (5.23) is neither more nor less than the noncommutative d-variable transfer function of input/state/output (i/s/o) multidimensional linear systems with evolution along a free semigroup. We shall show that in fact such a transfer function can also be expressed as a formal power series $\sum_{v\in\mathcal{F}_d}\widetilde{T}_v z^v$ where $\widetilde{T}_\lambda=\widetilde{D}$, $\widetilde{T}_{wg_j}=\widetilde{\mathbf{C}}\widetilde{\mathbf{A}}^w\widetilde{\mathbf{B}}_j$, and $\widetilde{\mathbf{A}}^w=\widetilde{\mathbf{A}}_{i_n}\widetilde{\mathbf{A}}_{i_{n-1}}\cdots\widetilde{\mathbf{A}}_{i_1}$ if $w=g_{i_n}g_{i_{n-1}}\cdots g_{i_1}$.

To do so, let us replace the operator $\widetilde{\mathbf{\Delta}}=\begin{bmatrix}\bigoplus_{i=1}^n S & 0\\ 0 & \widetilde{\Delta}\end{bmatrix}$ with

$$Z(z)=\begin{bmatrix} z_1 I_{\mathcal{H}} & & & & \\ & \begin{bmatrix} z_2 I_{r_1} & & \\ & \ddots & \\ & & z_{s+1}I_{r_s}\end{bmatrix} & & & \\ & & \begin{bmatrix} z_{s+2} & \cdots & z_{s+1+n_1}\\ \vdots & & \vdots \\ z_{s+2+n_1^2-n_1} & \cdots & z_{s+1+n_1^2}\end{bmatrix} & & \\ & & & \ddots & \\ & & & & \begin{bmatrix} z_{d-n_f^2+1} & \cdots & z_{d-n_f^2+n_f}\\ \vdots & & \vdots\\ z_{d-n_f+1} & \cdots & z_d\end{bmatrix}\end{bmatrix}$$

From this point on, we shall refer to the matrix $Z(z)$ as the *generalized structured noncommutative dynamics*. It should be noted that $Z(z)$ can be expanded as a finite sum

$$Z(z)=z_1\begin{bmatrix} I_{\mathcal{H}} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}+z_2\begin{bmatrix} 0 & 0 & \cdots & 0\\ 0 & I_{r_1} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}+\cdots+z_d\begin{bmatrix} 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}, \tag{5.24}$$

and hence $Z(z)\widetilde{\mathbf{A}}=\sum_{j=1}^d z_j\widetilde{\mathbf{A}}_j$ where $\widetilde{\mathbf{A}}_j=\begin{bmatrix} 0 & 0 & \cdots & 0\\ 0 & \underbrace{I}_{\text{(at the location of } z_j)} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\widetilde{\mathbf{A}}$ and the size of the identity operator I is determined from the context. Similarly, $Z(z)\widetilde{\mathbf{B}}=\sum_{j=1}^d z_j\widetilde{\mathbf{B}}_j$. Thus, one can rewrite the i/o map (5.23) as

$$\widetilde{y}=\left[\widetilde{\mathbf{D}}+\widetilde{\mathbf{C}}\left(I-\widetilde{\mathbf{\Delta}}\widetilde{\mathbf{A}}\right)^{-1}\widetilde{\mathbf{\Delta}}\widetilde{\mathbf{B}}\right]\cdot\widetilde{u}$$

$$= \left[\widetilde{\mathbf{D}} + \widetilde{\mathbf{C}}\left(I - \sum_{j=1}^{d} z_j \widetilde{\mathbf{A}}_j\right)^{-1} \sum_{j=1}^{d} z_j \widetilde{\mathbf{B}}_j\right] \cdot \widetilde{u} \quad \triangleq T_{\widetilde{\Sigma}}(z)\widetilde{u}. \tag{5.25}$$

Since the inverse $\left(I - \sum_{j=1}^{d} z_j \widetilde{\mathbf{A}}_j\right)^{-1}$ exists as a formal power series $\sum_{k=0}^{\infty}\left(\sum_{j=1}^{d} z_j \widetilde{\mathbf{A}}_j\right)^k$, we have the following

$$\begin{aligned} T_{\widetilde{\Sigma}}(z) &= \widetilde{\mathbf{D}} + \widetilde{\mathbf{C}}\left(I - \sum_{j=1}^{d} z_j \widetilde{\mathbf{A}}_j\right)^{-1} \sum_{j=1}^{d} z_j \widetilde{\mathbf{B}}_j \\ &= \widetilde{\mathbf{D}} + \widetilde{\mathbf{C}} \sum_{k=0}^{\infty}\left(\sum_{j=1}^{d} z_j \widetilde{\mathbf{A}}_j\right)^k \sum_{j=1}^{d} z_j \widetilde{\mathbf{B}}_j \\ &= \widetilde{\mathbf{D}} + \sum_{w \in \mathcal{F}_d} \sum_{j=1}^{d} \left(\widetilde{\mathbf{C}}\widetilde{\mathbf{A}}^w \widetilde{\mathbf{B}}_j\right) z^{w g_j} \quad \triangleq \sum_{v \in \mathcal{F}_d} \widetilde{T}_v z^v, \end{aligned} \tag{5.26}$$

where $\widetilde{T}_\lambda = \widetilde{\mathbf{D}}$, $\widetilde{T}_{wg_j} = \widetilde{\mathbf{C}}\widetilde{\mathbf{A}}^w\widetilde{\mathbf{B}}_j$, and $\widetilde{\mathbf{A}}^w = \widetilde{\mathbf{A}}_{i_n}\widetilde{\mathbf{A}}_{i_{n-1}} \cdots \widetilde{\mathbf{A}}_{i_1}$ if $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$. Thus, the i/o map from the input sequence $\widetilde{u} \in \bigoplus_{i=1}^{n_u} \ell_+^2(\mathbb{C})$ to the output sequence $\widetilde{y} \in \bigoplus_{i=1}^{n_y} \ell_+^2(\mathbb{C})$ can be expressed as a formal power series as required.

Conversely, suppose we are given a formal power series $T_{\widetilde{\Sigma}}(z) = \sum_{v \in \mathcal{F}_d} \widetilde{T}_v z^v$, where $z = (z_1, \ldots, z_d)$ the noncommuting d-indeterminants. Then the i/o operator $LFT_u(\widetilde{\mathbf{U}}, \widetilde{\mathbf{\Delta}})$ for the system $\widetilde{\Sigma}$ with time-varying perturbation $\widetilde{\Delta}$ is obtained by formally replacing z by disturbance operator[3] $\delta = (\delta_1, \ldots, \delta_d)$. Therefore, we have

$$T_{\widetilde{\Sigma}}(\delta) = D \otimes I_{\ell^2} + (C \otimes I_{\ell^2})\left(I - \sum_{j=1}^{d} A_j \otimes \delta_j\right)^{-1}\left(\sum_{j=1}^{d} B_j \otimes \delta_j\right), \tag{5.27}$$

which can also be expressed as

$$T_{\widetilde{\Sigma}}(\delta) = \sum_{v \in \mathcal{F}_d} \widetilde{T}_v \otimes \delta^v. \tag{5.28}$$

(See Appendix A for a brief discussion on tensor product and the proof.)

Thus, the above discussion gives a connection between the robust control theory and the noncommutative multidimensional linear system theory, which motivates the author to study the possibility to construct linear models representing the i/s/o d-D linear system described above. The discussion on modeling such a system and the corresponding transfer function will be presented in Chapter 6.

Remark 19. It is worth noting that the generalized structured noncommutative dynamics

[3] To simplify the exposition, let us rename $\delta = (S, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_s, \boldsymbol{\delta}^1_{n_1,n_1}, \ldots, \boldsymbol{\delta}^f_{n_f,n_f})$ by $(\delta_1, \ldots, \delta_d)$.

$Z(z)$ is a generalized version of the noncommutative Givone-Roesser (or, NCGR) dynamics (see Chapter 6 for more details on system modelling issue),

$$Z_d(z) = \mathrm{diag}\{z_1 I_{\mathcal{H}_1}, \ldots, z_d I_{\mathcal{H}_d}\},$$

and the noncommutative Fornasini-Marchesini (or, NCFM) dynamics,

$$Z_r(z) = \begin{bmatrix} z_1 I_{\mathcal{H}} & \cdots & z_d I_{\mathcal{H}} \end{bmatrix}.$$

Obviously, the NCGR dynamics is a particular case of the generalized structured noncommutative dynamics. Now if the transfer function has the form

$$T_\Sigma(z) = D + C(I - Z_r(z)A_c)^{-1} Z_r(z) B_c,$$

where

$$\begin{bmatrix} A_c & B_c \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix}$$

then the transfer function has the explicit representation $T_\Sigma(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$ where $T_\lambda = D$ and $T_{wg_j} = C(A_c)^w (B_c)_j$. Thus the transformation $(\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \widetilde{\mathbf{C}}, \widetilde{\mathbf{D}}) \mapsto (\widetilde{\mathbf{A}}_j, \widetilde{\mathbf{B}}_j, \widetilde{\mathbf{C}}, \widetilde{\mathbf{D}})$ as in (5.25) can be viewed as a transformation from a transfer function with generalized structured noncommutative dynamics to one with NCFM dynamics. In Chapter 6 we shall see this more explicitly in terms of state-space representations for the case of NCGR dynamics $Z_d(z)$. ▲

One interesting question arising here is: given a formal power series as in (5.26), can one write it in the form of the generalized NCGR model (with the generalized structured noncommutative dynamics $Z(z)$ specified up to the dimensions so the various blocks), or in the form of the NCGR model (with $Z_d(z)$), or in the form of the NCFM model (with $Z_r(z)$)—and with minimal state space dimension(s)? This question is the so-called *minimal realization problem* which we shall give a discussion on this issue in Chapter 8.

5.3 Notion of Time-axis

This Section is devoted to a discussion on how to formulate the "time-axis" for the i/s/o linear systems with evolution along the elements of a free semigroup, $\mathcal{F}_d$. We shall see that in fact the time-axis of such systems can be represented by a so-called *homogeneous tree with a root* in the graph theory literature, which is defined as follows:

Definition 28 (Homogeneous Tree). A homogeneous tree T with a root (or base point at λ) of order q is an infinite acyclic, connected graph (i.e., a connected graph which contains no cycles) such that every vertex of T has exactly $q+1$ branches except for a vertex λ at the bottom of a tree called a *root* which has only q branches.

Given two vertices on a graph, say v and w, a (v,w)-path is a sequence of vertices with adjacent vertices connected by a branch, beginning at v and ending at w. The distance between two vertices v and w, denoted by $d(v,w)$, is the number of branches along the unique path connecting v and w. A *path* through a tree, denoted by Γ^w is a path leading up and away from the root, and ending at w. A *level* in a tree is the collection of all vertices at a fixed distance from the root.

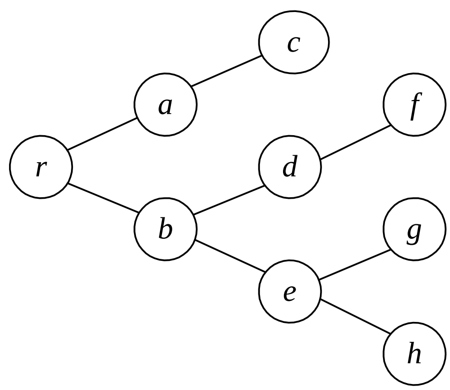

Figure 5.5: An example of a tree T

For example, let us consider the finite tree T depicted in Figure 5.5. A (b,g)-path is $\{b,e,g\}$ and the distance $d(b,g) = 2$. The path $\Gamma^f = \{r,b,d,f\}$. The levels of the tree are $\{r\},\{a,b\},\{c,d,e\}$ and $\{f,g,h\}$. For further discussion on graph theory, see e.g. [BCLF79, HHM00].

Now for a given free semigroup $\mathcal{F}_d$, one may represent $\mathcal{F}_d$ as a homogeneous tree T by labelling each vertex of T by a word $w \in \mathcal{F}_d$, where

- the root of T is labelled by λ, and
- each level of T is defined by the length of words, i.e. a level n of T consists of all words of length n ($|w| = n$).

There is a branch between v and w of distance $d(v,w) = 1$ if $v = g_j w$ for some $j \in \mathcal{I}_d$. Thus, there is a one-to-one correspondence between a free semigroup $\mathcal{F}_d$ and a homogeneous tree T with a root at λ of order d.

Given any word $w \in \mathcal{F}_d \setminus \{\lambda\}$, there always exists a path Γ^w defined as a set of all words $\widetilde{w}$ such that $v\widetilde{w} = w$ for some $v \in \mathcal{F}_d$, listed in order of increasing length. For instance, if w is given by $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$, then

$$\Gamma^w = \{\lambda, g_{i_1}, g_{i_2} g_{i_1}, \dots, g_{i_{n-1}} g_{i_{n-2}} \cdots g_{i_1}, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}\}. \tag{5.29}$$

For example, the path $\Gamma^{\{011\}}$ associated with a word $\{011\}$ is given by $\Gamma^{\{011\}} = \{\lambda, 1, 11, 011\}$. An example of the finite portion of a homogeneous tree of order 2 is illustrated in Figure 5.6.

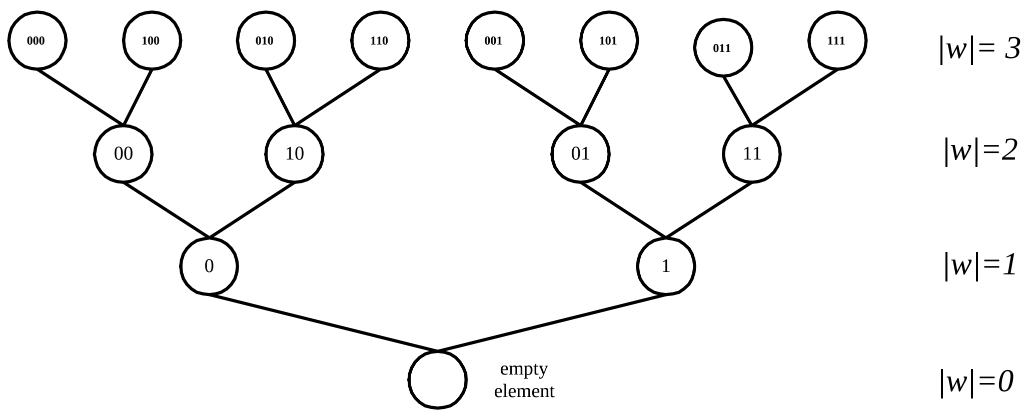

Figure 5.6: A finite portion of a homogeneous tree of order 2

Let x be a mapping $x : \mathcal{F}_d \to \mathcal{H}$ defined by $x : w \to x(w) \in \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space; i.e., we assign the value at each particular word by $x(w)$ which we shall call *the state* at the word w, and the Hilbert space $\mathcal{H}$ in this case is called the *state space*.

Suppose we are given a state at a particular word w, then the natural question arising here is whether or not one can find a control sequence $\{u(v)\}$ along the path Γ^w so that such a state can be reached from the zero initial state at λ. This property is called *w-reachability*[4] of the system, and the system is said to be *w-reachable* if there does exist a control sequence satisfying the reachability condition for arbitrary final state at w. We also have a notion of w-controllability, which is parallel with the notion of reachability but in the reverse direction.

Suppose the path associated with a word $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$ is given by

$$\Gamma^w = \{\lambda, g_{i_1}, g_{i_2} g_{i_1}, \dots, g_{i_{n-1}} g_{i_{n-2}} \cdots g_{i_1}, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}\},$$

[4]Notions of *length-n-reachability and length-n-controllability* will be given in Chapter 7.

and let us relabel each word contained in the path Γ^w as follows:

$$\begin{aligned}
\lambda &\mapsto \lambda' = g_{i_n} g_{i_{n-1}} \cdots g_{i_1} \\
g_{i_1} &\mapsto g'_{i_1} = g_{i_n} g_{i_{n-1}} \cdots g_{i_2} \\
g_{i_2} g_{i_1} &\mapsto g'_{i_2} g'_{i_1} = g_{i_n} g_{i_{n-1}} \cdots g_{i_3} \\
&\vdots \\
g_{i_n} g_{i_{n-1}} \cdots g_{i_2} &\mapsto g'_{i_n} g'_{i_{n-1}} \cdots g'_{i_2} = g_{i_n} \\
g_{i_n} g_{i_{n-1}} \cdots g_{i_2} g_{i_1} &\mapsto g'_{i_n} g'_{i_{n-1}} \cdots g'_{i_2} g'_{i_1} = \lambda.
\end{aligned}$$

Then the *reverse path* of Γ^w, denoted by ${}^w\Gamma$, is defined as

$$\begin{aligned}
{}^w\Gamma &= \{\lambda', g'_{i_1}, g'_{i_2} g'_{i_1}, \ldots, g'_{i_{n-1}} g'_{i_{n-2}} \cdots g'_{i_1}, g'_{i_n} g'_{i_{n-1}} \cdots g'_{i_1}\} \\
&= \{g_{i_n} g_{i_{n-1}} \cdots g_{i_1}, g_{i_n} g_{i_{n-1}} \cdots g_{i_2}, g_{i_n} g_{i_{n-1}} \cdots g_{i_3}, \ldots, g_{i_n}, \lambda\}. \qquad (5.30)
\end{aligned}$$

Thus, for given a word w, the system is said to be *w-controllable* if there does exist a control sequence $\{u(v)\}$ along the path ${}^w\Gamma$ so that a given state at w can be controlled to the state at λ within finite steps of iteration.

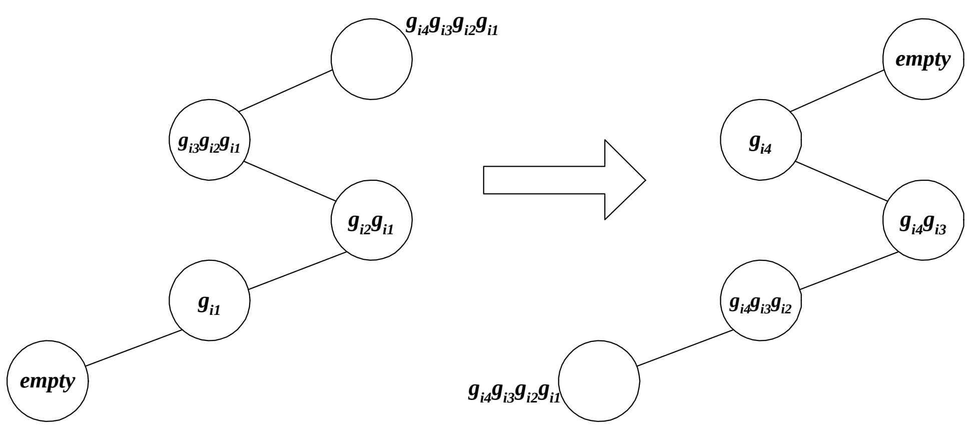

Figure 5.7: A path $\Gamma^{\{g_{i_4} g_{i_3} g_{i_2} g_{i_1}\}}$ and its reverse ${}^{\{g_{i_4} g_{i_3} g_{i_2} g_{i_1}\}}\Gamma$

For analysis purposes, we shall distinguish between the path Γ^w associated with a word w and its reverse ${}^w\Gamma$. We shall call Γ^w the *Future-path* and a homogeneous tree spanned by all possibilities of the Future-path is called the *Future-time*. Thus the homogeneous tree in Figure 5.6 is an example of the Future-time for 2D system in this setting. The *Past-time* is a homogeneous tree spanned by all possibilities of the *Past-path*, ${}^w\Gamma$. To make it more applicable, we connect the Past-time and the Future-time at the root labelled by λ, and we also use λ

as an index to identify whether a word belongs to the Past-time or to the Future-time. This construction leads to the notion of "time axis" for i/s/o linear systems with evolution along a free semigroup $\mathcal{F}_d$. The "time axis" for such systems when $d = 2$ is shown in Figure 5.8.

Definition 29. Let $\mathcal{T}_f \triangleq \{(w,\lambda) \in (\mathcal{F}_d \times \{\lambda\})\}$ and $\mathcal{T}_p \triangleq \{(\lambda, w) \in (\{\lambda\} \times \mathcal{F}_d \setminus \{\lambda\})\}$ denote respectively the homogeneous trees spanned by paths Γ^w and ${}^w\Gamma$ for all words in a free semigroup $\mathcal{F}_d$. Then the "time axis" for noncommutative d-D linear systems is represented by the set $\mathcal{T} = \mathcal{T}_p \cup \mathcal{T}_f \subset \mathcal{F}_d \times \mathcal{F}_d$ equipped with the partial ordering:

- $(w,\lambda) \succ (\lambda, v)$ for all $w \in \mathcal{F}_d, v \in \mathcal{F}_d \setminus \{\lambda\}$,
- $(w,\lambda) \succeq (w',\lambda)$ if and only if $w = \tilde{w}w'$ for some $\tilde{w} \in \mathcal{F}_d$,
- $(\lambda, v) \succeq (\lambda, v')$ if and only if $v' = \tilde{v}v$ for some $\tilde{v} \in \mathcal{F}_d$,

and the sets $\mathcal{T}_p$ and $\mathcal{T}_f$ are called the *Past-time* and the *Future-time*, respectively.

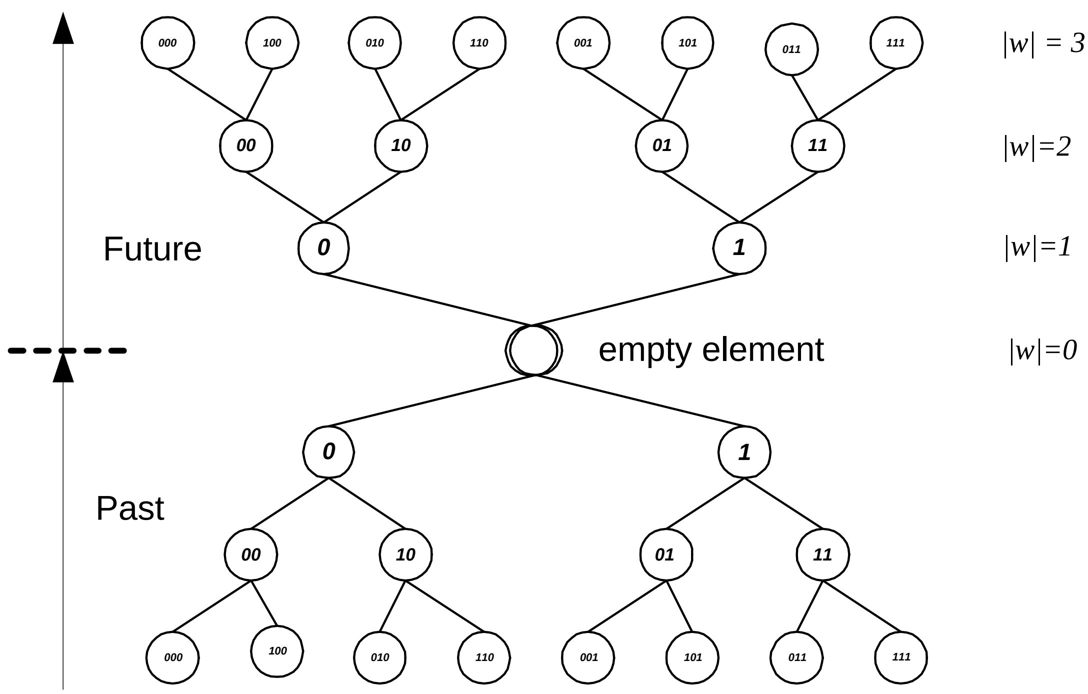

Figure 5.8: Time axis for the noncommutative 2D linear system

By using the concepts of Future and Past defined above, the notion of state in our setting can be stated as in the classical discrete-time 1D linear system: *the state is the summary of the past history of the system required to completely determine the future outputs from only a* *knowledge of future inputs*. Now we are in a position to establish the state-space formalism of an i/s/o linear system where the time-axis is given as in Definition 29. If we assume that the initial conditions $x_k(\lambda, v), k \in \mathcal{I}_d$ where $|v| = n$ are given, then the system update equations must generate the state sequence in such a way that the length of word v is reduced by one in each step of iteration (i.e., solve the system equations in the Past-time). When the states arrive at the time (λ, λ), the system update equations must generate the state sequence at every point $(w, \lambda), w \in \mathcal{F}_d$ such that the length of w is increasing by one in every step of recursive process.

Thus, from the above discussion, it is reasonable to define two sets of system equations: one for the Past-time and the other for the Future-time. For further details on system modelling and system update equations, see Chapter 6.

5.4 Some Noncommutative Systems in the Literature

The system we are dealing with is a multidimensional linear system with "time axis" equal to a homogeneous tree of order d generated by a free semigroup $\mathcal{F}_d$ on d letters. The resulting transfer function $T_\Sigma(z)$ is a formal power series $T_\Sigma(z) = \sum_{w \in \mathcal{F}_d} T_w z^w$ in the noncommuting indeterminants $z = (z_1, \ldots, z_d)$ (see Subsection 5.2.3). This setting should be distinguished from other settings in the system theory literature where free semigroups $\mathcal{F}_d$ or noncommuting power series come up. We mention here a few examples:

1. In automata theory (see [Arb69]), the state space and input space are taken to be a finite set and the time axis is the usual 1-dimensional $\mathbb{Z}_+$. In this context, if we let $\{g_1, \ldots, g_d\}$ be the set of admissible inputs, then the free semigroup arises as the space of possible input signals: a signal up to time n corresponds to a word $w = g_{i_1} \ldots g_{i_n}$ of length n. In our setting the free semigroup $\mathcal{F}_d$ arises as the *time axis* and the space of input signals is $\ell^2(\mathcal{F}_d, \mathcal{U})$. To get a parallel with the automata theory, we could consider the case where the input space $\mathcal{U}$ is a finite set rather than a Hilbert space. Then the space of input signals become functions defined on words (rather then a positive integer) with values in a finite set.

2. In the algebraic theory for time-varying systems presented in [FLR93], noncommutative Laurent series of the form $\sum_{\nu \geq \nu_0} T_\nu s^{-\nu}$ arise. Here the rules are different. There is only one variable or indeterminant s and the noncommutativity arises from the fact that the variable s does not commute with the coefficients T_ν; we are in the setting of a skew-field, or specifically, a differential field. In our setting, $z_1, \ldots, z_d$ do not commute with each other but do commute with the coefficients T_ν. To get an analogue of this skew-field setting in our context, we would have to consider a time-varying version of our system equations where the system matrix depends on the "time", i.e., on the current word w and

the system equations have the form

$$x_j(g_j w) = A_{j,1}(w)x_1(w) + \cdots + A_{j,d}(w)x_d(w) + B_j(w)u(w)$$
$$y(w) = C_1(w)x_1(w) + \cdots + C_d(w)x_d(w) + D(w)u(w).$$

We have no motivation to consider these generalizations at this time.

3. The closely related system to what we are considering here is a a so-called *discrete event system* (see, e.g. [RW89]) defined as follows: *A discrete event system is a dynamical system with a discrete state space and piecewise constant trajectories, where the time axis is the conventional one-dimension* $\mathbb{Z}$. At each time instant t_k at which state transitions

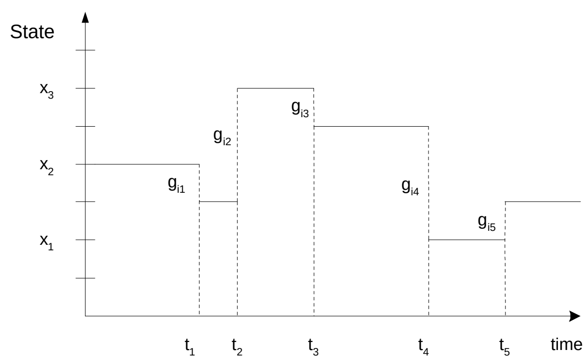

Figure 5.9: State trajectory for a Discrete Event System

occur, they assign an event, say $g_{i_k} \in \mathcal{F}$, the finite set of event labels. Then a system trajectory is defined as a sequential order of the events occurring along the time axis. For instance, if $\mathcal{F} = \{g_1, g_2, g_3\}$, then the state trajectory is one of the form $g_1g_3g_2g_3g_1 \cdots$. From the practical point of view, the state transitions occur randomly throughout the process. Thus one is able to assign the events to the process as long as the process is running. Consequently, the string consists of infinitely many events. It is of interest to consider a *partial trajectory* which contains a finite number of events rather than an infinite one. A typical state trajectory is depicted in Figure 5.9.

4. The subject of systems on a tree has been studied from several perspectives, e.g. Benveniste et al. [BNW94] studied a process of successive operations of filtering-and-decimation in the multiresolution signal processing. They formulated linear models of systems and stochastic processes on the *dyadic tree* and showed that such models provide a natural and powerful setting for multiscale modeling and processing. In addition, they also introduced a special class of transfer functions on a tree called *stationary transfer functions.*

Alpay and Volok [AV] have formulated a notion of time-varying point evaluation of an i/o operator along such a point in connection with the counterpart of the Hardy space for a class of stationary transfer functions defined by [BNW94].

The closely related i/s/o d-D linear systems with evolution along a free semigroup $\mathcal{F}_d$ were studied in [BVa, BVb] in connection with problems in operator theory (representations of the Cuntz algebra, multivariable generalization of Lax-Phillips scattering theory, and model theory for row contractions).

Chapter 6

Noncommutative Multidimensional Linear Models

This Chapter presents the mathematical models of input/state/output (i/s/o) multidimensional linear systems with evolution along a free semigroup (i.e., the time-axis represented by a homogeneous tree as described in Section 5.3). This class of systems may be regarded as a finite automaton in such a way that the word-length of data, say n, is arbitrarily large, but finite. There are two models discussed in this Chapter, namely a noncommutative d-D Fornasini-Marchesini (or, NCFM) and a noncommutative d-D Givone-Roesser (or, NCGR) linear models[1]. In fact, these two models have similar mathematical structures as the Fornasini-Marchesini (FM) and the Givone-Roesser (GR) models, which have already been discussed in Part 1 except that in this Part, we are dealing with a system whose transfer function is a formal power series in several noncommuting indeterminants, and hence we expect to obtain analogous results as those in Part 1.

This Chapter is organized as follows: We first introduce two mathematical models: the NCFM and the NCGR models, and then establish the identification between these models in Section 6.1. Since under suitable conditions, we can identify the NCFM with the NCGR, without loss of generality, we shall focus on the system equations of the NCGR model as well as its adjoint system in Section 6.2; the general solution of such system equations is also presented in this Section. The last Section is devoted to the noncommutative d-variable Z-transform analysis. Application of the Z-transform to the system equations yields the transfer function of several noncommuting indeterminants described by the NCGR model.

[1]Besides the NCFM and the NCGR models, we also have the generalized structured NCGR model with the structured dynamics $Z(z)$ (see Subsection 5.2.3); however, we do not consider this generalization at this time.

6.1 Operator Colligations and State-Space Formalisms

As in the classical discrete-time case, in order to fully understand the behavior of the systems in which we are interested, we need to analyze such systems in both the sequential (time) and the Z-transformed (frequency) domains. In this Section, we shall first formulate two state-space models representing the i/s/o d-D linear systems with evolution along a free semigroup $\mathcal{F}_d$. Such models are called noncommutative d-D Fornasini-Marchesini (NCFM) and noncommutative d-D Givone-Roesser (NCGR) linear models which have parallel mathematical structures as FM and GR models described in Part 1. As in the commutative case, both models are not independent; in fact, when certain conditions are imposed on the NCFM model, one can show that these models are equivalent to each other.

Now let us recall some terminology in operator theory that we shall use throughout this Part (see also Section 3.2 on page 33). A quadruple $\Sigma = (\mathcal{H}, \mathcal{U}, \mathcal{Y}, U)$ is said to be a *d-variable operator colligation* if there exist three Hilbert spaces $\mathcal{H}$ (the state space), $\mathcal{U}$ (the input space), and $\mathcal{Y}$ (the output space) together with a *connecting operator* U given by

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix} \tag{6.1}$$

where $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}), B \in \mathcal{L}(\mathcal{U}, \mathcal{H}), C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$.

In the control theory literature, the operators A, B, C, and D are called the *state operator*, the *input operator*, the *output operator*, and the *feedforward operator*, respectively. The colligation Σ is said to be *contractive*, *isometric*, *coisometric*, or *unitary* if the connecting operator U is respectively *contractive*, *isometric*, *coisometric*, or *unitary.* Associated with any d-variable operator colligation is a d-D linear system. Since our interest is focusing on an infinite automata system dealing with *letters* or *words* where the commutative property does not hold, this leads to the state-space formalism which is a so-called *noncommutative d-D linear model.*

6.1.1 Noncommutative d-D Givone-Roesser (NCGR) Model

The noncommutative d-D Givone-Roesser (NCGR) model has a connecting operator[2] U^{GR} of the form

$$U^{GR} \triangleq \begin{bmatrix} A^{GR} & B^{GR} \\ C^{GR} & D^{GR} \end{bmatrix} = \begin{bmatrix} A^{GR}_{1,1} & \cdots & A^{GR}_{1,d} & B^{GR}_1 \\ \vdots & \ddots & \vdots & \vdots \\ A^{GR}_{d,1} & \cdots & A^{GR}_{d,d} & B^{GR}_d \\ C^{GR}_1 & \cdots & C^{GR}_d & D^{GR} \end{bmatrix} : \begin{bmatrix} \bigoplus_{i=1}^d \mathcal{H}_i \\ \mathcal{U} \end{bmatrix} \mapsto \begin{bmatrix} \bigoplus_{i=1}^d \mathcal{H}_i \\ \mathcal{Y} \end{bmatrix} \tag{6.2}$$

[2]For notational convenience we here use the same superscript ^{GR} as in Chapter 3 since the corresponding operators behave in the similar way.

where $A_{ij}^{GR} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, $B_i^{GR} \in \mathcal{L}(\mathcal{U}, \mathcal{H}_i)$, $C_i^{GR} \in \mathcal{L}(\mathcal{H}_i, \mathcal{Y})$, and $D^{GR} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Here the state space $\mathcal{H}$ is decomposed into a fixed d-fold orthogonal direct-sum $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d$, and we shall call $\mathcal{H}_k$ the k-th *partial state space*.

Associated with the connecting operator U^{GR} is the NCGR model:

Past-time:

$$\Sigma_p^{GR} \triangleq \begin{cases} x_1(\lambda, vg_1^{-1}) & = \sum_{k=1}^d A_{1,k}^{GR} x_k(\lambda, v) + B_1^{GR} u(\lambda, v) \\ \vdots & \quad\vdots \\ x_d(\lambda, vg_d^{-1}) & = \sum_{k=1}^d A_{d,k}^{GR} x_k(\lambda, v) + B_d^{GR} u(\lambda, v) \\ y(\lambda, v) & = \sum_{k=1}^d C_k^{GR} x_k(\lambda, v) + D^{GR} u(\lambda, v) \end{cases} \tag{6.3}$$

Future-time:

$$\Sigma_f^{GR} \triangleq \begin{cases} x_1(g_1 w, \lambda) & = \sum_{k=1}^d A_{1,k}^{GR} x_k(w, \lambda) + B_1^{GR} u(w, \lambda) \\ \vdots & \quad\vdots \\ x_d(g_d w, \lambda) & = \sum_{k=1}^d A_{d,k}^{GR} x_k(w, \lambda) + B_d^{GR} u(w, \lambda) \\ y(w, \lambda) & = \sum_{k=1}^d C_k^{GR} x_k(w, \lambda) + D^{GR} u(w, \lambda). \end{cases} \tag{6.4}$$

For compactness, the systems Σ_p^{GR} and Σ_f^{GR}, respectively can be rewritten as

$$\begin{bmatrix} x(\lambda, vg^{-1}) \\ y(\lambda, v) \end{bmatrix} = U^{GR} \begin{bmatrix} x(\lambda, v) \\ u(\lambda, v) \end{bmatrix} = \begin{bmatrix} A^{GR} & B^{GR} \\ C^{GR} & D^{GR} \end{bmatrix} \begin{bmatrix} x(\lambda, v) \\ u(\lambda, v) \end{bmatrix}, \tag{6.5}$$

and

$$\begin{bmatrix} x(gw, \lambda) \\ y(w, \lambda) \end{bmatrix} = U^{GR} \begin{bmatrix} x(w, \lambda) \\ u(w, \lambda) \end{bmatrix} = \begin{bmatrix} A^{GR} & B^{GR} \\ C^{GR} & D^{GR} \end{bmatrix} \begin{bmatrix} x(w, \lambda) \\ u(w, \lambda) \end{bmatrix}, \tag{6.6}$$

where

$$x(\lambda, vg^{-1}) = \begin{bmatrix} x_1(\lambda, vg_1^{-1}) \\ \vdots \\ x_d(\lambda, vg_d^{-1}) \end{bmatrix}, x(\lambda, v) = \begin{bmatrix} x_1(\lambda, v) \\ \vdots \\ x_d(\lambda, v) \end{bmatrix}, x(gw, \lambda) = \begin{bmatrix} x_1(g_1 w, \lambda) \\ \vdots \\ x_d(g_d w, \lambda) \end{bmatrix}, x(w; \lambda) = \begin{bmatrix} x_1(w, \lambda) \\ \vdots \\ x_d(w, \lambda) \end{bmatrix}$$

Remark 20. In fact, the system equations for the Past-time (6.3) and those for the Future-time (6.4) are derived from the same system. Suppose that we are given a word, say $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$ and use the Future-time system equation (6.6) to update the state $x(v, \lambda)$, and to generate the output sequence $\{y(v, \lambda)\}$ for $v \in \Gamma^w$ (see page 110 for the definition of a path) from a given $x(\lambda, \lambda)$ and $u(v, \lambda)$. We then set $u(\lambda, v') = u(v, \lambda)$, $x(\lambda, v') = x(v, \lambda)$, and $y(\lambda, v') = y(v, \lambda)$, where $v' \in^w \Gamma$, to obtain a trajectory $(u(\lambda, v'), x(\lambda, v'), y(\lambda, v'))$ along the reverse path ${}^w\Gamma$ (see (5.30) on page 111).

One can verify that this trajectory along ${}^{w}\Gamma$ satisfies the Past-time system equations (6.3). In this way, the Past-time system equations (6.3) follow from the Future-time system equations (6.4) via the reverse path change of variable $(v, \lambda) \mapsto (\lambda, v')$ taking Γ^{w} to ${}^{w}\Gamma$.

For example, the state at a word $w = g_1 g_2 g_2$ along the path Γ^w is

$$\begin{aligned} x_1(g_1g_2g_2,\lambda) &= A_{11}x_1(g_2g_2,\lambda) + A_{12}x_2(g_2g_2,\lambda) + B_1u(g_2g_2,\lambda) \\ &= A_{11}x_1(g_2g_2,\lambda) + A_{12}A_{21}x_1(g_2,\lambda) + B_1u(g_2g_2,\lambda) + A_{12}B_2u(g_2,\lambda) \\ &\quad + A_{12}A_{22}B_2u(\lambda,\lambda) + A_{12}A_{22}A_{21}x_1(\lambda,\lambda) + A_{12}A_{22}A_{22}x_2(\lambda,\lambda). \end{aligned}$$

By letting

$$\lambda \mapsto g_1g_2g_2, \quad g_2 \mapsto g_1g_2, \quad g_2g_2 \mapsto g_1, \quad g_1g_2g_2 \mapsto \lambda,$$

and setting $u(\lambda, v') = u(v, \lambda)$, $x(\lambda, v') = x(v, \lambda)$ as above, we have

$$\begin{aligned} x_1(\lambda,\lambda) = A_{11}x_1(\lambda,g_1) &+ A_{12}A_{21}x_1(\lambda,g_1g_2) + B_1u(\lambda,g_1) + A_{12}B_2u(\lambda,g_1g_2) \\ &+ A_{12}A_{22}B_2u(\lambda,g_1g_2g_2) + A_{12}A_{22}A_{21}x_1(\lambda,g_1g_2g_2) + A_{12}A_{22}A_{22}x_2(\lambda,g_1g_2g_2) \end{aligned}$$

which is identical to $x_1(\lambda, \lambda)$ computed by the Past-time system equations. ▲

6.1.2 Noncommutative d-D Fornasini-Marchesini (NCFM) Model

For the case of a noncommutative d-D Fornasini-Marchesini (NCFM) linear model, the connecting operator[3] U^{FM} has the form

$$U^{FM} \triangleq \begin{bmatrix} A^{FM} & B^{FM} \\ C^{FM} & D^{FM} \end{bmatrix} = \begin{bmatrix} A_1^{FM} & B_1^{FM} \\ \vdots & \vdots \\ A_d^{FM} & B_d^{FM} \\ C^{FM} & D^{FM} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \mapsto \begin{bmatrix} \bigoplus_1^d \mathcal{H} \\ \mathcal{Y} \end{bmatrix} \tag{6.7}$$

where $A_j^{FM} \in \mathcal{L}(\mathcal{H}, \oplus_1^d \mathcal{H}), B_j^{FM} \in \mathcal{L}(\mathcal{U}, \oplus_1^d \mathcal{H}), C^{FM} \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, and $D^{FM} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$.

Associated with this connecting operator U^{FM} is the NCFM model:

Past-time:

$$\Sigma_p^{FM} \triangleq \begin{cases} x(\lambda, v) &= \sum_{k=1}^d A_k^{FM} x(\lambda, vg_k) + \sum_{k=1}^d B_k^{FM} u(\lambda, vg_k) \\ y(\lambda, v) &= C^{FM} x(\lambda, v) + D^{FM} u(\lambda, v) \end{cases} \tag{6.8}$$

[3]Note also that the operators A_j, B_j, C and D in this Section behave in the similar way as those in the classical FM model discussed in Chapter 3. Therefore, we shall use the same superscript ^{FM} for NCFM as well.

Future-time:

$$\Sigma_f^{FM} \triangleq \begin{cases} x(g_1 w, \lambda) &= A_1^{FM} x(w,\lambda) + B_1^{FM} u(w,\lambda) \\ \quad\vdots & \quad\vdots \\ x(g_d w, \lambda) &= A_d^{FM} x(w,\lambda) + B_d^{FM} u(w,\lambda) \\ y(w,\lambda) &= C^{FM} x(w,\lambda) + D^{FM} u(w,\lambda) \end{cases} \tag{6.9}$$

For notational convenience, let us write $x(gv,\lambda) = \begin{bmatrix} x(g_1 v,\lambda) \\ \vdots \\ x(g_d v,\lambda) \end{bmatrix}$. Then, the system Σ_f^{FM} becomes:

$$\begin{bmatrix} x(gw,\lambda) \\ y(w,\lambda) \end{bmatrix} = U^{FM} \begin{bmatrix} x(w,\lambda) \\ u(w,\lambda) \end{bmatrix} = \begin{bmatrix} A^{FM} & B^{FM} \\ C^{FM} & D^{FM} \end{bmatrix} \begin{bmatrix} x(w,\lambda) \\ u(w,\lambda) \end{bmatrix}. \tag{6.10}$$

Remark 21. As in the NCGR case, the Past-time system equations (6.8) can be derived as a consequence of the Future-time system equations (6.9) under the reverse-path change of variable: $\Gamma^w \mapsto {}^w\Gamma$. In the NCFM case, when considering all possible paths, one adds the update increments coming in from all d immediate predecessor locations; whereas, in the NCGR case, each state component at a given location in the Past has a unique immediate predecessor which influences its value. ▲

6.1.3 Identification between NCGR and NCFM Models

As in the commutative case discussed in Part 1, we here investigate the connection between the NCGR and the NCFM models. In fact, one will see that the results here are analogous to the ones in Section 3.5. That is, the NCGR can be embedded into the NCFM in a natural way; on the other hand, the dimension of the state-space in general is enlarged when we embed the NCFM model into the NCGR model. However, it is possible to preserve the dimension of the state-space if we impose certain assumptions on the NCFM model.

Embedding NCGR to NCFM

Let us first consider the case when $d = 2$, for simplicity. The system equations for the Past-time 2D NCGR model are given by:

$$\Sigma_p^{GR} = \begin{cases} \begin{bmatrix} x_1(\lambda, v g_1^{-1}) \\ x_2(\lambda, v g_2^{-1}) \end{bmatrix} &= \begin{bmatrix} A_{1,1}^{GR} & A_{1,2}^{GR} \\ A_{2,1}^{GR} & A_{2,2}^{GR} \end{bmatrix} \begin{bmatrix} x_1(\lambda, v) \\ x_2(\lambda, v) \end{bmatrix} + \begin{bmatrix} B_1^{GR} \\ B_2^{GR} \end{bmatrix} u(\lambda, v) \\ y(\lambda, v) &= \begin{bmatrix} C_1^{GR} & C_2^{GR} \end{bmatrix} \begin{bmatrix} x_1(\lambda, v) \\ x_2(\lambda, v) \end{bmatrix} + D^{GR} u(\lambda, v). \end{cases} \tag{6.11}$$

By changing variables, the system (6.11) is equivalent to

$$\Sigma_p^{GR} = \begin{cases} x_1(\lambda, v) & = A_{1,1}^{GR} x_1(\lambda, vg_1) + A_{1,2}^{GR} x_2(\lambda, vg_1) + B_1^{GR} u(\lambda, vg_1) \\ x_2(\lambda, v) & = A_{2,1}^{GR} x_1(\lambda, vg_2) + A_{2,2}^{GR} x_2(\lambda, vg_2) + B_2^{GR} u(\lambda, vg_2) \\ y(\lambda, v) & = C_1^{GR} x_1(\lambda, v) + C_2^{GR} x_2(\lambda, v) + D^{GR} u(\lambda, v). \end{cases} \tag{6.12}$$

Define $x(\lambda, v) := \begin{bmatrix} x_1(\lambda, v) \\ x_2(\lambda, v) \end{bmatrix}$. Then (6.12) becomes:

$$\begin{cases} x(\lambda, v) & = \begin{bmatrix} A_{1,1}^{GR} & A_{1,2}^{GR} \\ 0 & 0 \end{bmatrix} x(\lambda, vg_1) + \begin{bmatrix} 0 & 0 \\ A_{2,1}^{GR} & A_{2,2}^{GR} \end{bmatrix} x(\lambda, vg_2) \\ & + \begin{bmatrix} B_1^{GR} \\ 0 \end{bmatrix} u(\lambda, vg_1) + \begin{bmatrix} 0 \\ B_2^{GR} \end{bmatrix} u(\lambda, vg_2) \\ y(\lambda, v) & = \begin{bmatrix} C_1^{GR} & C_2^{GR} \end{bmatrix} x(\lambda, v) + D^{GR} u(\lambda, v), \end{cases} \tag{6.13}$$

which is exactly the system equations for the Past-time 2D NCFM model (6.8) where we set

$$A_1^{FM} = \begin{bmatrix} A_{1,1}^{GR} & A_{1,2}^{GR} \\ 0 & 0 \end{bmatrix}, \quad A_2^{FM} = \begin{bmatrix} 0 & 0 \\ A_{2,1}^{GR} & A_{2,2}^{GR} \end{bmatrix}, \quad B_1^{FM} = \begin{bmatrix} B_1^{GR} \\ 0 \end{bmatrix}, \quad B_2^{FM} = \begin{bmatrix} 0 \\ B_2^{GR} \end{bmatrix},$$

$$C^{FM} = C^{GR} = \begin{bmatrix} C_1^{GR} & C_2^{GR} \end{bmatrix}, \quad D^{FM} = D^{GR}.$$

For the Future-time, the system equations are described by:

$$\Sigma_f^{GR} = \begin{cases} \begin{bmatrix} x_1(g_1 w, \lambda) \\ x_2(g_2 w, \lambda) \end{bmatrix} & = \begin{bmatrix} A_{1,1}^{GR} & A_{1,2}^{GR} \\ A_{2,1}^{GR} & A_{2,2}^{GR} \end{bmatrix} \begin{bmatrix} x_1(w, \lambda) \\ x_2(w, \lambda) \end{bmatrix} + \begin{bmatrix} B_1^{GR} \\ B_2^{GR} \end{bmatrix} u(w, \lambda) \\ y(w, \lambda) & = \begin{bmatrix} C_1^{GR} & C_2^{GR} \end{bmatrix} \begin{bmatrix} x_1(w, \lambda) \\ x_2(w, \lambda) \end{bmatrix} + D^{GR} u(w, \lambda). \end{cases} \tag{6.14}$$

Let us define $x(w, \lambda) := \begin{bmatrix} x_1(w, \lambda) \\ x_2(w, \lambda) \end{bmatrix}$. Then Σ_f^{GR} in (6.14) becomes:

$$x(g_1 w, \lambda) = \begin{bmatrix} x_1(g_1 w, \lambda) \\ x_2(g_1 w, \lambda) \end{bmatrix} = \begin{bmatrix} x_1(g_1 w, \lambda) \\ 0 \end{bmatrix} = \begin{bmatrix} A_{1,1}^{GR} & A_{1,2}^{GR} \\ 0 & 0 \end{bmatrix} x(w, \lambda) + \begin{bmatrix} B_1^{GR} \\ 0 \end{bmatrix} u(w, \lambda), \tag{6.15}$$

$$x(g_2 w, \lambda) = \begin{bmatrix} x_1(g_2 w, \lambda) \\ x_2(g_2 w, \lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ x_2(g_2 w, \lambda) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ A_{2,1}^{GR} & A_{2,2}^{GR} \end{bmatrix} x(w, \lambda) + \begin{bmatrix} 0 \\ B_2^{GR} \end{bmatrix} u(w, \lambda), \tag{6.16}$$

and the output equation is

$$y(w,\lambda) = \begin{bmatrix} C_1^{GR} & C_2^{GR} \end{bmatrix} x(w,\lambda) + D^{GR} u(w,\lambda). \tag{6.17}$$

Thus, we arrive at the Future-time 2D NCFM model where we set A_i^{FM}, B_i^{FM} $(i = 1, 2)$, C^{FM}, and D^{FM} as in the Past-time case.

This embedding can be generalized to the case when $d > 2$ by introducing the orthogonal projection $P_k : \mathcal{H} \mapsto \mathcal{H}$ with image equal to $\mathcal{H}_k$ identified as a subspace of $\mathcal{H}$ for $k = 1, \ldots, d$. By setting $x(\lambda, v) := \begin{bmatrix} x_1(\lambda,v) \\ \vdots \\ x_d(\lambda,v) \end{bmatrix}$, and $x(w,\lambda) := \begin{bmatrix} x_1(w,\lambda) \\ \vdots \\ x_d(w,\lambda) \end{bmatrix}$, we have

$$\begin{aligned} x(\lambda, v) &= \sum_{k=1}^{d} P_k A^{GR} x(\lambda, v g_k) + \sum_{k=1}^{d} P_k B^{GR} u(\lambda, v g_k), \\ y(\lambda, v) &= C^{GR} x(\lambda, v) + D^{GR} u(\lambda, v), \\ x(g_k w, \lambda) &= P_k A^{GR} x(w,\lambda) + P_k B^{GR} u(w,\lambda) \quad \text{for } k \in \mathcal{I}_d, \\ y(w,\lambda) &= C^{GR} x(w,\lambda) + D^{GR} u(w,\lambda). \end{aligned}$$

Finally, we let

$$A_k^{FM} = P_k A^{GR}, \quad B_k^{FM} = P_k B^{GR} \text{ for } k = 1, \ldots, d, \quad C^{FM} = C^{GR}, \quad D^{FM} = D^{GR},$$

and hence we arrive at the general d-D NCFM formalism as required.

Embedding NCFM to NCGR

To embed the NCFM model into the NCGR formalism, we need to construct the Hilbert spaces $\mathcal{H}_k$ for $k = 1, \ldots, d$, such that the direct sum (not necessarily orthogonal) $\dot{+}_{k=1}^{d} \mathcal{H}_k = \mathcal{H}$. To do so, let us assume that

$$\operatorname{im} \begin{bmatrix} A_j^{FM} & B_j^{FM} \end{bmatrix} \cap \operatorname{im} \begin{bmatrix} A_k^{FM} & B_k^{FM} \end{bmatrix} = \{0\}, \quad k \neq j \tag{6.18}$$

and define $\mathcal{H}_k$ so that $\operatorname{im} \begin{bmatrix} A_k^{FM} & B_k^{FM} \end{bmatrix} \subset \mathcal{H}_k$. Set

$$\begin{aligned} A_{i,j}^{GR} &= P_i A_i^{FM} \big|_{\mathcal{H}_j} : \mathcal{H}_j \mapsto \mathcal{H}_i, \qquad & B_i^{GR} &= P_i B_i^{FM} : \mathcal{U} \mapsto \mathcal{H}_i, \\ C_j^{GR} &= C^{FM} \big|_{\mathcal{H}_j} : \mathcal{H}_j \mapsto \mathcal{Y}, \qquad & D^{GR} &= D^{FM} : \mathcal{U} \mapsto \mathcal{Y}. \end{aligned}$$

Thus, for each $k = 1, \ldots, d$,

$$\begin{aligned} P_k x(\lambda, v) = x_k(\lambda, v) &= P_k \left[\sum_{\ell=1}^{d} A_\ell^{FM} x(\lambda, vg_\ell) + \sum_{\ell=1}^{d} B_\ell^{FM} u(\lambda, vg_\ell) \right] \\ &= P_k A_k^{FM} x(\lambda, vg_k) + P_k B_k^{FM} u(\lambda, vg_k) \\ &= \sum_{j=1}^{d} P_k A_k^{FM}\big|_{\mathcal{H}_j} x_j(\lambda, vg_k) + P_k B_k^{FM} u(\lambda, vg_k) \\ &= \sum_{j=1}^{d} A_{k,j}^{GR} x_j(\lambda, vg_k) + B_k^{GR} u(\lambda, vg_k), \end{aligned} \tag{6.19}$$

and also,

$$\begin{aligned} P_k x(g_k w, \lambda) = x_k(g_k w, \lambda) &= P_k \left[A_k^{FM} x(w, \lambda) + B_k^{FM} u(w, \lambda) \right] \\ &= \sum_{j=1}^{d} P_k A_k^{FM}\big|_{\mathcal{H}_j} x_j(w, \lambda) + P_k B_k^{FM} u(w, \lambda) \\ &= \sum_{j=1}^{d} A_{k,j}^{GR} x_j(w, \lambda) + B_k^{GR} u(w, \lambda). \end{aligned} \tag{6.20}$$

Clearly, the state equations (6.19) and (6.20) are the k-th state equation of the Past-time (6.3) and of the Future-time (6.4) NCGR models, respectively.

For the output equations, we have

$$\begin{aligned} y(\lambda, v) &= C^{FM} x(\lambda, v) + D^{FM} u(\lambda, v) \\ &= \sum_{j=1}^{d} C^{FM}\big|_{\mathcal{H}_j} x_j(\lambda, v) + D^{FM} u(\lambda, v) \\ &= \sum_{j=1}^{d} C_j^{GR} x_j(\lambda, v) + D^{GR} u(\lambda, v), \end{aligned} \tag{6.21}$$

and

$$\begin{aligned} y(w, \lambda) &= C^{FM} x(w, \lambda) + D^{FM} u(w, \lambda) \\ &= \sum_{j=1}^{d} C^{FM}\big|_{\mathcal{H}_j} x_j(w, \lambda) + D^{FM} u(w, \lambda) \\ &= \sum_{j=1}^{d} C_j^{GR} x_j(w, \lambda) + D^{GR} u(w, \lambda), \end{aligned} \tag{6.22}$$

Thus, these form the output equations for the Past-time and for the Future-time of the NCGR linear models as required.

6.2 The Adjoint System and the General Response of the NCGR Systems

In this Section, we shall focus on the system equations of the i/s/o d-D linear system described by the NCGR model. We then construct the adjoint system, and investigate the general response of the NCGR linear model. The results from this Section can be obtained in the similar way for the i/s/o system described by the NCFM model since one can identify the NCGR model with the NCFM model under suitable conditions as we have already discussed in Section 6.1.3. Since the system in which we are interested now is described in the form of the NCGR model, we shall drop the superscript ^{GR} for notational convenience.

6.2.1 The Adjoint Systems

Recall that the NCGR linear model is described by:

Past-time:

$$\begin{bmatrix} x_1(\lambda, vg_1^{-1}) \\ \vdots \\ x_d(\lambda, vg_d^{-1}) \\ y(\lambda, v) \end{bmatrix} = U \begin{bmatrix} x_1(\lambda, v) \\ \vdots \\ x_d(\lambda, v) \\ u(\lambda, v) \end{bmatrix} = \begin{bmatrix} A_{1,1} & \cdots & A_{1,d} & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{d,1} & \cdots & A_{d,d} & B_d \\ C_1 & \cdots & C_d & D \end{bmatrix} \begin{bmatrix} x_1(\lambda, v) \\ \vdots \\ x_d(\lambda, v) \\ u(\lambda, v) \end{bmatrix}, \tag{6.23}$$

Future-time:

$$\begin{bmatrix} x_1(g_1 w, \lambda) \\ \vdots \\ x_d(g_d w, \lambda) \\ y(w, \lambda) \end{bmatrix} = U \begin{bmatrix} x_1(w, \lambda) \\ \vdots \\ x_d(w, \lambda) \\ u(w, \lambda) \end{bmatrix} = \begin{bmatrix} A_{1,1} & \cdots & A_{1,d} & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{d,1} & \cdots & A_{d,d} & B_d \\ C_1 & \cdots & C_d & D \end{bmatrix} \begin{bmatrix} x_1(w, \lambda) \\ \vdots \\ x_d(w, \lambda) \\ u(\lambda, v) \end{bmatrix}. \tag{6.24}$$

Let us consider the Past-time system first. The adjoint system Σ_p^* is a system in which its trajectories (u_*, x_*, y_*) are characterized as those $(\mathcal{U}^* \times \mathcal{H} \times \mathcal{Y}^*)$-valued functions on $\mathcal{F}_d$ satisfying the adjoint pairing relation:

$$\begin{aligned} &\langle x_1(\lambda, vg_1^{-1}), x_{*1}(\lambda, vg_1^{-1})\rangle + \cdots + \langle x_d(\lambda, vg_d^{-1}), x_{*d}(\lambda, vg_d^{-1})\rangle + \langle y(\lambda, v), u_*(\lambda, v)\rangle \\ &\qquad = \langle x_1(\lambda, v), x_{*1}(\lambda, v)\rangle + \cdots + \langle x_d(\lambda, v), x_{*d}(\lambda, v)\rangle + \langle u(\lambda, v), y_*(\lambda, v)\rangle \end{aligned} \tag{6.25}$$

which is equivalent to

$$\left\langle \begin{bmatrix} x_1(\lambda, vg_1^{-1}) \\ \vdots \\ x_d(\lambda, vg_d^{-1}) \\ y(\lambda, v) \end{bmatrix}, \begin{bmatrix} x_{*1}(\lambda, vg_1^{-1}) \\ \vdots \\ x_{*d}(\lambda, vg_d^{-1}) \\ u_*(\lambda, v) \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x_1(\lambda, v) \\ \vdots \\ x_d(\lambda, v) \\ u(\lambda, v) \end{bmatrix}, \begin{bmatrix} x_{*1}(\lambda, v) \\ \vdots \\ x_{*d}(\lambda, v) \\ y_*(\lambda, v) \end{bmatrix} \right\rangle. \tag{6.26}$$

By substituting (6.23) into (6.26), we have

$$\left\langle \begin{bmatrix} x_1(\lambda, v) \\ \vdots \\ x_d(\lambda, v) \\ u(\lambda, v) \end{bmatrix}, U^* \begin{bmatrix} x_{*1}(\lambda, vg_1^{-1}) \\ \vdots \\ x_{*d}(\lambda, vg_d^{-1}) \\ u_*(\lambda, v) \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x_1(\lambda, v) \\ \vdots \\ x_d(\lambda, v) \\ u(\lambda, v) \end{bmatrix}, \begin{bmatrix} x_{*1}(\lambda, v) \\ \vdots \\ x_{*d}(\lambda, v) \\ y_*(\lambda, v) \end{bmatrix} \right\rangle \tag{6.27}$$

in which we deduce that trajectories (u_*, x_*, y_*) for the adjoint system Σ_p^* are characterized by

$$\begin{bmatrix} x_{*1}(\lambda, v) \\ \vdots \\ x_{*d}(\lambda, v) \\ y_*(\lambda, v) \end{bmatrix} = U^* \begin{bmatrix} x_{*1}(\lambda, vg_1^{-1}) \\ \vdots \\ x_{*d}(\lambda, vg_d^{-1}) \\ u_*(\lambda, v) \end{bmatrix}. \tag{6.28}$$

Thus, the equations of the Past-time adjoint system Σ_p^* are

$$\Sigma_p^* \triangleq \begin{cases} x_{*1}(\lambda, v) &= \sum_{j=1}^d (A_{j,1})^* x_{*j}(\lambda, vg_j^{-1}) + (C_1)^* u_*(\lambda, v) \\ & \vdots \\ x_{*d}(\lambda, v) &= \sum_{j=1}^d (A_{j,d})^* x_{*j}(\lambda, vg_j^{-1}) + (C_d)^* u_*(\lambda, v) \\ y_*(\lambda, v) &= \sum_{j=1}^d (B_j)^* x_{*j}(\lambda, vg_j^{-1}) + (D)^* u_*(\lambda, v). \end{cases} \tag{6.29}$$

The equations of the Future-time adjoint system Σ_f^* can be constructed in the similar way and are given by

$$\Sigma_f^* \triangleq \begin{cases} x_{*1}(w, \lambda) &= \sum_{j=1}^d (A_{j,1})^* x_{*j}(g_j w, \lambda) + (C_1)^* u_*(w, \lambda) \\ & \vdots \\ x_{*d}(w, \lambda) &= \sum_{j=1}^d (A_{j,d})^* x_{*j}(g_j w, \lambda) + (C_d)^* u_*(w, \lambda) \\ y_*(w, \lambda) &= \sum_{j=1}^d (B_j)^* x_{*j}(g_j w, \lambda) + (D)^* u_*(w, \lambda), \end{cases} \tag{6.30}$$

or in the vector-matrix form

$$\begin{bmatrix} x_{*1}(w,\lambda) \\ \vdots \\ x_{*d}(w,\lambda) \\ y_*(w,\lambda) \end{bmatrix} = U^* \begin{bmatrix} x_{*1}(g_1 w,\lambda) \\ \vdots \\ x_{*d}(g_d w,\lambda) \\ u_*(w,\lambda) \end{bmatrix}, \tag{6.31}$$

where the adjoint pairing relation in this case is given by

$$\begin{aligned} &\langle x_1(g_1 w,\lambda), x_{*1}(g_1 w,\lambda)\rangle + \cdots + \langle x_d(g_d w,\lambda), x_{*d}(g_d w,\lambda)\rangle + \langle y(w,\lambda), u_*(w,\lambda)\rangle \\ &\quad = \langle x_1(w,\lambda), x_{*1}(w,\lambda)\rangle + \cdots + \langle x_d(w,\lambda), x_{*d}(w,\lambda)\rangle + \langle u(w,\lambda), y_*(w,\lambda)\rangle. \end{aligned} \tag{6.32}$$

An i/s/o linear system is said to be *conservative* provided that a $(\mathcal{U},\mathcal{H},\mathcal{Y})$-valued function (u,x,y) is a trajectory of such a system if and only if (y,x,u) is a trajectory of its adjoint system. Now we shall verify that the system is conservative if and only if the connecting operator U is unitary. To see this, let us first consider the Past-time system equations (6.23) and substitute (y,x,u) by (u_*,x_*,y_*). Thus, we have

$$\begin{bmatrix} x_{*1}(\lambda, vg_1^{-1}) \\ \vdots \\ x_{*d}(\lambda, vg_d^{-1}) \\ u_*(\lambda,v) \end{bmatrix} = U \begin{bmatrix} x_{*1}(\lambda,v) \\ \vdots \\ x_{*d}(\lambda,v) \\ y_*(\lambda,v) \end{bmatrix} = UU^* \begin{bmatrix} x_{*1}(\lambda, vg_1^{-1}) \\ \vdots \\ x_{*d}(\lambda, vg_d^{-1}) \\ u_*(\lambda,v) \end{bmatrix}. \tag{6.33}$$

Also, from (6.28),

$$\begin{bmatrix} x_1(\lambda,v) \\ \vdots \\ x_d(\lambda,v) \\ u(\lambda,v) \end{bmatrix} = U^* \begin{bmatrix} x_1(\lambda, vg_1^{-1}) \\ \vdots \\ x_d(\lambda, vg_d^{-1}) \\ y(\lambda,v) \end{bmatrix} = U^*U \begin{bmatrix} x_1(\lambda,v) \\ \vdots \\ x_d(\lambda,v) \\ u(\lambda,v) \end{bmatrix}. \tag{6.34}$$

Similarly, for the Future-time system equations (6.24),

$$\begin{bmatrix} x_{*1}(g_1 w,\lambda) \\ \vdots \\ x_{*d}(g_d w,\lambda) \\ u_*(w,\lambda) \end{bmatrix} = U \begin{bmatrix} x_{*1}(w,\lambda) \\ \vdots \\ x_{*d}(w,\lambda) \\ y_*(w,\lambda) \end{bmatrix} = UU^* \begin{bmatrix} x_{*1}(g_1 w,\lambda) \\ \vdots \\ x_{*d}(g_d w,\lambda) \\ u_*(w,\lambda) \end{bmatrix}, \tag{6.35}$$

and from (6.31),

$$\begin{bmatrix} x_1(w,\lambda) \\ \vdots \\ x_d(w,\lambda) \\ u(w,\lambda) \end{bmatrix} = U^* \begin{bmatrix} x_1(g_1 w,\lambda) \\ \vdots \\ x_d(g_d w,\lambda) \\ y(w,\lambda) \end{bmatrix} = U^*U \begin{bmatrix} x_1(\lambda,v) \\ \vdots \\ x_d(\lambda,v) \\ u(\lambda,v) \end{bmatrix}. \tag{6.36}$$

From the above relations, it is clear that the connecting operator U is unitary since $UU^* = U^*U = I$. If an i/s/o d-D linear system is conservative (i.e. U is unitary), then the adjoint pairings (6.25) and (6.32) collapse to the energy balance relations:

$$\sum_{k=1}^{d} \|x_k(\lambda, vg_k^{-1})\|^2 + \|y(\lambda,v)\|^2 = \sum_{k=1}^{d} \|x_k(\lambda,v)\|^2 + \|u(\lambda,v)\|^2 \quad \text{for } \Sigma_p, \tag{6.37}$$

$$\sum_{k=1}^{d} \|x_{*k}(\lambda, vg_k^{-1})\|^2 + \|u_*(\lambda,v)\|^2 = \sum_{k=1}^{d} \|x_{*k}(\lambda,v)\|^2 + \|y_*(\lambda,v)\|^2 \quad \text{for } \Sigma_p^*, \tag{6.38}$$

$$\sum_{k=1}^{d} \|x_k(g_k w,\lambda)\|^2 + \|y(w,\lambda)\|^2 = \sum_{k=1}^{d} \|x_k(w,\lambda)\|^2 + \|u(w,\lambda)\|^2 \quad \text{for } \Sigma_f, \tag{6.39}$$

$$\sum_{k=1}^{d} \|x_{*k}(g_k w,\lambda)\|^2 + \|u_*(w,\lambda)\|^2 = \sum_{k=1}^{d} \|x_{*k}(w,\lambda)\|^2 + \|y_*(w,\lambda)\|^2 \quad \text{for } \Sigma_f^*. \tag{6.40}$$

Moreover, the equations of the Past-time adjoint system Σ_p^* (6.29), and those of the Future-time adjoint system Σ_f^* (6.30) become:

$$\Sigma_{p,b}^* \triangleq \begin{cases} x_1(\lambda,v) & = \sum_{j=1}^{d} (A_{j,1})^* x_j(\lambda, vg_j^{-1}) + (C_1)^* y(\lambda,v) \\ & \vdots \\ x_d(\lambda,v) & = \sum_{j=1}^{d} (A_{j,d})^* x_j(\lambda, vg_j^{-1}) + (C_d)^* y(\lambda,v) \\ u(\lambda,v) & = \sum_{j=1}^{d} (B_j)^* x_j(\lambda, vg_j^{-1}) + (D)^* y(\lambda,v), \end{cases} \tag{6.41}$$

$$\Sigma_{f,b}^* \triangleq \begin{cases} x_1(w,\lambda) & = \sum_{j=1}^{d} (A_{j,1})^* x_j(g_j w,\lambda) + (C_1)^* y(w,\lambda) \\ & \vdots \\ x_d(w,\lambda) & = \sum_{j=1}^{d} (A_{j,d})^* x_j(g_j w\lambda) + (C_d)^* y(w,\lambda) \\ u(w,\lambda) & = \sum_{j=1}^{d} (B_j)^* x_j(g_j w\lambda) + (D)^* y(w,\lambda). \end{cases} \tag{6.42}$$

We shall call (6.41) (resp., (6.42)) the backward-time system equations for the Past-time (resp., the Future-time) NCGR linear model.

6.2.2 The General Response of the NCGR Systems

In this Section, we are seeking the general response of the i/s/o d-D linear system described by the NCGR model. Suppose now that we are given the initial conditions $x_k(\lambda, v)$, $k \in \mathcal{I}_d$ and for all v such that $|v| = n$, say $v = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$. Then one can solve the system equations

$$\Sigma_p^{GR} \triangleq \begin{cases} x_1(\lambda, v g_1^{-1}) & = \sum_{k=1}^d A_{1,k} x_k(\lambda, v) + B_1 u(\lambda, v) \\ \quad \vdots & \qquad \vdots \\ x_d(\lambda, v g_d^{-1}) & = \sum_{k=1}^d A_{d,k} x_k(\lambda, v) + B_d u(\lambda, v) \\ \quad y(\lambda, v) & = \sum_{k=1}^d C_k x_k(\lambda, v) + D u(\lambda, v) \end{cases} \tag{6.43}$$

recursively as follows:

$$\begin{aligned} x_{i_1}(\lambda, g_{i_n} \cdots g_{i_2}) &= \sum_{k=1}^d A_{i_1,k} x_k(\lambda, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}) + B_{i_1} u(\lambda, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}) \\ x_{i_2}(\lambda, g_{i_n} \cdots g_{i_3}) &= \sum_{i_1,k=1}^d A_{i_2,i_1} A_{i_1,k} x_k(\lambda, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}) \\ &\quad + \sum_{i_1=1}^d A_{i_2,i_1} B_{i_1} u(\lambda, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}) + B_{i_2} u(\lambda, g_{i_n} \cdots g_{i_2}) \\ x_{i_3}(\lambda, g_{i_n} \cdots g_{i_4}) &= \sum_{i_2,i_1,k=1}^d A_{i_3,i_2} A_{i_2,i_1} A_{i_1,k} x_k(\lambda, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}) \\ &\quad + \sum_{i_2,i_1=1}^d A_{i_3,i_2} A_{i_2,i_1} B_{i_1} u(\lambda, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}) \\ &\quad + \sum_{i_2=1}^d A_{i_3,i_2} B_{i_2} u(\lambda, g_{i_n} \cdots g_{i_2}) + B_{i_3} u(\lambda, g_{i_n} \cdots g_{i_3}) \\ &\ \ \vdots \\ x_{i_n}(\lambda, \lambda) &= \sum_{i_{n-1},\ldots,i_1,k=1}^d A_{i_n,i_{n-1}} \cdots A_{i_2,i_1} A_{i_1,k} x_k(\lambda, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}) \\ &\quad + \sum_{i_{n-1},\ldots,i_1=1}^d A_{i_n,i_{n-1}} \cdots A_{i_2,i_1} B_{i_1} u(\lambda, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}) \\ &\quad + \cdots + \sum_{i_{n-1}=1}^d A_{i_n,i_{n-1}} B_{i_{n-1}} u(\lambda, g_{i_n} g_{i_{n-1}}) + B_{i_n} u(\lambda, g_{i_n}). \end{aligned} \tag{6.44}$$

To condense notation, let us embed the NCGR system matrices A and B into the NCFM system matrices $\underline{A}$ and $\underline{B}$ respectively by using the orthogonal projection $P_k : \mathcal{H} \mapsto \mathcal{H}$ from $\mathcal{H}$ onto $\mathcal{H}_k$ identified as a subspace of $\mathcal{H}$ for $k = 1, \dots, d$ as before. We then set

$$\underline{A}_k = P_k A, \quad \underline{B}_k = P_k B \tag{6.45}$$

i.e.,

$$\underline{A}_k = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ A_{k,1} & \cdots & A_{k,d} \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \underline{B}_k = \begin{bmatrix} 0 \\ \vdots \\ B_k \\ \vdots \\ 0 \end{bmatrix}.$$

Then by simple algebra, one can verify that the general response (6.44) collapses to

$$x(\lambda, \lambda) = \sum_{v:|v|=n} \underline{A}^v x(\lambda, v) + \sum_{k=1}^{d} \sum_{v:|v|<n} \underline{A}^v \underline{B}_k u(\lambda, v g_k), \tag{6.46}$$

where $\underline{A}^v = \underline{A}_{i_n} \underline{A}_{i_{n-1}} \cdots \underline{A}_{i_1}$ if $v = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$.

Example 12. Suppose that we are given all initial conditions $x_k(\lambda, v)$ for $k = 1, 2$ and $v \in \mathcal{F}_2$ such that $|v| = 3$. Then, for $i_3 \in \{1, 2\}$, we have

$$\begin{aligned} x_{i_3}(\lambda, \lambda) = & \sum_{i_2, i_1, k=1}^{2} A_{i_3,i_2} A_{i_2,i_1} A_{i_1,k} x_k(\lambda, g_{i_3} g_{i_2} g_{i_1}) + \sum_{i_2, i_1=1}^{2} A_{i_3,i_2} A_{i_2,i_1} B_{i_1} u(\lambda, g_{i_3} g_{i_2} g_{i_1}) \\ & + \sum_{i_2=1}^{2} A_{i_3,i_2} B_{i_2} u(\lambda, g_{i_3} g_{i_2}) + B_{i_3} u(\lambda, g_{i_3}) \\ = & A_{i_3,1} A_{11}^2 x_1(\lambda, g_{i_3} g_1 g_1) + A_{i_3,1} A_{11} A_{12} x_2(\lambda, g_{i_3} g_1 g_1) + A_{i_3,1} A_{12} A_{21} x_1(\lambda, g_{i_3} g_1 g_2) \\ & + A_{i_3,1} A_{12} A_{22} x_2(\lambda, g_{i_3} g_1 g_2) + A_{i_3,2} A_{21} A_{11} x_1(\lambda, g_{i_3} g_2 g_1) + A_{i_3,2} A_{21} A_{12} x_2(\lambda, g_{i_3} g_2 g_1) \\ & + A_{i_3,2} A_{22} A_{21} x_1(\lambda, g_{i_3} g_2 g_2) + A_{i_3,2} A_{22}^2 x_2(\lambda, g_{i_3} g_2 g_2) + A_{i_3,1} A_{11} B_1 u(\lambda, g_{i_3} g_1 g_1) \\ & + A_{i_3,1} A_{12} B_2 u(\lambda, g_{i_3} g_1 g_2) + A_{i_3,2} A_{21} B_1 u(\lambda, g_{i_3} g_2 g_1) + A_{i_3,2} A_{22} B_2 u(\lambda, g_{i_3} g_2 g_2) \\ & + A_{i_3,1} B_1 u(\lambda, g_{i_3} g_1) + A_{i_3,2} B_2 u(\lambda, g_{i_3} g_2) + B_{i_3} u(\lambda, g_{i_3}). \qquad \diamondsuit \end{aligned}$$

Suppose we now arrive at the state $x(\lambda, \lambda)$ as shown in (6.46). Then one can continue the recursive process using the Future-time system update equations (see (6.4) on page 118) to get the general response for any w, say $w = g_{j_m} g_{j_{m-1}} \cdots g_{j_1}$ in the Future-time. From (6.4), we have

$$x_{j_1}(g_{j_1}, \lambda) = \sum_{i_n=1}^{d} A_{j_1,i_n} x_{i_n}(\lambda, \lambda) + B_{j_1} u(\lambda, \lambda)$$

$$\begin{aligned}
x_{j_2}(g_{j_2}g_{j_1},\lambda) &= \sum_{\substack{k=1\\k\neq j_1}}^{d} A_{j_2,k}x_k(g_{j_1},\lambda) + A_{j_2,j_1}\sum_{i_n=1}^{d} A_{j_1,i_n}x_{i_n}(\lambda,\lambda) + A_{j_2,j_1}B_{j_1}u(\lambda,\lambda) + B_{j_2}u(g_{j_1},\lambda)\\
x_{j_3}(g_{j_3}g_{j_2}g_{j_1},\lambda) &= \sum_{\substack{k=1\\k\neq j_2}}^{d} A_{j_3,k}x_k(g_{j_2}g_{j_1},\lambda) + A_{j_3,j_2}\sum_{\substack{k=1\\k\neq j_1}}^{d} A_{j_2,k}x_k(g_{j_1},\lambda)\\
&\quad + A_{j_3,j_2}A_{j_2,j_1}\sum_{i_n=1}^{d} A_{j_1,i_n}x_{i_n}(\lambda,\lambda) + A_{j_3,j_2}A_{j_2,j_1}B_{j_1}u(\lambda,\lambda)\\
&\quad + A_{j_3,j_2}B_{j_2}u(g_{j_1},\lambda) + B_{j_3}u(g_{j_2}g_{j_1},\lambda)\\
&\ \vdots\\
x_j(g_jw,\lambda) &= IC_j + BC_j + B_ju(g_{j_m}g_{j_{m-1}}\cdots g_{j_1},\lambda) + A_{j,j_m}B_{j_m}u(g_{j_{m-1}}\cdots g_{j_1},\lambda)\\
&\quad + A_{j,j_m}A_{j_m,j_{m-1}}B_{j_{m-1}}u(g_{j_{m-2}}\cdots g_{j_1},\lambda) + \cdots\\
&\quad + A_{j,j_m}A_{j_m,j_{m-1}}\cdots A_{j_3,j_2}B_{j_2}u(g_{j_1},\lambda)\\
&\quad + A_{j,j_m}A_{j_m,j_{m-1}}\cdots A_{j_3,j_2}A_{j_2,j_1}B_{j_1}u(\lambda,\lambda) \qquad (6.47)
\end{aligned}$$

where

$$IC_j = A_{j,j_m}A_{j_m,j_{m-1}}\cdots A_{j_3,j_2}A_{j_2,j_1}\sum_{i_n=1}^{d} A_{j_1,i_n}x_{i_n}(\lambda,\lambda)$$

and
$$\begin{aligned}
BC_j &= \sum_{\substack{k=1\\k\neq j_m}}^{d} A_{j,k}x_k(g_{j_m}\cdots g_{j_1},\lambda) + A_{j,j_m}\sum_{\substack{k=1\\k\neq j_{m-1}}}^{d} A_{j_m,k}x_k(g_{j_{m-1}}\cdots g_{j_1},\lambda)\\
&\quad + A_{j,j_m}A_{j_m,j_{m-1}}\sum_{\substack{k=1\\k\neq j_{m-2}}}^{d} A_{j_{m-1},k}x_k(g_{j_{m-2}}\cdots g_{j_1},\lambda) + \cdots\\
&\quad + A_{j,j_m}A_{j_m,j_{m-1}}\cdots A_{j_3,j_2}\sum_{\substack{k=1\\k\neq j_1}}^{d} A_{j_2,k}x_k(g_{j_1},\lambda).
\end{aligned}$$

For convenience, we assume that the boundary conditions $BC_j = 0$. In fact, this assumption is reasonable since upon this assumption, there exists the one-to-one correspondence between the particular word and the state. In other words, suppose the word is given by $w = g_{j_m}g_{j_{m-1}}\cdots g_{j_1} \neq \lambda$, then the state corresponding to this particular word w is $x_{j_m}(w,\lambda)$ where $x_j(w,\lambda) = 0$ for all $j \neq j_m \in \mathcal{I}_d$.

By embedding the system matrix A of the NCGR model into the NCFM system matrix $\underline{A}$ as before, one can check that the general response (6.47) is

$$x(gw,\lambda) - \sum_{k=1}^{d} \underline{A}^{g_k w} x(\lambda,\lambda)$$
$$= \begin{bmatrix} B_1 & A_{1,j_m}B_{j_m} & \cdots & A_{1,j_m}\cdots A_{j_2,j_1}B_{j_1} \\ \vdots & \vdots & & \vdots \\ B_d & A_{d,j_m}B_{j_m} & \cdots & A_{d,j_m}\cdots A_{j_2,j_1}B_{j_1} \end{bmatrix} \cdot \begin{bmatrix} u(g_{j_m}g_{j_{m-1}}\cdots g_{j_1},\lambda) \\ u(g_{j_{m-1}}\cdots g_{j_1},\lambda) \\ \vdots \\ u(g_{j_1},\lambda) \\ u(\lambda,\lambda) \end{bmatrix}. \quad (6.48)$$

6.3 Frequency domain analysis

To maintain an analogue with the commutative case as described in Part 1, it is more convenient to introduce the *noncommutative d-variable Z-transform* which considerably simplifies the analysis of the i/s/o d-D linear systems with evolution along a free semigroup. For convenience, we shall refer to the noncommutative d-variable Z-transform as the *Z-transform*.

For any Hilbert space $\mathcal{H}$, we denote by $\ell^2((\mathcal{F}_d \times \mathcal{F}_d),\mathcal{H})$ the set of all $\mathcal{H}$-valued function $(w,v) \mapsto f(w,v)$ on $\mathcal{F}_d \times \mathcal{F}_d$ where

$$\|f\|^2_{\ell^2((\mathcal{F}_d\times\mathcal{F}_d),\mathcal{H})} \triangleq \sum_{(w,v)\in\mathcal{F}_d\times\mathcal{F}_d} \|f(w,v)\|^2_{\mathcal{H}} < \infty. \quad (6.49)$$

For any function $f(w,v) \in \ell^2((\mathcal{F}_d \times \mathcal{F}_d),\mathcal{H})$, the Z-transform of f is defined by:

$$f^{\wedge(\mathcal{F}_d\times\mathcal{F}_d)}(z,\xi) \triangleq \sum_{(w,v)\in(\mathcal{F}_d\times\mathcal{F}_d)} f(w,v) z^w \xi^v, \quad (6.50)$$

where $z = (z_1,\ldots,z_d)$ and $\xi = (\xi_1,\ldots,\xi_d)$ are two d-tuples of noncommutative indeterminants. Here we write

$$z^w = z_{j_m}\cdots z_{j_1} \text{ if } w = g_{j_m}g_{j_{m-1}}\cdots g_{j_1} \text{ for any } j_k \in \mathcal{I}_d,$$

and

$$\xi^v = \xi_{i_n}\cdots\xi_{i_1} \text{ if } v = g_{i_n}g_{i_{n-1}}\cdots g_{i_1} \text{ for any } i_\ell \in \mathcal{I}_d.$$

From this definition, the Z-transform is indeed a formal power series mapping from the space $\ell^2((\mathcal{F}_d\times\mathcal{F}_d),\mathcal{H})$ onto the space $L^2((\mathcal{F}_d\times\mathcal{F}_d),\mathcal{H})$, where $L^2((\mathcal{F}_d\times\mathcal{F}_d),\mathcal{H})$ denotes the set of all such formal power series $f^{\wedge(\mathcal{F}_d\times\mathcal{F}_d)}(z,\xi)$ in the frequency domain.

Application of the Z-transform to the Future-time system (6.4) where the Cartesian product

$(\mathcal{F}_d \times \mathcal{F}_d)$ is replaced by $\mathcal{T}_f = (\mathcal{F}_d \times \{\lambda\})$ yields

$$\begin{aligned}\sum_{(w,\lambda)\in\mathcal{T}_f} x_k(g_k w,\lambda) z^w &= \sum_{j=1}^{d} \sum_{(w,\lambda)\in\mathcal{T}_f} A_{k,j} x_j(w,\lambda) z^w + \sum_{(w,\lambda)\in\mathcal{T}_f} B_k u_k(w,\lambda) z^w \\ &= \sum_{j=1}^{d} A_{k,j} x_j^{\wedge\mathcal{T}_f}(z,0) + B_k u^{\wedge\mathcal{T}_f}(z,0). \end{aligned} \tag{6.51}$$

Multiply both sides of (6.51) by z_k, we have

$$\sum_{(w,\lambda)\in\mathcal{T}_f} x_k(g_k w,\lambda) z^{g_k w} = z_k \sum_{j=1}^{d} A_{k,j} x_j^{\wedge\mathcal{T}_f}(z,0) + z_k B_k u^{\wedge\mathcal{T}_f}(z,0). \tag{6.52}$$

Let us consider the left hand side of (6.52) for a moment. Note that $\sum_{(w,\lambda)\in\mathcal{T}_f} x_k(g_k w,\lambda) z^{g_k w}$ is equivalent to $\sum_{w:|w|=0}^{\infty} x_k(g_k w,\lambda) z^{g_k w}$, and as we always assume that the boundary conditions are all zero (i.e., $x_k(g_j w) = 0$ for all $w \in \mathcal{F}_d$ unless $k = j$), we have

$$\begin{aligned}\sum_{w:|w|=0}^{\infty} x_k(g_k w,\lambda) z^{g_k w} &= \sum_{\tilde{w}:|\tilde{w}|=1}^{\infty} x_k(\tilde{w},\lambda) z^{\tilde{w}} \\ &= \sum_{\tilde{w}:|\tilde{w}|=0}^{\infty} x_k(\tilde{w},\lambda) z^{\tilde{w}} - x_k(\lambda,\lambda) \\ &= x_k^{\wedge\mathcal{T}_f}(z,0) - x_k(\lambda,\lambda).\end{aligned}$$

Thus the Z-transform of the k-th state equation (6.51) becomes:

$$x_k^{\wedge\mathcal{T}_f}(z,0) = z_k \sum_{j=1}^{d} A_{k,j} x_j^{\wedge\mathcal{T}_f}(z,0) + z_k B_k u^{\wedge\mathcal{T}_f}(z,0) + x_k(\lambda,\lambda). \tag{6.53}$$

More compactly, let us introduce the matrix $Z_d(z) = \begin{bmatrix} z_1 I_{\mathcal{H}_1} & & \\ & \ddots & \\ & & z_d I_{\mathcal{H}_d} \end{bmatrix}$, where $z_i z_j \neq z_j z_i$ unless $i = j$. Hence, the Z-transform of the state equations (6.4) is

$$x^{\wedge\mathcal{T}_f}(z,0) \triangleq \begin{bmatrix} x_1^{\wedge\mathcal{T}_f}(z,0) \\ \vdots \\ x_d^{\wedge\mathcal{T}_f}(z,0) \end{bmatrix} = (I - Z_d(z)A)^{-1} Z_d(z) B u^{\wedge\mathcal{T}_f}(z,0) + (I - Z_d(z)A)^{-1} x(\lambda,\lambda), \tag{6.54}$$

where $u^{\wedge\mathcal{T}_f}(z,0) \in L^2((\mathcal{F}_d \times \{\lambda\}), \mathcal{H})$.

Application of the Z-transform to the output equation of the Future-time system yields

$$\begin{aligned} y^{\wedge \mathcal{T}_f}(z,0) &= Cx^{\wedge \mathcal{T}_f}(z,0) + Du^{\wedge \mathcal{T}_f}(z,0) \\ &= \left[C\,(I - Z_d(z)A)^{-1}\, Z_d(z)B + D\right] \cdot u^{\wedge \mathcal{T}_f}(z,0) + C(I - Z_d(z)A)^{-1}x(\lambda,\lambda) \\ &\triangleq T_{\Sigma_f}(z) \cdot u^{\wedge \mathcal{T}_f}(z,0) + C(I - Z_d(z)A)^{-1}x(\lambda,\lambda), \end{aligned} \tag{6.55}$$

where $T_{\Sigma_f}(z)$ is called the *noncommutative Givone-Roesser (* NCGR*) transfer function* for the Future-time system. Note that $(I - Z_d(z)A)^{-1}$ does exist as a geometric series $\sum_{j=0}^{\infty}(Z_d(z)A)^j$, then the transfer function $T_{\Sigma_f}(z)$ admits a formal power series representation as follows:

$$(I - Z_d(z)A)^{-1} = \sum_{j=0}^{\infty}(Z_d(z)A)^j = \sum_{j=0}^{\infty}\left(\sum_{k=1}^{d} z_k \underline{A}_k\right)^j = \sum_{w\in\mathcal{F}_d} \underline{A}^w z^w. \tag{6.56}$$

Also, $Z_d(z)B = \sum_{k=1}^{d} \underline{B}_k z_k$ and therefore, the transfer function T_{Σ_f} can also be rewritten as

$$\begin{aligned} T_{\Sigma_f}(z) &= D + C\,(I - Z_d(z)A)^{-1}\, Z_d(z)B \\ &= D + C\left[\sum_{j=0}^{\infty}\left(\sum_{k=1}^{d} \underline{A}_k z_k\right)^j\right] \cdot \sum_{k=1}^{d} \underline{B}_k z_k \\ &= D + \sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d} (C\underline{A}^w \underline{B}_k)\, z^{wg_k} \quad := \sum_{v\in\mathcal{F}_d} T_v z^v, \end{aligned} \tag{6.57}$$

where $T_\lambda = D$ and $T_{wg_k} = C\underline{A}^w\underline{B}_k$. See Section 5.1.2 for discussion on a formal power series.

Similarly, let us consider the k-th backward-time system equation for the Past-time NCGR linear system (see (6.41) on page 127). Application of the Z-transform to (6.41) over all the time set $\mathcal{T}_p = (\{\lambda\} \times \mathcal{F}_d \setminus \{\lambda\})$ yields,

$$\sum_{(\lambda,v)\in\mathcal{T}_p} x_k(\lambda,v)\xi^v = \sum_{j=1}^{d}\sum_{(\lambda,v)\in\mathcal{T}_p}(A_{j,k})^* x_j(\lambda, vg_j^{-1})\xi^v + \sum_{(\lambda,v)\in\mathcal{T}_p}(C_k)^* y(\lambda,v)\xi^v$$

$$\text{Or,} \quad x_k^{\wedge\mathcal{T}_p}(0,\xi) = \sum_{j=1}^{d}\sum_{(\lambda,v)\in\mathcal{T}_p}(A_{j,k})^* x_j(\lambda, vg_j^{-1})\xi^v + (C_k)^* y^{\wedge\mathcal{T}_p}(0,\xi). \tag{6.58}$$

Note that

$$\sum_{j=1}^{d}\sum_{(\lambda,v)\in\mathcal{T}_p}(A_{j,k})^* x_j(\lambda, vg_j^{-1})\xi^v = \sum_{j=1}^{d}\sum_{v:|v|=1}^{\infty}(A_{j,k})^* x_j(\lambda, vg_j^{-1})\xi^v$$

$$
\begin{aligned}
&= \sum_{j=1}^{d} (A_{j,k})^* \sum_{v:|v|=1}^{\infty} x_j(\lambda, v g_j^{-1}) \xi^{v g_j^{-1}} \cdot \xi_j \\
&= \sum_{j=1}^{d} (A_{j,k})^* \sum_{\tilde{v}:|\tilde{v}|=0}^{\infty} x_j(\lambda, \tilde{v}) \xi^{\tilde{v}} \cdot \xi_j \\
&= \sum_{j=1}^{d} (A_{j,k})^* \left[\sum_{\tilde{v}:|\tilde{v}|=1}^{\infty} x_j(\lambda, \tilde{v}) \xi^{\tilde{v}} + x_j(\lambda, \lambda) \right] \cdot \xi_j \\
&= \sum_{j=1}^{d} (A_{j,k})^* \left[\sum_{(\lambda,\tilde{v}) \in \mathcal{T}_p} x_j(\lambda, \tilde{v}) \xi^{\tilde{v}} + x_j(\lambda, \lambda) \right] \cdot \xi_j \\
&= \sum_{j=1}^{d} (A_{j,k})^* \left[x_j^{\wedge \mathcal{T}_p}(0, \xi) + x_j(\lambda, \lambda) \right] \cdot \xi_j .
\end{aligned}
$$

By substituting the above expression into (6.58), we have

$$
\begin{aligned}
x_k^{\wedge \mathcal{T}_p}(0, \xi) &= \sum_{j=1}^{d} (A_{j,k})^* \left[x_j^{\wedge \mathcal{T}_p}(0, \xi) + x_j(\lambda, \lambda) \right] \cdot \xi_j + (C_k)^* y^{\wedge \mathcal{T}_p}(0, \xi) \\
&= \sum_{j=1}^{d} (A_{j,k})^* \xi_j \left[x_j^{\wedge \mathcal{T}_p}(0, \xi) + x_j(\lambda, \lambda) \right] + (C_k)^* y^{\wedge \mathcal{T}_p}(0, \xi).
\end{aligned}
$$

Or in more compact form,

$$
x^{\wedge \mathcal{T}_p}(0, \xi) = (I - A^* Z_d(\xi))^{-1} C^* y^{\wedge \mathcal{T}_p}(0, \xi) + (I - A^* Z_d(\xi))^{-1} A^* Z_d(\xi) x(\lambda, \lambda), \tag{6.59}
$$

where $Z_d(\xi) = \mathrm{diag}(\xi_1 I_{\mathcal{H}_1}, \ldots, \xi_d I_{\mathcal{H}_d})$.

We now apply the Z-transform to the backward-time output equation for the Past-time system and hence, we get

$$
\begin{aligned}
u^{\wedge \mathcal{T}_p}(0, \xi) &= \sum_{j=1}^{d} B_j^* \xi_j \left[x_j^{\wedge \mathcal{T}_p}(0, \xi) + x_j(\lambda, \lambda) \right] + D^* y^{\wedge \mathcal{T}_p}(0, \xi) \\
&= B^* Z_d(\xi) x^{\wedge \mathcal{T}_p}(0, \xi) + D^* y^{\wedge \mathcal{T}_p}(0, \xi) + B^* Z_d(\xi) x(\lambda, \lambda).
\end{aligned} \tag{6.60}
$$

Substituting $x^{\wedge \mathcal{T}_p}(0, \xi)$ from (6.59) into the Z-transform of the output equation (6.60), we have

$$
\begin{aligned}
u^{\wedge \mathcal{T}_p}(0, \xi) &= \left[B^* Z_d(\xi)(I - A^* Z_d(\xi))^{-1} C^* + D^* \right] \cdot y^{\wedge \mathcal{T}_p}(0, \xi) + B^* Z_d(\xi)(I - A^* Z_d(\xi))^{-1} x(\lambda, \lambda) \\
&\triangleq T_{\Sigma_{p,b}^*}(\xi) \cdot y^{\wedge \mathcal{T}_p}(0, \xi) + B^* Z_d(\xi)(I - A^* Z_d(\xi))^{-1} x(\lambda, \lambda),
\end{aligned} \tag{6.61}
$$

where $T_{\Sigma_{p,b}^*}(\xi)$ is called the *noncommutative Givone-Roesser (*NCGR*) adjoint transfer function* for the Past-time system.

6.4 Conclusion

This Chapter presents the mathematical linear models for the input/state/output (i/s/o) d-D linear systems with evolution along a free semigroup, namely a noncommutative d-D Givone-Roesser (NCGR) model and a noncommutative d-D Fornasini-Marchesini (NCFM) model. These models are not completely independent; in fact, one can identify one model with the other if the appropriate assumptions are imposed on the NCFM model. We also establish the adjoint systems of the Future-time and the Past-time linear systems which are regarded as systems whose state equations iterate backward in time. Application of the noncommutative d-variable Z-transform yields the transfer functions of i/s/o systems which in the form of a formal power series in noncommuting d-indeterminants.

Chapter 7

Reachability, Controllability, and Observability

This Chapter presents the fundamental concepts of reachability, controllability and observability having been used extensively in conjunction with the state-space analysis for the classical discrete-time linear system. It is well-known that the controllability and the reachability involve the influence of the input signal on the state vector; while the observability deals with the influence of the state vector on the output. These concepts can be extended and applied to the analysis and design procedure of the i/s/o linear systems where the time-axis represented by a free semigroup $\mathcal{F}_d$.

In the state-space analysis, it is known that there are many choices of states in the mathematical models representing the same quantities of the physical system. As a result, these mathematical models describe the behavior and convey the same information of the same physical system even though they have different system matrices $\{A, B, C, D\}$. Sometimes the given state-space representation is much more difficult to implement in the physical aspects. One has a choice to choose state-space representation in an alternative way as long as it preserves full information of the system. For instance, one may choose a new state by changing the state-space coordinates as $\tilde{x} = Sx$, where S is an arbitrary invertible matrix. The Future-time system (6.6) is then described by

$$\begin{bmatrix} \tilde{x}(gw, \lambda) \\ y(w, \lambda) \end{bmatrix} = \tilde{U} \begin{bmatrix} \tilde{x}(w, \lambda) \\ u(w, \lambda) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} \tilde{x}(w, \lambda) \\ u(w, \lambda) \end{bmatrix}, \tag{7.1}$$

where

$$\tilde{A} = SAS^{-1}, \tilde{B} = SB, \tilde{C} = CS^{-1}, \tilde{D} = D \tag{7.2}$$

The transformation matrix, S is called the *similarity transformation*, and the Future-time system

(6.6) and (7.1) are said to be *similar* to each other. Sometimes we may write:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \overset{S}{\sim} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}, \tag{7.3}$$

if (7.2) holds. This result is applicable similarly for the Past-time system.

Since the similarity transformation is an alternative way to convert the original state to a new one without changing information of a system, the question arising here is that if two systems are similar to each other (i.e., there exists an invertible matrix S satisfying the similarity transformation (7.2)), do they still provide the same transfer function? The answer is YES and it follows from the following Lemma. It should be noted here that in general the transformation must commute with the NCGR dynamics $Z_d(z) = \mathrm{diag}\{z_1 I_{\mathcal{H}_1}, \ldots, z_d I_{\mathcal{H}_d}\}$ where $z_i z_j \neq z_j z_i$ unless $i = j$. This leads to the definition of the *admissible similarity transformation.*

Definition 30 (Admissible Transformation). For a given NCGR dynamics Z_d, a set $\mathcal{S}$ is called the *commutative matrix set* with respect to Z_d if each member in this set commutes with Z_d, i.e.

$$\mathcal{S} = \left\{ S \in \mathbb{C}^{n \times n} : S Z_d(\cdot) = Z_d(\cdot) S \right\}$$

where $n = \sum_{i=1}^{d} \dim\,(\mathcal{H}_i)$.

Since $Z_d(\cdot)$ is a diagonal matrix, any matrix $S \in \mathcal{S}$ admits block diagonal structure. In addition, a matrix $S \in \mathcal{S}$ is said to be an *admissible transformation* if S is invertible.

Lemma 7.1. *The transfer function is invariant under the admissible similarity transformation*

Proof. Suppose that $\tilde{T}_{\Sigma_f}$ be transfer function of the similar Future-time system (7.1). Then, it follows that for any $S \in \mathcal{S}$

$$\begin{aligned} \tilde{T}_{\Sigma_f}(z) &= \tilde{C} \left(I - Z_d(z) \tilde{A} \right)^{-1} Z_d(z) \tilde{B} + \tilde{D} \\ &= C S^{-1} \left(I - Z_d(z) S A S^{-1} \right)^{-1} Z_d(z) S B + D \\ &= C (I - Z_d(z) A)^{-1} Z_d(z) B + D \quad = \ T_{\Sigma_f}(z). \end{aligned}$$

Likewise, let $\tilde{T}_{\Sigma^*_{p,b}}$ be an adjoint transfer function of similar Past-time system. Then, it follows that

$$\begin{aligned} \tilde{T}_{\Sigma^*_{p,b}}(\xi) &= \tilde{B}^* Z_d(\xi) \left(I - \tilde{A}^* Z_d(\xi) \right)^{-1} \tilde{C}^* + \tilde{D}^* \\ &= B^* S^* Z_d(\xi) \left(I - (S^{-1})^* A^* S^* Z_d(\xi) \right)^{-1} (S^{-1})^* C^* + D^* \\ &= B^* Z_d(\xi) \left(I - A^* Z_d(\xi) \right)^{-1} C^* + D^* \quad = \ T_{\Sigma^*_{p,b}}(\xi) \end{aligned}$$

∎

This Lemma establishes the fact that the transfer function is invariant with respect to the similarity transformation. Conversely, if two collections of system matrices, say $\{A_i, B_i, C_i, D_i\}$ for $i = 1, 2$, have the same transfer function, it is not always the case that these system matrices are similar to each other. In this case, the concepts of controllability, reachability and observability of a system play an important role to verify whether or not two given collections of system matrices are similar to each other.

7.1 Reachability

In this Section, two notions of reachability: $\{w\}$-reachability and length-n-reachability, are introduced. Suppose the word w is given. Then one may ask if there does exist a control sequence such that the state sequence starting from the zero initial conditions will reach the desired state at "time w". We shall call this property as $\{w\}$-*reachability*. There is another concept of reachability called *length-n-reachability* which is somewhat weaker than the $\{w\}$-reachability. The length-n-reachability means that for given $n \in \mathbb{Z}^+$, there does exist a control sequence such that the system generates the state sequence starting from the zero initial conditions so that the span of state-values on words of length n is the whole space. The precise definitions are provided as follows:

Definition 31 ($\{w\}$-Reachability). Let w be a given word, say $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}, i_k \in \mathcal{I}_d$. Then the system is said to be $\{w\}$-*reachable* if, for zero initial and boundary conditions, and any desired state $x^d \in \mathcal{H}_{i_n}$, there exists a control sequence $\{u(v, \lambda)\}_{\tilde{v}v = w : \tilde{v} \neq \lambda}$ along the path Γ^w so that the system starts at the zero initial condition $x(\lambda, \lambda) = 0$, it generates a state sequence $\{x_k(g_k v, \lambda)\}_{|v| < n}$ satisfying $x_{i_n}(w, \lambda) = x^d$.

Definition 32 (Length-n-Reachability). Given $n \in \mathbb{Z}^+$, the system is said to be *length-n-reachable* if for zero initial and boundary conditions, and any desired state $x^d_{i_n} \in \mathcal{H}_{i_n}$, for each $i_n \in \mathcal{I}_d$, there exists a control sequence $\{u(v, \lambda)\}_{|v| < n}$ so that the system starts at the zero initial conditions $x(\lambda, \lambda) = 0$, it generates a state sequence $\{x_k(g_k v, \lambda)\}_{|v| < n}$ satisfying $x^d_{i_n} = \sum_{\substack{w' \in \mathcal{F}_d, \\ |w'| = n-1}} x_{i_n}(g_{i_n} w', \lambda)$ for all $i_n \in \mathcal{I}_d$.

Definition 33 (Reachability). The system is said to be *reachable* if it is length-n-reachable for some $n \in \mathbb{Z}^+$.

By the definition of $\{w\}$-reachability, it is of interest for us to find a condition such that the system starts at the zero initial condition $x(\lambda, \lambda) = 0$ and then it will reach at the desired state x^d at a given word w. Thus, in this case, we consider the system update equations for the

Future-time which are given by

$$\Sigma_f = \begin{cases} x_k(g_k w, \lambda) & = \sum_{j=1}^{d} A_{k,j} x_j(w,\lambda) + B_k u(w,\lambda) \quad \text{for } k = 1, \ldots, d \\ y(w,\lambda) & = \sum_{j=1}^{d} C_j x_j(w,\lambda) + D u(w,\lambda) \end{cases}$$

and recall that the general response of this state equation is (see (6.48))

$$x(gw,\lambda) - \sum_{k=1}^{d} \underline{A}^{g_k w} x(\lambda,\lambda) = \begin{bmatrix} B_1 & A_{1,i_n} B_{i_n} & \cdots & A_{1,i_n} \cdots A_{i_2,i_1} B_{i_1} \\ \vdots & \vdots & & \vdots \\ B_d & A_{d,i_n} B_{i_n} & \cdots & A_{d,i_n} \cdots A_{i_2,i_1} B_{i_1} \end{bmatrix} \cdot \begin{bmatrix} u(g_{i_n} g_{i_{n-1}} \cdots g_{i_1}, \lambda) \\ u(g_{i_{n-1}} \cdots g_{i_1}, \lambda) \\ \vdots \\ u(g_{i_1}, \lambda) \\ u(\lambda,\lambda) \end{bmatrix}. \tag{7.4}$$

Let x^d be the desired state corresponding to the given word, $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$. Then the $\{w\}$-reachability property implies that

$$\begin{aligned} x^d = x_{i_n}(w,\lambda) &= \begin{bmatrix} B_{i_n} & A_{i_n,i_{n-1}} B_{i_{n-1}} & \cdots & A_{i_n,i_{n-1}} \cdots A_{i_2,i_1} B_{i_1} \end{bmatrix} \cdot \begin{bmatrix} u(g_{i_{n-1}} \cdots g_{i_1}, \lambda) \\ \vdots \\ u(g_{i_1}, \lambda) \\ u(\lambda,\lambda) \end{bmatrix} \\ &= \operatorname{row}_{\substack{\tilde{v}v = w \text{ given} \\ \tilde{v} = g_{i_n} g_{i_{n-1}} \cdots g_{i_{n-\ell}} \neq \lambda \\ \ell = 0,1,\ldots,n-1}} [A_{i_n,i_{n-1}} \cdots A_{i_{n+1-\ell},i_{n-\ell}} B_{i_{n-\ell}}] \cdot \operatorname{col}_{v \in \Gamma^w \setminus \{w\}} [u(v,\lambda)] \\ &\triangleq \mathcal{R}_w^{i_n} \cdot \operatorname{col}_{v \in \Gamma^w \setminus \{w\}} [u(v,\lambda)] \end{aligned} \tag{7.5}$$

where Γ^w is a path corresponding to the given word $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$, and $\mathcal{R}_w^{i_n}$ is called the *$\{w\}$-reachability matrix*. It is clear from (7.5) that $x^d \in \operatorname{im}(\mathcal{R}_w^{i_n})$.

If $\operatorname{rank}(\mathcal{R}_w^{i_n}) < \dim(\mathcal{H}_{i_n})$ we cannot possibly find a solution for every $x_k(g_k \tilde{w}, \lambda)$ for $k \in \mathcal{I}_d$ such that $x^d = x_{i_n}(w,\lambda)$ for given x^d with zero initial and boundary conditions. Hence the system is not $\{w\}$-reachable. Therefore, by the definition of $\{w\}$-reachability, the system Σ_f is $\{w\}$-reachable whenever the $\{w\}$-reachability matrix $\mathcal{R}_w^{i_n}$ is full row rank, i.e. $\operatorname{rank}(\mathcal{R}_w^{i_n}) = \dim(\mathcal{H}_{i_n})$. Equivalently, $\operatorname{im}(\mathcal{R}_w^{i_n}) = \mathcal{H}_{i_n}$. These results lead to the following Theorem.

Theorem 7.2. *Given a word $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$ where $i_k \in \mathcal{I}_d$. Then the following statements are equivalent:*

1. *A system is $\{w\}$-reachable,*

2. $\mathrm{im}(\mathcal{R}_w^{i_n}) = \mathcal{H}_{i_n}$,

3. $\mathrm{rank}(\mathcal{R}_w^{i_n}) = \dim(\mathcal{H}_{i_n})$.

Now we let x_{i_n} denote the span of states at all words of length n starting with a letter g_k given by $x_{i_n} := \sum_{w':|w'|=n-1} x_{i_n}(g_{i_n}w', \lambda)$ and assume that all boundary conditions $\{x_k(g_\ell w)\} = 0$ whenever $k \neq \ell$. Thus we have

$$\begin{aligned} x_{i_n} - IC_{i_n} &= B_{i_n}u(\overbrace{g_1g_1\cdots g_1}^{n-1}, \lambda) + B_{i_n}u(\overbrace{g_1g_1\cdots g_2}^{n-1}, \lambda) + \cdots + B_{i_n}u(\overbrace{g_dg_d\cdots g_d}^{n-1}, \lambda) \\ &\quad + A_{i_n,1}B_1u(\overbrace{g_1g_1\cdots g_1}^{n-2}, \lambda) + \cdots + A_{i_n,1}B_1u(\overbrace{g_dg_d\cdots g_d}^{n-2}, \lambda) \\ &\quad + A_{i_n,2}B_2u(\overbrace{g_1g_1\cdots g_1}^{n-2}, \lambda) + \cdots + A_{i_n,d}A_{d,d}^{n-3}A_{d,d-1}B_{d-1}u(\lambda, \lambda) \\ &\quad + A_{i_n,d}A_{d,d}^{n-2}B_du(\lambda, \lambda) \end{aligned} \tag{7.6}$$

where

$$\begin{aligned} IC_{i_n} &= A_{i_n,1}A_{1,1}^{n-2}\sum_{j=1}^{d}A_{1,j}x_j(\lambda,\lambda) + A_{i_n,1}A_{1,1}^{n-3}A_{1,2}\sum_{j=1}^{d}A_{2,j}x_j(\lambda,\lambda) + \cdots \\ &\quad + A_{i_n,i_n-1}A_{i_n,i_n}^{n-2}\sum_{j=1}^{d}A_{i_n,j}x_j(\lambda,\lambda) + A_{i_n,i_n}^{n-1}\sum_{j=1}^{d}A_{i_n,j}x_j(\lambda,\lambda). \end{aligned}$$

Let $x_{i_n}^d$ be the desired state such that $x_{i_n}^d = \sum_{\substack{w'\in\mathcal{F}_d,\\ |w'|=n-1}} x_{i_n}(g_{i_n}w', \lambda) = x_{i_n}$. Then (7.6) can be written in a condense notation as

$$x_{i_n}^d = \mathrm{row}_{\substack{w:w=g_{i_n}\tilde{w}\\ |\tilde{w}|=n-1}}\left[\mathcal{R}_w^{i_n}\right] \cdot \mathrm{col}_{v:|v|<n}\left[u(v,\lambda)\right] \triangleq \mathcal{R}_n^{i_n} \cdot \mathrm{col}_{v:|v|<n}\left[u(v,\lambda)\right] \tag{7.7}$$

Since some columns in $\mathcal{R}_n^{i_n}$ are identical, the rank condition does not change if such columns are dropped. By using the same notation for convenience, the matrix $\mathcal{R}_n^{i_n}$ corresponding to each $x_{i_n}, i_n \in \mathcal{I}_d$ is given by

$$\mathcal{R}_n^{i_n} = \begin{bmatrix} B_{i_n} & A_{i_n,1}B_1 & \cdots & A_{i_n,d}B_d & A_{i_n,1}A_{1,1}B_1 & \cdots & A_{i_n,d}A_{d,d}^{n-3}A_{d,d-1}B_{d-1} & A_{i_n,d}A_{d,d}^{n-2}B_d \end{bmatrix},$$

where $\mathcal{R}_n^{i_n}$ is called the *length-n-reachability matrix with respect to a letter g_{i_n}*. Thus (7.7) implies that $x_{i_n}^d \in \mathrm{im}(\mathcal{R}_n^{i_n}) \subseteq \mathcal{H}_{i_n}$. If $\mathrm{im}(\mathcal{R}_n^{i_n}) = \mathcal{H}_{i_n}$, it implies that for all words of length n starting with a letter g_{i_n}, the desired state can be reached from the zero initial conditions. By using the same argument as in the $\{w\}$-reachability, one can conclude that the system is length-n-reachable if and only if $\mathrm{im}(\mathcal{R}_n^{i_n}) = \mathcal{H}_{i_n}$ for all $i_n \in \mathcal{I}_d$. Equivalently, for each i_n, the matrix $\mathcal{R}_n^{i_n}$ is full row rank, i.e. $\mathrm{rank}(\mathcal{R}_n^{i_n}) = \dim(\mathcal{H}_{i_n})$.

Let us denote by $\mathcal{R}_n$ the *length-n-reachability matrix* which is a diagonal block matrix defined as follows:

$$\mathcal{R}_n = \mathrm{diag}\,\{\mathcal{R}_n^1, \mathcal{R}_n^2, \ldots, \mathcal{R}_n^d\} = \begin{bmatrix} \mathcal{R}_n^1 & 0 & \cdots & 0 \\ 0 & \mathcal{R}_n^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{R}_n^d \end{bmatrix}. \tag{7.8}$$

Hence, the condition such that $\mathrm{rank}(\mathcal{R}_n^{i_n}) = \dim(\mathcal{H}_{i_n})$ for all $i_n \in \mathcal{I}_d$ is equivalent to $\mathrm{rank}(\mathcal{R}_n) = \dim(\mathcal{H})$. Now let X_n^R denote the space such that (7.7) holds for all $i_n \in \mathcal{I}_d$, i.e.

$$X_n^R = \{x \in \mathcal{H} \mid x_{i_n} = \mathcal{R}_n^{i_n} \cdot \mathrm{col}_{v:|v|<n}[u(v, \lambda)], \forall i_n \in \mathcal{I}_d\}.$$

We shall call X_n^R the *reachability subspace.* Clearly, $X_n^R \subseteq \mathcal{H}$. If $X_n^R = \mathcal{H}$, then every state $x_{i_n}(g_{i_n}\tilde{w}, \lambda) \in \mathcal{H}_{i_n}$, where $|\tilde{w}| = n - 1$ is given and for all $i_n \in \mathcal{I}_d$, can be reached from the zero initial condition which implies also that the system is length-n-reachable. These results lead to the following Theorem.

Theorem 7.3. *Let n be a positive integer. Then the following statements are equivalent:*

1. *A system is length-n-reachable,*
2. *for each $i_n \in \mathcal{I}_d$, $\mathcal{R}_n^{i_n}$ is full row rank, i.e. $\mathrm{im}(\mathcal{R}_n^{i_n}) = \mathcal{H}_{i_n}$,*
3. *the length-n-reachability matrix $\mathcal{R}_n$ given in (7.8) is full rank, i.e. $\mathrm{im}(\mathcal{R}_n) = \bigoplus_{i_n=1}^d \mathcal{H}_{i_n}$,*
4. *the reachability subspace $X_n^R = \mathcal{H}$.*

Obviously, if the system is $\{w\}$-reachable, then it is also length-n-reachable for $n = |w|$. Note that if there exists at least one $j \in \mathcal{I}_d$ such that $\mathrm{rank}(\mathcal{R}_n^j)$ is not full row rank, the system is unreachable and the state x_j is said to be an unreachable mode. In addition, such a system can be decomposed into reachable and unreachable subspaces using a linear transformation operator such as an admissible transformation S in Definition 30. To be more precise, suppose that $\mathrm{rank}(\mathcal{R}_n^j) = r_j < n_j := \dim(\mathcal{H}_j)$. Then there exists nonsingular matrices $S_j \in \mathcal{S} \subset \mathbb{C}^{n_j \times n_j}$ such that

$$S_j \mathcal{R}_n^j = \begin{bmatrix} \tilde{\mathcal{R}}_n^j \\ 0 \end{bmatrix}, \qquad \text{where } \tilde{\mathcal{R}}_n^j \text{ has } r_j \text{ rows},$$

i.e.,

$$S_j \cdot \begin{bmatrix} B_j & A_{j,1}B_1 & \cdots & A_{j,d}B_d & A_{j,1}A_{1,1}B_1 & \cdots & A_{j,d}A_{d,d}^{n-3}A_{d,d-1}B_{d-1} & A_{j,d}A_{d,d}^{n-2}B_d \end{bmatrix}$$
$$= \begin{bmatrix} \tilde{B}_j & \tilde{A}_{j,1}\tilde{B}_1 & \cdots & \tilde{A}_{j,d}\tilde{B}_d & \tilde{A}_{j,1}\tilde{A}_{1,1}\tilde{B}_1 & \cdots & \tilde{A}_{j,d}\tilde{A}_{d,d}^{n-3}\tilde{A}_{d,d-1}\tilde{B}_{d-1} & \tilde{A}_{j,d}\tilde{A}_{d,d}^{n-2}\tilde{B}_d \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

This implies that $S_jB_j = \begin{bmatrix} \tilde{B}_j \\ 0 \end{bmatrix}$, and for each $k = 1, \ldots, d$, $S_jA_{j,k}B_k = \begin{bmatrix} \widetilde{A_{j,k}}\tilde{B}_k \\ 0 \end{bmatrix}$. Thus, there exists a nonsingular matrix $S_k \in \mathcal{S} \subset \mathbb{C}^{n_k \times n_k}$ such that

$$S_jA_{j,k}B_k = (S_jA_{j,k}S_k^{-1})(S_kB_k) = \begin{bmatrix} \tilde{A}_{j,k} & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \tilde{B}_k \\ 0 \end{bmatrix},$$

where $*$ is arbitrary.

Note that the results for $A_{j,k}A_{k,\ell} \cdots B_d$ terms can be obtained in the similar way. Then we set $S = \text{diag}\{S_1, \ldots, S_d\}$ to be the admissible transformation which decomposes the realization $\{A, B, C, D\}$ into unreachable and reachable submatrices corresponding to unreachable and reachable subspaces, respectively.

Remark 22. If n is arbitrary, we shall write $\mathcal{R} := \text{diag}\{\mathcal{R}^1, \ldots, \mathcal{R}^d\}$ rather than $\mathcal{R}_n$ for some fixed n and $\mathcal{R}$ itself is called the *reachability matrix*. For each $k \in \mathcal{I}_d$, $\mathcal{R}^k$ is an infinite row matrix and we consider it acting on columns of infinite length but with finite support (all but finitely many entries are zero, but which finitely many and how many depends on the particular vector). Thus the reachability means that this infinite matrix $\mathcal{R}$ has full rank. ▲

Example 13. Suppose $\mathcal{F} = \{0, 1\}$ and the desired states is such that the word-length $|w| = n = 3$ where $w \in \mathcal{F}_2$. Then one can verify that

$$\begin{bmatrix} x_1(000, \lambda) \\ x_2(100, \lambda) \end{bmatrix} = \begin{bmatrix} B_1 & A_{11}B_1 & A_{11}^2B_1 \\ B_2 & A_{21}B_1 & A_{21}A_{11}B_1 \end{bmatrix} \begin{bmatrix} u(00, \lambda) \\ u(0, \lambda) \\ u(\lambda) \end{bmatrix}$$

$$\begin{bmatrix} x_1(001, \lambda) \\ x_2(101, \lambda) \end{bmatrix} = \begin{bmatrix} B_1 & A_{11}B_1 & A_{11}A_{12}B_2 \\ B_2 & A_{21}B_1 & A_{21}A_{12}B_2 \end{bmatrix} \begin{bmatrix} u(01, \lambda) \\ u(1, \lambda) \\ u(\lambda) \end{bmatrix}$$

$$\begin{bmatrix} x_1(010, \lambda) \\ x_2(110, \lambda) \end{bmatrix} = \begin{bmatrix} B_1 & A_{12}B_2 & A_{12}A_{21}B_1 \\ B_2 & A_{22}B_2 & A_{22}A_{21}B_1 \end{bmatrix} \begin{bmatrix} u(10, \lambda) \\ u(0, \lambda) \\ u(\lambda) \end{bmatrix}$$

$$\begin{bmatrix} x_1(011,\lambda) \\ x_2(111,\lambda) \end{bmatrix} = \begin{bmatrix} B_1 & A_{12}B_2 & A_{12}A_{22}B_2 \\ B_2 & A_{22}B_2 & A_{22}^2B_2 \end{bmatrix} \begin{bmatrix} u(11,\lambda) \\ u(1,\lambda) \\ u(\lambda) \end{bmatrix}$$

This system is length-3-reachable if and only if $\mathcal{R}_3 = \mathrm{diag}\{\mathcal{R}_3^1, \mathcal{R}_3^2\}$ is of full rank, where

$$\mathcal{R}_3^1 = \begin{bmatrix} B_1 & A_{11}B_1 & A_{12}B_2 & A_{11}^2B_1 & A_{11}A_{12}B_2 & A_{12}A_{21}B_1 & A_{12}A_{22}B_2 \end{bmatrix}$$
$$\mathcal{R}_3^2 = \begin{bmatrix} B_2 & A_{21}B_1 & A_{22}B_2 & A_{21}A_{11}B_1 & A_{21}A_{12}B_2 & A_{22}A_{21}B_1 & A_{22}^2B_2 \end{bmatrix}$$

Now if we are interested in the reachability of a system at a particular word, say $w = 110$, then the system is $\{110\}$-reachable if and only if $\mathcal{R}_{\{110\}} = \begin{bmatrix} B_2 & A_{22}B_2 & A_{22}A_{21}B_1 \end{bmatrix}$ is of full row rank. $\diamondsuit$

7.2 Controllability

The concept of controllability is parallel with the concept of reachability but in the reverse direction of the path. The concept of reachability concerns whether or not the given state x^d in the Future can be reached from the initial state at the Present time $x(\lambda,\lambda)$ within finite number of steps of iteration. In contrast, the concept of controllability concerns whether or not the given state x^d in the Future can be controlled back to the initial state $x(\lambda,\lambda)$ at the Present time within finite number of steps of iteration. Since the trajectory $(u(v,\lambda), x(v,\lambda), y(v,\lambda))$ along the path Γ^w is identical to the trajectory $(u(\lambda,v'), x(\lambda,v'), y(\lambda,v'))$ along the reverse path ${}^w\Gamma$ (by the change of varible $(v,\lambda) \mapsto (\lambda,v')$), the concept of controllability may be viewed as the the reachability of the system at the Present-time λ from the zero initial state $x(\lambda, gw)$ given in the Past. For further discussion on this construction, we refer to Section 5.3 and Remark 20.

There are also two notions of controllability: $\{w\}$-controllability and length-n-controllability. A $\{w\}$-controllability means that for a given word w, there exists a control sequence $\{u(\lambda,v)\}_{v\tilde{v}=w}$ such that the system starts at the word w, it generates a state sequence until it reaches the initial state $x(\lambda,\lambda)$ within finite steps. Instead, if we are given $n \in \mathbb{Z}^+$ the length of a word rather than a specified word, and asked whether or not there exists a control sequence $\{u(\lambda,v)\}_{|v|\le n}$ to control any state of length n to the initial state within finite steps of iteration, this is a concept of length-n-controllability. The precise definitions are provided as follows:

Definition 34 ($\{w\}$-Controllability). Let w be a given word, say $w = g_{i_n}g_{i_{n-1}}\cdots g_{i_1} \neq \lambda$, where $i_k \in \mathcal{I}_d$. Then the system is said to be $\{w\}$-*controllable* if, for zero initial conditions $x(\lambda,w), |w| = n$, there exists a control sequence $\{u(\lambda,v)\}_{v\tilde{v}=w}$ along the path ${}^w\Gamma$ so that the system starting at the word w, can generate a state sequence $\{x_k(\lambda, vg_k^{-1})\}_{v\tilde{v}=w}$ until it reaches the state $x_{i_1}(\lambda,\lambda)$.

Definition 35 (Length-n-Controllability). Given $n \in \mathbb{Z}^+$, the system is said to be *length-n-controllable* if for zero initial conditions $x_k(\lambda, w) = 0$, for each $k \in \mathcal{I}_d$ and for all $w \in \mathcal{F}_d$ such that $|w| = n$, there exists a control sequence $\{u(\lambda, v)\}_{|v| \leq n}$ so that the system starting at any word of length n can generate a state sequence $\{x_k(\lambda, vg_k^{-1})\}_{|v|<n}$ until it reaches the state $x(\lambda, \lambda)$.

Definition 36 (Controllability). The system is said to be *controllable* if it is length-n-controllable for some $n \in \mathbb{Z}^+$.

Recall that the Past-time system equations are given by

$$\Sigma_p = \begin{cases} x_k(\lambda, wg_k^{-1}) & = \sum_{j=1}^d A_{k,j} x_j(\lambda, w) + B_k u(\lambda, w) \quad \text{for } k = 1, \dots, d \\ y(\lambda, w) & = \sum_{j=1}^d C_j x_j(\lambda, w) + D u(\lambda, w). \end{cases}$$

Suppose that the initial conditions at the level n are all zero, i.e. $x_k(\lambda, w) = 0$ for all $k \in \mathcal{I}_d$ and for all $w \in \mathcal{F}_d, |w| = n$. Then the general solution of this system (see (6.46) on page 129) collapses to

$$x(\lambda, \lambda) = \sum_{k=1}^{d} \sum_{w:|w|<n} \underline{A}^w \underline{B}_k u(\lambda, wg_k) \tag{7.9}$$

where $\underline{A}^w = \underline{A}_{i_n} \underline{A}_{i_{n-1}} \cdots \underline{A}_{i_1}$ if $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$. It is easy to check that the solution in (7.9) can be expressed as

$$x(\lambda, \lambda) = \begin{bmatrix} \mathcal{C}_n^1 & 0 & \cdots & 0 \\ 0 & \mathcal{C}_n^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{C}_n^d \end{bmatrix} \cdot \begin{bmatrix} u_n^1 \\ u_n^2 \\ \vdots \\ u_n^d \end{bmatrix} \triangleq \mathcal{C}_n \cdot \operatorname{col}_{\substack{v:|v|<n \\ k=1,\dots,d}} [u(\lambda, g_k v)] \tag{7.10}$$

where $u_n^k := u(\lambda, g_k v)$ for all $v \in \mathcal{F}_d, |v| < n$, and

$$\mathcal{C}_n^k := \begin{bmatrix} B_k & A_{k,1}B_1 & \cdots & A_{k,d}B_d & A_{k,1}A_{1,1}B_1 & \cdots & A_{k,d}A_{d,d}^{n-1}B_d \end{bmatrix}.$$

We shall refer to $\mathcal{C}_n^k$ and $\mathcal{C}_n$ as the *length-n-controllability matrix with respect to a letter g_k* and the *length-n-controllability matrix*, respectively. Clearly, the matrix $\mathcal{C}_n$ is full rank if and only if for each $k \in \mathcal{I}_d$, $\mathcal{C}_n^k$ is full rank.

Now let X_n^C be the space such that (7.10) holds, i.e.

$$X_n^C = \{x \in \mathcal{H} \mid x(\lambda, \lambda) = \mathcal{C}_n \cdot \operatorname{col}_{v:|v| \leq n}[u(\lambda, v)]\}$$

where X_n^C is called the *controllability subspace*. Obviously, $X_n^C \subseteq \mathcal{H}$. If $X_n^C = \mathcal{H}$, then every state $x_k(\lambda, w) \in \mathcal{H}_k$, where $w \in \mathcal{F}_d, |w| = n$ can be transferred to the state $x(\lambda, \lambda)$ which implies that the system is length-n-controllable. Thus, we have

Theorem 7.4. *Let n be a positive integer. Then the following statements are equivalent:*

1. *A system is length-n-controllable,*
2. *for each $k \in \mathcal{I}_d$, $\mathcal{C}_n^k$ is full row rank, i.e.* $\mathrm{im}(\mathcal{C}_n^k) = \mathcal{H}_k$,
3. *the length-n-controllability matrix $\mathcal{C}_n$ given in* (7.10) *is full rank, i.e.,* $\mathrm{im}(\mathcal{C}_n) = \bigoplus_{k=1}^d \mathcal{H}_k$,
4. *the controllability subspace $X_n^C = \mathcal{H}$.*

If n is arbitrary, we shall write $\mathcal{C} := \mathrm{diag}\{\mathcal{C}^1, \ldots, \mathcal{C}^d\}$ rather than $\mathcal{C}_n$ for some fixed n, and call $\mathcal{C}$ as the *controllability matrix*, which is in fact an infinite matrix. This fact is analogous to the reachability matrix $\mathcal{R}$ (see Remark 22). We should note that if there exists at least one $j \in \mathcal{I}_d$ such that $\mathrm{rank}(\mathcal{C}_n^j)$ is not of full row rank, the system is uncontrollable and the state x_j is said to be an uncontrollable mode. In addition, such a system can be decomposed into controllable and uncontrollable subspaces using an admissible transformation S.

Suppose we are interested in a specified word, say $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$ and consider only state along the path ${}^w\Gamma$ (i.e., $x(\lambda, v) = 0$ unless $v \in {}^w\Gamma$), then from the explicit formula for $x_{i_n}(\lambda, \lambda)$ in (6.44), we have:

$$
\begin{aligned}
x_{i_n}(\lambda,\lambda) &= A_{i_n,i_{n-1}} \cdots A_{i_2,i_1} B_{i_1} u(\lambda, w) + \cdots + A_{i_n,i_{n-1}} B_{i_{n-1}} u(\lambda, g_{i_n} g_{i_{n-1}}) + B_{i_n} u(\lambda, g_{i_n}) \\
&= \begin{bmatrix} B_{i_n} & A_{i_n,i_{n-1}} B_{i_{n-1}} & \cdots & A_{i_n,i_{n-1}} \cdots A_{i_2,i_1} B_{i_1} \end{bmatrix} \cdot \begin{bmatrix} u(\lambda, g_{i_n}) \\ u(\lambda, g_{i_n} g_{i_{n-1}}) \\ \vdots \\ u(\lambda, g_{i_n} g_{i_{n-1}} \cdots g_{i_1}) \end{bmatrix} \\
&= \mathrm{row}_{\substack{v\tilde{v}=w \text{ given} \\ \tilde{v}=g_{i_{n-\ell}} \cdots g_{i_1} \\ \text{for } \ell=1,\ldots,n-1; \\ \text{if } \tilde{v}=\lambda, \text{ set } \ell=n}} \begin{bmatrix} A_{i_n,i_{n-1}} A_{i_{n-1},i_{n-2}} \cdots A_{i_{n-\ell+2},i_{n-\ell+1}} B_{i_{n-\ell+1}} \end{bmatrix} \cdot \mathrm{col}_{v \in ({}^w\Gamma \setminus \{\lambda\})} [u(\lambda, v)] \\
&\triangleq \mathcal{C}_w^{i_n} \cdot \mathrm{col}_{v \in ({}^w\Gamma \setminus \{\lambda\})} [u(\lambda, v)],
\end{aligned} \tag{7.11}
$$

where ${}^w\Gamma$ is a reverse path of Γ^w corresponding to the given word $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$, and $\mathcal{C}_w^{i_n}$ is called the $\{w\}$-*controllability matrix*. It is clear from (7.11) that $x_{i_n}(\lambda, \lambda) \in \mathrm{im}(C_w^{i_n})$. By using the similar argument as in the $\{w\}$-reachability case, we obtain the following Theorem.

Theorem 7.5. *Given a word $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$, where $i_k \in \mathcal{I}_d$. Then the following statements are equivalent:*

1. *A system is $\{w\}$-controllable,*

2. $\mathrm{im}(\mathcal{C}_w^{i_n}) = \mathcal{H}_{i_n}$,

3. $\mathrm{rank}(\mathcal{C}_w^{i_n}) = \dim(\mathcal{H}_{i_n})$.

Clearly, if the system is $\{w\}$-controllable, then it is also length-n-controllable.

Remark 23. As mentioned previously, the notions of controllability and of reachability are parallel to each other. In fact the system generating a state sequence from the zero initial condition $x(\lambda, \lambda)$ to the desired state $x(w, \lambda)$ along the path Γ^w is the same as the system generating a state sequence from the zero initial condition $x(\lambda, w)$ to the state $x(\lambda, \lambda)$ along the reverse path ${}^w\Gamma$. It is obvious from analysis that the controllability matrix $\mathcal{C}$, *etc.* is identical to the reachability matrix $\mathcal{R}$, *etc.* Thus, one may use interchangeably between the notion of $\{w\}$-reachability (resp., reachability) and that of $\{w\}$-controllability (resp., controllability). ▲

7.3 Observability

The observability property of a system involves the influence of the state vector on the output. It is of interest to investigate to what extent it is possible to reconstruct the state x when the output sequence $\{y(w, \lambda)\}$ is known. Often one is able to measure the output sequence and prescribe the known input, whereas the state variable is hidden. This Section presents two concepts of observability which act in the dual manner of the controllability property. The $\{w\}$-observability means that for given word w, one is able to completely determine the initial condition $x(\lambda, \lambda)$ from the measurement data $\{y(v, \lambda)\}$ for all v along the path Γ^w. Rather than considering the particular word, we have the notion of *length-n-observability* which is somewhat weaker than the $\{w\}$-observability. Given $n \in \mathbb{Z}^+$, the length-n-observability implies that every initial condition $x(\lambda, \lambda)$ is completely determined from an output sequence $\{y(v, \lambda)\}$ for all possible v such that $|v| \leq n$. The precise definitions are provided as follows:

Definition 37 ($\{w\}$-observability). Let w be a given word, say $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$, $i_k \in \mathcal{I}_d$. Then the system is said to be *$\{w\}$-observable* if, for zero input and zero boundary conditions, there exists an output sequence $\{y(v, \lambda)\}_{\tilde{v}v=w}$ such that any initial condition $x_{i_1}(\lambda, \lambda)$, corresponding to a given word, w, can be completely determined.

Definition 38 (Length-n-observability). Given $n \in \mathbb{Z}_+$, then the system is said to be *length-n-observable* if for zero input and zero boundary conditions, every initial condition $x(\lambda, \lambda)$ can be determined from an output sequence $\{y(v, \lambda)\}_{|v| \leq n}$.

Definition 39 (Observability). The system is said to be *observable* if it is length-n-observable for some $n \in \mathbb{Z}^+$.

Consider the unforced input/state/output d-D linear system (i.e., the system with zero input sequence). Suppose we are given a word $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1} \neq \lambda$. Then, from the Future-time system equations, we have

$$
\begin{aligned}
y(\lambda,\lambda) &= \sum_{k=1}^{d} C_k x_k(\lambda,\lambda) \\
y(g_{i_1},\lambda) &= \sum_{\substack{k=1\\k\neq i_1}}^{d} C_k x_k(g_{i_1},\lambda) + C_{i_1} \sum_{k=1}^{d} A_{i_1,k} x_k(\lambda,\lambda) \\
y(g_{i_2}g_{i_1},\lambda) &= \sum_{\substack{k=1\\k\neq i_2}}^{d} C_k x_k(g_{i_2}g_{i_1},\lambda) + C_{i_2} \sum_{\substack{k=1\\k\neq i_1}}^{d} A_{i_2,k} x_k(g_{i_1},\lambda) + C_{i_2}A_{i_2,i_1} \sum_{k=1}^{d} A_{i_1,k} x_k(\lambda,\lambda) \\
y(g_{i_3}g_{i_2}g_{i_1},\lambda) &= \sum_{\substack{k=1\\k\neq i_3}}^{d} C_k x_k(g_{i_3}g_{i_2}g_{i_1},\lambda) + C_{i_3} \sum_{\substack{k=1\\k\neq i_2}}^{d} A_{i_3,k} x_k(g_{i_2}g_{i_1},\lambda) \\
&\quad + C_{i_3}A_{i_3,i_2} \sum_{\substack{k=1\\k\neq i_1}}^{d} A_{i_2,k} x_k(g_{i_1},\lambda) + C_{i_3}A_{i_3,i_2}A_{i_2,i_1} \sum_{k=1}^{d} A_{i_1,k} x_k(\lambda,\lambda) \\
&\ \vdots \\
y(w,\lambda) &= \sum_{\substack{k=1\\k\neq i_n}}^{d} C_k x_k(g_{i_n}g_{i_{n-1}}\cdots g_{i_1},\lambda) + C_{i_n} \sum_{\substack{k=1\\k\neq i_{n-1}}}^{d} A_{i_n,k} x_k(g_{i_{n-1}}\cdots g_{i_1},\lambda) \\
&\quad + \cdots + C_{i_n}A_{i_n,i_{n-1}}A_{i_{n-1},i_{n-2}}\cdots A_{i_3,i_2} \sum_{\substack{k=1\\k\neq i_1}}^{d} A_{i_2,k} x_k(g_{i_1},\lambda) \\
&\quad + C_{i_n}A_{i_n,i_{n-1}}A_{i_{n-1},i_{n-2}}\cdots A_{i_2,i_1} \sum_{k=1}^{d} A_{i_1,k} x_k(\lambda,\lambda) \qquad (7.12)
\end{aligned}
$$

Suppose now that the boundary conditions $x_k(g_\ell \tilde{w},\lambda) = 0$ unless $k = \ell$. Then, the output equation (7.12) becomes

$$
\begin{aligned}
y(w,\lambda) &= \begin{cases} \sum_{k=1}^{d} C_k x_k(\lambda,\lambda), & \text{if } w = \lambda; \\ C_{i_n}A_{i_n,i_{n-1}}A_{i_{n-1},i_{n-2}}\cdots A_{i_2,i_1} \sum_{k=1}^{d} A_{i_1,k} x_k(\lambda,\lambda), & \text{if } w = g_{i_n}g_{i_{n-1}}\cdots g_{i_1} \neq \lambda, \end{cases} \\
&\triangleq \begin{cases} \sum_{k=1}^{d} y_\lambda^k(\lambda,\lambda), & \text{if } w = \lambda; \\ \sum_{k=1}^{d} y^k(w,\lambda), & \text{if } w = g_{i_n}g_{i_{n-1}}\cdots g_{i_1} \neq \lambda. \end{cases} \qquad (7.13)
\end{aligned}
$$

where $y_\lambda^k(\lambda,\lambda) = C_k x_k(\lambda,\lambda)$ and $y^k(w,\lambda) = C_{i_n}A_{i_n,i_{n-1}}A_{i_{n-1},i_{n-2}}\cdots A_{i_2,i_1}A_{i_1,k}x_k(\lambda,\lambda)$.

Consider now when $x_k(\lambda,\lambda) = 0$ unless $k = i_1$, i.e. $x(\lambda,\lambda) = \begin{bmatrix} 0 & \cdots & x_{i_1}^\top(\lambda,\lambda) & \cdots & 0 \end{bmatrix}^\top$. Then the output equation (7.13) collapses to

$$y(w,\lambda) = \begin{cases} y_\lambda^{i_1}(\lambda,\lambda) = C_{i_1} x_{i_1}(\lambda,\lambda), & \text{if } w = \lambda; \\ y^{i_1}(w,\lambda) = C_{i_n} A_{i_n,i_{n-1}} A_{i_{n-1},i_{n-2}} \cdots A_{i_2,i_1} A_{i_1,i_1} x_{i_1}(\lambda,\lambda), & \text{if } w \neq \lambda. \end{cases} \tag{7.14}$$

Therefore, the output sequence $\{y(v,\lambda)\}_{\tilde{v}v=w}$ along the path Γ^w corresponding to the given word, $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$, is given by:

$$\begin{bmatrix} y(\lambda,\lambda) \\ y(g_{i_1},\lambda) \\ \vdots \\ y(g_{i_n} g_{i_{n-1}} \cdots g_{i_1},\lambda) \end{bmatrix} = \begin{bmatrix} C_{i_1} \\ C_{i_1} A_{i_1,i_1} \\ \vdots \\ C_{i_n} A_{i_n,i_{n-1}} A_{i_{n-1},i_{n-2}} \cdots A_{i_2,i_1} A_{i_1,i_1} \end{bmatrix} x_{i_1}(\lambda,\lambda) \tag{7.15}$$

Or in the condensed notation,

$$\mathbb{Y}_w = \mathcal{O}_w^{i_1} \cdot x_{i_1}(\lambda,\lambda)$$

where $\mathbb{Y}_w = \mathrm{col}_{v \in \Gamma^w}[y(v)]$, $\mathcal{O}_w^{i_1} = \mathrm{col}_{\substack{\tilde{v}v = w \text{ given} \\ \tilde{v} = g_{i_n} g_{i_{n-1}} \cdots g_{i_{n-\ell}} \\ \text{for } \ell = 0,1,\ldots,n-1; \\ \text{if } \tilde{v} = \lambda, \text{ set } \ell = -1}} \left[C_{i_{n-\ell-1}} A_{i_{n-\ell-1},i_{n-\ell-2}} \cdots A_{i_2,i_1} A_{i_1,i_1} \right]$, and C_{i_0} is defined as C_{i_1}. The matrix $\mathcal{O}_w^{i_1}$ is called the $\{w\}$-*observability matrix* corresponding to the given word $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$.

Now if $\mathrm{rank}(\mathcal{O}_w^{i_1}) < \dim(\mathcal{H}_{i_1})$ the initial state $x_{i_1}(\lambda,\lambda)$ cannot be observed from the output sequence $\{y(v,\lambda)\}$, and hence the system is not $\{w\}$-observable. Thus, the system is $\{w\}$-observable if by definition $\mathrm{rank}(\mathcal{O}_w^{i_1}) = \mathrm{rank}(\mathcal{H}_{i_1})$. In particular, for the system to be $\{w\}$-observable, the rank condition must hold if the output sequence is set to be a zero vector. Thus, this implies that $\ker(\mathcal{O}_w^{i_1}) = \{0\}$. This observation leads to the following Theorem.

Theorem 7.6. *Given a word $w = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$, where $i_k \in \mathcal{I}_d$. Then the following statements are equivalent:*

1. *A system is $\{w\}$-observable,*

2. $\ker(\mathcal{O}_w^{i_1}) = \{0\} \in \mathcal{H}_{i_1}$,

3. $\mathrm{rank}(\mathcal{O}_w^{i_1}) = \dim(\mathcal{H}_{i_1})$.

Suppose now that one can measure all output sequences $\{y(v,\lambda)\}$ described in the preceding paragraph for all words with length less than or equal to n, and define

$$\mathbb{Y}^k \triangleq \mathcal{O}_n^k x_k(\lambda,\lambda)$$

where

$$\mathcal{O}_n^k = \begin{bmatrix} C_k \\ C_1 A_{1,k} \\ \vdots \\ C_d A_{d,k} \\ C_1 A_{1,1} A_{1,k} \\ \vdots \\ C_d A_{d,d}^{n-1} A_{d,k} \end{bmatrix} \tag{7.16}$$

and we shall refer to it as the *length-n-observability matrix with respect to a letter g_k.*

This yields,

$$\mathrm{col}_{v:|v|\leq n}[y(v,\lambda)] = \begin{bmatrix} \mathbb{Y}^1 \\ \mathbb{Y}^2 \\ \vdots \\ \mathbb{Y}^d \end{bmatrix} = \begin{bmatrix} \mathcal{O}_n^1 & 0 & \cdots & 0 \\ 0 & \mathcal{O}_n^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{O}_n^d \end{bmatrix} \begin{bmatrix} x_1(\lambda,\lambda) \\ x_2(\lambda,\lambda) \\ \vdots \\ x_d(\lambda,\lambda) \end{bmatrix}. \tag{7.17}$$

To condense notation, we shall write (7.17) as $\mathbb{Y} = \mathcal{O}_n \cdot x(\lambda,\lambda)$, where $\mathcal{O}_n$ is called the *length-n-observability matrix.* In order to determine every initial condition $x(\lambda,\lambda)$, $\mathrm{rank}(\mathcal{O}_n)$ must be of full column rank, i.e., $\mathrm{rank}(\mathcal{O}_n) = \dim(\mathcal{H})$. Equivalently, $\mathrm{rank}(\mathcal{O}_n^k) = \dim(\mathcal{H}_k)$ for each $k \in \mathcal{I}_d$. The system is length-n-observable if the condition (7.17) holds for all $\mathbb{Y}$. In particular, it is of interest to consider when $\mathbb{Y} = 0$. Thus, (7.17) collapses to

$$0 = \begin{bmatrix} \mathcal{O}_n^1 & 0 & \cdots & 0 \\ 0 & \mathcal{O}_n^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{O}_n^d \end{bmatrix} x(\lambda,\lambda), \tag{7.18}$$

i.e., $x(\lambda,\lambda) \in \ker(\mathcal{O}_n)$. Now let X_n^O be the space such that (7.18) holds, i.e.

$$X_n^O = \{x \in \mathcal{H} \mid \mathcal{O}_n \cdot x = 0\}.$$

We shall call X_n^O the *observability subspace.* Clearly, $X_n^O \subseteq \mathcal{H}$. If there is only a trivial solution $x \equiv 0 \in \mathcal{H}$ contained in X_n^O for given n, then the system is length-n-observable. Otherwise, the matrix $\mathcal{O}_n$ degenerates rank; i.e., there does exist a vector $x \neq 0$ so that $\mathcal{O}_n \cdot x = 0$. This implies that the individual state x_k cannot be completely determined by the measurement data. In fact, the measurement data provides only the information of the linear combination of state $x_k, k \in \mathcal{I}_d$ rather than the individual x_k. Thus, the system is not length-n-observable. This result leads to the following theorem.

Theorem 7.7. *Let n be a positive integer. Then the following statements are equivalent:*

1. *A system is length-n-observable,*
2. *for each $k \in \mathcal{I}_d$, $\mathcal{O}_n^k$ is full column rank, i.e.* $\ker(\mathcal{O}_n^k) = \{0\} \in \mathcal{H}_k$.
3. *the length-n-observability matrix $\mathcal{O}_n$ given in* (7.17) *is full column rank, i.e.*

$$\ker(\mathcal{O}_n) = \{0\} \in \mathcal{H} = \bigoplus_{i_1=1}^{d} \mathcal{H}_{i_1},$$

4. *the observability subspace* $X_n^O = \{0\}$.

Obviously, if the system is $\{w\}$-observable, then it is also length-n-observable for $n = |w|$. If there exists at least one $j \in \mathcal{I}_d$ such that $\text{rank}(\mathcal{O}_w^j)$ is not of full column rank, the system is unobservable and the state x_j is said to be an unobservable mode. In addition, such a system can be decomposed into observable and unobservable subspaces using an admissible transformation in Definition 30. Since the notion of observability is a dual concept of the reachability, the reader is referred to Section 7.1 for further discussion on this issue.

Example 14. Suppose $\mathcal{F} = \{0, 1\}$. Then one can check that

$$\begin{bmatrix} y(\lambda,\lambda) \\ y(0,\lambda) \\ y(1,\lambda) \\ y(00,\lambda) \\ y(01,\lambda) \\ y(10,\lambda) \\ y(11,\lambda) \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_1A_{11} & C_1A_{12} \\ C_2A_{21} & C_2A_{22} \\ C_1A_{11}^2 & C_1A_{11}A_{12} \\ C_1A_{12}A_{21} & C_1A_{12}A_{22} \\ C_2A_{21}A_{11} & C_2A_{21}A_{12} \\ C_2A_{22}A_{21} & C_2A_{22}^2 \end{bmatrix} \begin{bmatrix} x_1(\lambda,\lambda) \\ x_2(\lambda,\lambda) \end{bmatrix}$$

Hence, $\mathcal{O}_2^1 = \begin{bmatrix} C_1 \\ C_1A_{11} \\ C_2A_{21} \\ C_1A_{11}^2 \\ C_1A_{12}A_{21} \\ C_2A_{21}A_{11} \\ C_2A_{22}A_{21} \end{bmatrix}$, and $\mathcal{O}_2^2 = \begin{bmatrix} C_2 \\ C_1A_{12} \\ C_2A_{22} \\ C_1A_{11}A_{12} \\ C_1A_{12}A_{22} \\ C_2A_{21}A_{12} \\ C_2A_{22}^2 \end{bmatrix}$ and the length-2-observability matrix, $\mathcal{O}_2 = \text{diag}\{\mathcal{O}_w^1, \mathcal{O}_w^2\}$. This system is length-2-observable if and only if $\mathcal{O}_2$ is of full rank. Suppose we are interested in the observability of a system at a particular word, say $w = 11$, then the system is $\{11\}$-observable if and only if $\mathcal{O}_{\{11\}} = \begin{bmatrix} C_2 \\ C_2A_{21} \\ C_2A_{22}A_{11} \end{bmatrix}$ is of full column rank. $\diamondsuit$

In the next example, the matrices A, B, and C are adopted from [BD99].

Example 15. Suppose the realization matrices $\{A, B, C\}$ of 2-D Roesser system are given by:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0.5 & -0.4 \\ 0.05 & 0.2 \end{bmatrix} & \begin{bmatrix} 0.25 & 0.2 \\ 0.15 & 0.05 \end{bmatrix} \\ \begin{bmatrix} 0.1 & -0.2 \\ 0.2 & -0.4 \end{bmatrix} & \begin{bmatrix} -0.3 & 0.35 \\ -1.0 & 0.9 \end{bmatrix} \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix}$$

Then the length-3-reachability matrix, $\mathcal{C}_3 = \text{diag}\{\mathcal{C}_3^1, \mathcal{C}_3^2\}$ where,

$$\begin{aligned} \mathcal{C}_3^1 &= \begin{bmatrix} B_1 & A_{11}B_1 & A_{12}B_2 & A_{11}^2B_1 & A_{11}A_{12}B_2 & A_{12}A_{21}B_1 & A_{12}A_{22}B_2 \end{bmatrix} \\ &= \begin{bmatrix} 1.00 & 4.00 & 0.50 & 1.20 & 0.65 & 0.65 & 0.23 & 0.36 & 0.225 & 0.225 & 0.065 & 0.00 & 0.26 & 0.26 \\ 0.00 & 2.00 & 0.05 & 0.60 & 0.25 & 0.25 & 0.035 & 0.18 & 0.0825 & 0.0825 & 0.025 & 0.00 & 0.10 & 0.10 \end{bmatrix} \\ \mathcal{C}_3^2 &= \begin{bmatrix} B_2 & A_{21}B_1 & A_{22}B_2 & A_{21}A_{11}B_1 & A_{21}A_{12}B_2 & A_{22}A_{21}B_1 & A_{22}^2B_2 \end{bmatrix} \\ &= \begin{bmatrix} 1.00 & 1.00 & 0.10 & 0.00 & 0.40 & 0.40 & 0.04 & 0.00 & 0.015 & 0.015 & 0.04 & 0.00 & 0.16 & 0.16 \\ 2.00 & 2.00 & 0.20 & 0.00 & 0.80 & 0.80 & 0.08 & 0.00 & 0.03 & 0.03 & 0.08 & 0.00 & 0.32 & 0.32 \end{bmatrix} \end{aligned}$$

and the observability matrix, $\mathcal{O} = \text{diag}\{\mathcal{O}_2^1, \mathcal{O}_2^2\}$ where,

$$\mathcal{O}_2^1 = \begin{bmatrix} 1.0000 & -2.0000 \\ 0.4000 & -0.8000 \\ 0.2000 & -0.4000 \\ 0.1600 & -0.3200 \\ 0.0150 & -0.0300 \\ 0.0800 & -0.1600 \\ 0.0800 & -0.1600 \end{bmatrix}, \quad \text{and } \mathcal{O}_2^2 = \begin{bmatrix} 0 & 1.0000 \\ -0.0500 & 0.1000 \\ -1.0000 & 0.9000 \\ -0.0200 & 0.0400 \\ -0.0850 & 0.0725 \\ -0.0100 & 0.0200 \\ -0.6000 & 0.4600 \end{bmatrix}$$

Since $\text{rank}(\mathcal{C}_3^1) = 2$ but $\text{rank}(\mathcal{C}_3^2) = 1$, the system is unreachable. Also, since rank $(\mathcal{O}_2^1) = 1$ but $\text{rank}(\mathcal{O}_2^2) = 2$, the system is unobservable. In addition, the state x_1 is called a reachable-unobservable mode; on the other hand the state x_2 is called an unreachable-observable mode. $\diamondsuit$

Remark 24. If n is arbitrary, we shall write $\mathcal{O} := \mathrm{diag}\{\mathcal{O}^1, \ldots, \mathcal{O}^d\}$ rather than $\mathcal{O}_n$ for some fixed n, and $\mathcal{O}$ is called the *observability matrix.* For each $k \in \mathcal{I}_d$, $\mathcal{O}^k$ is an infinite column matrix corresponding to an infinite column output vector $\{y(w, \lambda)\}$ but with finite support. Thus the observability means that this infinite matrix $\mathcal{O}$ has full rank. ▲

Remark 25. Suppose we start with a collection of the NCGR system matrices $\{A, B, C, D\}$ and embed it into the NCFM model with system matrices $\{\underline{A}_1, \ldots, \underline{A}_d, \underline{B}_1, \ldots, \underline{A}_d, C, D\}$ where we set $\underline{A}_k = P_k A^{GR}$, $\underline{B}_k = P_k B^{GR}$ for $k = 1, \ldots, d$, then the observability and controllability matrices in this case are given respectively by

$$\underline{\mathcal{O}} = \begin{bmatrix} C \\ C\underline{A}_1 \\ \vdots \\ C\underline{A}_d \\ C\underline{A}_1\underline{A}_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} C_1 & \cdots & C_d \\ C_1A_{1,1} & \cdots & C_1A_{1,d} \\ \vdots & & \vdots \\ C_dA_{d,1} & \cdots & C_dA_{d,d} \\ C_1A_{1,1}^2 & \cdots & C_1A_{1,1}A_{1,d} \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{O}^1 & \cdots & \mathcal{O}^d \end{bmatrix}. \tag{7.19}$$

and

$$\begin{aligned} \underline{\mathcal{C}} &= \begin{bmatrix} \underline{B}_1 & \cdots & \underline{B}_d & \underline{A}_1\underline{B}_1 & \cdots & \underline{A}_1\underline{B}_d & \underline{A}_2\underline{B}_1 & \cdots & \underline{A}_2\underline{B}_d & \cdots \end{bmatrix} \\ &= \begin{bmatrix} B_1 & \cdots & 0 & A_{1,1}B_1 & \cdots & A_{1,d}B_d & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \\ 0 & \cdots & B_d & 0 & \cdots & 0 & \cdots & A_{d,1}B_1 & \cdots & A_{d,d}B_d & \cdots \end{bmatrix} \end{aligned} \tag{7.20}$$

where the matrices with underline can be viewed as the original NCFM system matrices or as the NCGR system matrices embedded into the NCGR model. Thus, to distinguish between $\mathcal{O}$ (resp., $\mathcal{C}$) and $\underline{\mathcal{O}}$ (resp., $\underline{\mathcal{C}}$) we shall call $\underline{\mathcal{O}}$ (resp. $\underline{\mathcal{C}}$) as the NCFM-observability matrix (resp. the NCFM-controllability matrix).

When one views the NCGR system as an NCFM system, then the NCFM-observability means that the NCFM-observability matrix

$$\underline{\mathcal{O}} = \begin{bmatrix} \mathcal{O}^1 & \cdots & \mathcal{O}^d \end{bmatrix}$$

be injective. This implies that each $\mathcal{O}^k$ for $k = 1, \ldots, d$ is also injective. Conversely, if each $\mathcal{O}^k$ is injective individually, it does not guarantee that $\underline{\mathcal{O}}$ be injective. Therefore, for a NCGR system, the NCFM-observability implies the NCGR observability but in general not conversely. To demonstrate this fact, let us consider Example 15 for a moment. It is clear that

$$\mathrm{rank}(\underline{\mathcal{O}}_2) = \mathrm{rank}\left(\begin{bmatrix} \mathcal{O}_2^1 & \mathcal{O}_2^2 \end{bmatrix}\right) = 2;$$

while $\operatorname{rank}(\mathcal{O}_2^2) = 1$. Evidently, the NCGR system is unobservable even though $\underline{\mathcal{O}}_2$ is full rank.

Now let us consider the NCFM-controllability matrix $\underline{\mathcal{C}}$ given in (7.20). Since the rank condition of matrix does not change when columns are interchanged, one can observe that

$$\operatorname{rank}(\underline{\mathcal{C}}) = \operatorname{rank}\left(\begin{bmatrix} \mathcal{C}^1 & & \\ & \ddots & \\ & & \mathcal{C}^d \end{bmatrix}\right). \tag{7.21}$$

We also note that the NCGR system is NCGR-controllable if $\mathcal{C}^k$ is surjective for each k. If we view the NCGR system Σ^{GR} as an NCFM system, then the system Σ^{GR} is NCGR-controllable is equivalent to Σ^{GR} being NCFM-controllable. ▲

Since the NCFM-observability and NCGR-observability matrices are different, we shall establish the Similarity theory for each case separately.

Theorem 7.8 (Similarity Theorem for NCFM Model). *Given two* NCFM*-realizations*

$$\{\underline{A}_1, \ldots, \underline{A}_d, \underline{B}_1, \ldots, \underline{B}_d, C, D\} \quad \text{and} \quad \{\underline{\widetilde{A}}_1, \ldots, \underline{\widetilde{A}}_d, \underline{\widetilde{B}}_1, \ldots, \underline{\widetilde{B}}_d, \widetilde{C}, \widetilde{D}\}.$$

If both realizations are similar to each other, then they have an identical transfer function. Conversely, if both realizations are NCFM*-controllable and* NCFM*-observable, and such that*

$$D + C\left(I - \sum_{k=1}^{d} z_k \underline{A}_k\right)^{-1} \sum_{k=1}^{d} z_k \underline{B}_k = \widetilde{D} + \widetilde{C}\left(I - \sum_{k=1}^{d} z_k \underline{\widetilde{A}}_k\right)^{-1} \sum_{k=1}^{d} z_k \underline{\widetilde{B}}_k,$$

then they both are similar to each other.

Proof. The proof of the first part is provided in Lemma 7.1. Now suppose that two realizations are given such that they both have an identical transfer function, i.e.,

$$D + C\left(I - \sum_{k=1}^{d} z_k \underline{A}_k\right)^{-1} \sum_{k=1}^{d} z_k \underline{B}_k = \widetilde{D} + \widetilde{C}\left(I - \sum k = 1^d z_k \underline{\widetilde{A}}_k\right)^{-1} \sum_{k=1}^{d} z_k \underline{\widetilde{B}}_k$$

which can also be expressed as

$$D + \sum_{w \in \mathcal{F}_d} \sum_{k=1}^{d} \left(C \underline{A}^w \underline{B}_k\right) z^{w g_k} = \widetilde{D} + \sum_{w \in \mathcal{F}_d} \sum_{k=1}^{d} \left(\widetilde{C} \underline{\widetilde{A}}^w \underline{\widetilde{B}}_k\right) z^{w g_k}.$$

By equating coefficients of $z^{g_k w}$-terms of the above expression, we have the following:

$$D = \widetilde{D}, \text{ and } C \underline{A}^w \underline{B}_k = \widetilde{C} \underline{\widetilde{A}}^w \underline{\widetilde{B}}_k \text{ for all } w \in \mathcal{F}_d \text{ and } k = 1, \ldots, d. \tag{7.22}$$

Define the similarity transformation operator $S \in \mathcal{S}$ by

$$S : \underline{A}^w \underline{B}_k u(\lambda, wg_k) \mapsto \widetilde{\underline{A}}^w \widetilde{\underline{B}}_k u(\lambda, wg_k).$$

We have to verify that S is well-defined, one-to-one and onto.

Well-defined: Let $\xi = \operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [\underline{A}^w \underline{B}_k] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)] = 0$. Then,

$$\begin{aligned} 0 = \operatorname{col}_{w\in\mathcal{F}_d} [C\underline{A}^w]\, \xi &= \operatorname{col}_{w\in\mathcal{F}_d} [C\underline{A}^w] \cdot \operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [\underline{A}^w \underline{B}_k] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)] \\ &= \operatorname{col}_{w\in\mathcal{F}_d} \left[\widetilde{C}\widetilde{\underline{A}}^w\right] \cdot \operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} \left[\widetilde{\underline{A}}^w \widetilde{\underline{B}}_k\right] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)]. \end{aligned} \tag{7.23}$$

But the system is NCFM-observable; i.e., $\operatorname{col}_{w\in\mathcal{F}_d} \left[\widetilde{C}\widetilde{\underline{A}}^w\right]$ is of full rank. Hence, (7.23) implies that

$$\operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} \left[\widetilde{\underline{A}}^w \widetilde{\underline{B}}_k\right] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)] = S\xi = 0.$$

Onto: We have to show that for given $\widetilde{\xi} \in \widetilde{\mathcal{H}}$, there exists $\xi \in \mathcal{H}$ such that $S\xi = \widetilde{\xi}$. Since the system is NCFM-controllable, there does exist a control sequence $\operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)]$ so that

$$\widetilde{\xi} = \operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} \left[\widetilde{\underline{A}}^w \widetilde{\underline{B}}_k\right] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)].$$

Now set $\xi = \operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [\underline{A}^w \underline{B}_k] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)]$. Then by definition, $S\xi = \widetilde{\xi}$.

One-to-one: Suppose $\xi = \operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [\underline{A}^w \underline{B}_k] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)]$ such that $S\xi = 0$, i.e.

$$\operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} \left[\widetilde{\underline{A}}^w \widetilde{\underline{B}}_k\right] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)] = 0.$$

Then $\operatorname{col}_{w\in\mathcal{F}_d} \left[\widetilde{C}\widetilde{\underline{A}}^w\right] \cdot \operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} \left[\widetilde{\underline{A}}^w \widetilde{\underline{B}}_k\right] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)] = 0$. The conditions (7.22) imply that

$$\operatorname{col}_{w\in\mathcal{F}_d} [C\underline{A}^w] \cdot \operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [\underline{A}^w \underline{B}_k] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)] = 0. \tag{7.24}$$

But the system is NCFM-observable; i.e., $\operatorname{col}_{w\in\mathcal{F}_d} [C\underline{A}^w]$ is of full rank. Hence, (7.24) implies that

$$\operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [\underline{A}^w \underline{B}_k] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, wg_k)] = \xi = 0$$

Thus, S is well-defined, one-to-one and onto.

For any control sequence $\{u(\lambda, wg_k)\} \in \mathcal{U}$, we define ξ as

$$\xi = \operatorname{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\ldots,d}} [\underline{A}^w \underline{B}_k] \cdot \operatorname{col}_{\substack{w\in\mathcal{F}_d \\ k=1,\ldots,d}} [u(\lambda, wg_k)]$$

which we can rewrite as

$$\xi = \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \underline{A}^w \underline{B}_k u(\lambda, wg_k).$$

Thus, for such ξ,

$$\begin{aligned}
S\underline{A}_j\xi &= S\underline{A}_j \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \underline{A}^w \underline{B}_k u(\lambda, wg_k) \\
&= S \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \underline{A}^{g_j w} \underline{B}_k u(\lambda, wg_k) \\
&= \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \widetilde{\underline{A}}^{g_j w} \widetilde{\underline{B}}_k u(\lambda, wg_k) \\
&= \widetilde{\underline{A}}_j S \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \underline{A}^w \underline{B}_k u(\lambda, wg_k) = \widetilde{\underline{A}}_j S\xi
\end{aligned}$$

Since ξ is arbitrary, one can deduce that $S\underline{A}_j = \widetilde{\underline{A}}_j S$ or $\widetilde{\underline{A}}_j = S\underline{A}_j S^{-1}$ for $j = 1, \ldots, d$.

From the preceding result, we also have

$$S \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \underline{A}^w \underline{B}_k u(\lambda, wg_k) = \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \widetilde{\underline{A}}^w S\underline{B}_k u(\lambda, wg_k).$$

On the other hand, by the definition of S,

$$S \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \underline{A}^w \underline{B}_k u(\lambda, wg_k) = \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \widetilde{\underline{A}}^w \widetilde{\underline{B}}_k u(\lambda, wg_k).$$

This implies that $\widetilde{\underline{B}}_k = S\underline{B}_k$ for $k = 1, \ldots, d$.

Now for each $\xi \in \mathcal{H}$,

$$\begin{aligned}
\widetilde{C}S\xi &= \widetilde{C}S \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \underline{A}^w \underline{B}_k u(\lambda, wg_k) = \widetilde{C} \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \widetilde{\underline{A}}^w \widetilde{\underline{B}}_k u(\lambda, wg_k) \\
&= C \sum_{w\in\mathcal{F}_d} \sum_{k=1}^{d} \underline{A}^w \underline{B}_k u(\lambda, wg_k) = C\xi
\end{aligned}$$

Hence, $\widetilde{C} = CS^{-1}$.

It follows from these arguments that such an S is an admissible similarity transformation and this completes the proof. ■

Theorem 7.9 (Similarity Theorem for NCGR Model). *Given two* NCGR*-realizations* $\{A, B, C, D\}$ *and* $\{\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\}$*. If both realizations are similar to each other, then they have an identical transfer function. Conversely, if both realizations are* NCGR*-controllable and* NCGR*-observable, and such that*

$$D + C\left(I - Z_d(z)A\right)^{-1} Z_d(z)B = \widetilde{D} + \widetilde{C}\left(I - Z_d(z)\widetilde{A}\right)^{-1} Z_d(z)\widetilde{B},$$

then they both are similar to each other.

Proof. The proof of the first part is provided in Lemma 7.1. Now suppose that two realizations are given and such that they both have an identical transfer function, i.e.,

$$D + C\left(I - Z_d(z)A\right)^{-1} Z_d(z)B = \widetilde{D} + \widetilde{C}\left(I - Z_d(z)\widetilde{A}\right)^{-1} Z_d(z)\widetilde{B}.$$

Recall that the transfer function can be rewritten in terms of the NCFM system matrices as (see (6.57) for details)

$$T_\Sigma(z) = D + \sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\left(C\underline{A}^w\underline{B}_k\right) z^{wg_k}, \tag{7.25}$$

where $\underline{A}_k = P_kA$, and $\underline{B}_k = P_kB$ (see (6.45) on page 129 for the definition of the orthogonal projection, P_k).

Likewise, by assumption,

$$T_\Sigma(z) = \widetilde{D} + \sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\left(\widetilde{C}\widetilde{\underline{A}}^w\widetilde{\underline{B}}_k\right) z^{wg_k}. \tag{7.26}$$

By equating coefficients of z^{wg_k}-terms of (7.25) and (7.26), we have the following:

$$D = \widetilde{D}, \text{ and } C\underline{A}^w\underline{B}_k = \widetilde{C}\widetilde{\underline{A}}^w\widetilde{\underline{B}}_k \text{ for all } w\in\mathcal{F}_d \text{ and } k = 1,\ldots,d, \tag{7.27}$$

which is identical to

$$D = \widetilde{D}, \text{ and } C_{i_n}A_{i_n,i_{n-1}}\cdots A_{i_1,k}B_k = \widetilde{C}_{i_n}\widetilde{A}_{i_n,i_{n-1}}\cdots\widetilde{A}_{i_1,k}\widetilde{B}_k \text{ for all } i_j, k\in\mathcal{I}_d, \tag{7.28}$$

if $w = g_{i_n}g_{i_{n-1}}\cdots g_{i_1}$. Define the similarity transformation operator $S_{i_n}\in\mathcal{S}$ by

$$S_{i_n} : A_{i_n,i_{n-1}}\cdots A_{i_1,k}B_ku(\lambda, wg_k) \mapsto \widetilde{A}_{i_n,i_{n-1}}\cdots\widetilde{A}_{i_1,k}\widetilde{B}_ku(\lambda, wg_k)$$

We have to verify that S_{i_n} is well-defined, onto and one-to-one. First let us recall that the controllability matrix with respect to a letter g_k is (see page 144)

$$\mathcal{C}^k = \begin{bmatrix} B_k & A_{k,1}B_1 & \cdots & A_{k,d}B_d & A_{k,1}A_{1,1}B_1 & \cdots & A_{k,d}A_{d,d}B_d & \cdots \end{bmatrix},$$

and the observability matrix with respect to a letter g_{i_n} is (see page 149)

$$\mathcal{O}^{i_n} = \begin{bmatrix} C_{i_n} \\ C_1 A_{1,i_n} \\ \vdots \\ C_d A_{d,i_n} \\ C_1 A_{1,1} A_{1,i_n} \\ \vdots \end{bmatrix}$$

Well-defined: Let $u := \operatorname{col}_{\substack{w \in \mathcal{F}_d \\ k=1,\ldots,d}} [u(\lambda, wg_k)]$ and define $\xi := \mathcal{C}^k u = 0$. Then, it follows that

$$0 = \mathcal{O}^{i_n} \cdot \xi = \mathcal{O}^{i_n} \cdot \mathcal{C}^k u = \widetilde{\mathcal{O}}^{i_n} \cdot \widetilde{\mathcal{C}}^k u, \tag{7.29}$$

where each block-matrix entry in $\widetilde{\mathcal{O}}^{i_n}$ is the same as that in $\mathcal{O}^{i_n}$ with $\widetilde{\ }$ (similarly for $\widetilde{\mathcal{C}}^k$). Since the system is NCGR-observable, i.e. $\widetilde{\mathcal{O}}^{i_n}$ is full rank. Hence, (7.29) implies that $\widetilde{\mathcal{C}}^k u = S_{i_n}\xi = 0$.

Onto: Since the system is NCGR-controllable, there exists a control sequence $\{u(\lambda, wg_k)\}$ so that $\widetilde{\xi} = \widetilde{\mathcal{C}}^k u$. Set $\xi = \mathcal{C}^k u$ and hence by definition of S_{i_n}, we have $S_{i_n}\xi = \widetilde{\xi}$.

One-to-one: Suppose $\xi = \mathcal{C}^k u$ such that $S_{i_n}\xi = \widetilde{\mathcal{C}}^k u = 0$. Then, $\widetilde{\mathcal{O}}^{i_n} \cdot \widetilde{\mathcal{C}}^k u = 0$. The conditions (7.28) implies that $\mathcal{O}^{i_n} \cdot \mathcal{C}^k u = 0$. It follows that $\mathcal{C}^k u = \xi = 0$ since $\mathcal{O}^{i_n}$ is full rank.

Thus, for each $i_n = 1, \ldots, d$, S_{i_n} is well-defined, onto and one-to-one. Then the admissible transformation matrix S is defined as a diagonal block-matrix

$$S = \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_d \end{bmatrix}$$

From now, it is more convenient to view the NCGR system matrices in terms of NCFM system matrices (i.e., $\underline{A}_j = P_j A$ and $\underline{B}_j = P_j B$). For any control sequence $\{u(\lambda, wg_k)\} \in \mathcal{U}$, we define ξ as

$$\xi = \sum_{w \in \mathcal{F}_d} \sum_{k=1}^{d} \underline{A}^w \underline{B}_k u(\lambda, wg_k).$$

Thus, for such ξ,

$$\begin{aligned} SA\xi = SA\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\underline{A}^w\underline{B}_k u(\lambda,wg_k) &= S\left(\sum_{j=1}^{d}\underline{A}_j\right)\left(\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\underline{A}^w\underline{B}_k u(\lambda,wg_k)\right) \\ &= S\sum_{j=1}^{d}\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\underline{A}^{g_jw}\underline{B}_k u(\lambda,wg_k) \\ &= \sum_{j=1}^{d}\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\widetilde{\underline{A}}^{g_jw}\widetilde{\underline{B}}_k u(\lambda,wg_k) \\ &= \left(\sum_{j=1}^{d}\widetilde{\underline{A}}_j\right)S\left(\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\underline{A}^w\underline{B}_k u(\lambda,wg_k)\right) \\ &= \widetilde{A}S\left(\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\underline{A}^w\underline{B}_k u(\lambda,wg_k)\right) = \widetilde{A}S\xi \end{aligned}$$

Hence, $SA = \widetilde{A}S$ or $\widetilde{A} = SAS^{-1}$.

From the preceding result, we also have

$$S\left(\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\underline{A}^w\underline{B}_k u(\lambda,wg_k)\right) = \sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\widetilde{\underline{A}}^w S\underline{B}_k u(\lambda,wg_k).$$

On the other hand, by the definition of S,

$$S\left(\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\underline{A}^w\underline{B}_k u(\lambda,wg_k)\right) = \sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\widetilde{\underline{A}}^w\widetilde{\underline{B}}_k u(\lambda,wg_k).$$

This implies that $\widetilde{B} = SB$.

Now for each $\xi\in\mathcal{H}$,

$$\begin{aligned} \widetilde{C}S\xi = \widetilde{C}S\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\underline{A}^w\underline{B}_k u(\lambda,wg_k) &= \widetilde{C}\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\widetilde{\underline{A}}^w\widetilde{\underline{B}}_k u(\lambda,wg_k) \\ = C\sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\underline{A}^w\underline{B}_k u(\lambda,wg_k) = C\xi & \end{aligned}$$

Hence, $\widetilde{C} = CS^{-1}$.

It follows from these arguments that such an S is an admissible similarity transformation and this completes the proof. ■

7.4 Conclusion

This Chapter establishes the notions of similarity, reachability, controllability, and observability via the system equations of the NCGR model; while these notions appeared in [BD99] were defined based on the LFT realization of linear systems with structured uncertainty. Evidently, our results presented here give precise state-space interpretation of results of [BD99] and also justify their terminology.

Chapter 8

Minimal Realization and Stability

This Chapter concerns with the so-called *minimal realization problem* and the stability issue for the i/s/o d-D linear systems where the "time-axis" is a free semigroup[1]. Suppose we are given system matrices $\{A, B, C, D\}$, one can easily compute the transfer function of the system in the form of a formal power series $T_\Sigma(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$, where the coefficients T_v are uniquely determined by system matrices. Conversely, given such a transfer function, the realization problem is to find system matrices $\{A, B, C, D\}$ defining the linear system equations described by the NCFM model, or the NCGR model. If we require in addition that the size of the system matrix A has to be the smallest one among other realizations, this problem is then called the *minimal realization problem*. This issue will be discussed in Section 8.1. A good reference on realization theory for the classical 1D discrete-time linear system using the operator theoretical approach is [FFGK98].

Section 8.2 is devoted to the stability issue which is a crucial property of the control system. We introduce the notions of finite ℓ^2-gain system, exponential stability and asymptotic stability, and also establish the so-called *Lyapunov theory* for noncommutative linear systems. The reader will see that the results here are analogous to those in the classical case.

8.1 Minimal Realization Theory

Realization theory in the classical control literature has been studied for over four decades. It provides a connection between linear system equations and the corresponding input/output map (or transfer function). From the practical point of view, the output sequence can be measured whenever the known input sequence is fed through the system, and hence the input/output map can be obtained easily via the experimental data. Our concern here is once we know the input/output map, how can we find the set of system matrices $\{A, B, C, D\}$ defining the linear

[1] one can also view the time axis as a homogeneous tree of order d with a root.

system equations with the smallest dimension of the corresponding state-space which realizes the given input/output map? This is a well-known problem in the classical case and we often refer to it as the *minimal realization problem*[2]. In fact, a system realization of the classical discrete-time linear system is minimal if and only if such a system is controllable and observable.

After the two-dimensional linear models were introduced in the seventies, the theory of multidimensional (d-D) linear systems has been developed based on the classical one, and this includes the realization theory as well. C. Beck [Bec01] established a connection between realization theory results for formal power series and the concept of minimality, which is developed for systems with uncertainty operator Δ represented by the so-called *Linear Fractional Transformation*. This can also be formulated in terms of linear system equations of the i/s/o linear systems with evolution along the elements of a free semigroup $\mathcal{F}_d$; however, as in [BD99], Beck did not provide the state-space interpretation in her work.

Rather, we here establish the realization theory systematically based on the linear system models described in Chapter 6. We first formulate the system equations in such a way that its transfer function has the form of the recognizable series. Then by applying the results of M. Schützenberger [Sch61] and M. Fliess [Fli74], one can conclude that the Hankel matrix $\mathbb{H}_{TR}$ associated with such systems has finite rank. Therefore, for any linear systems whose transfer function can be expressed as a formal power series, if the associated Hankel matrix has finite rank, then such systems are recognizable, and hence one can solve the minimal realization problem.

From this fact, we shall begin this Section with the formulation of a linear system model which we shall refer to it as the *recognizable system* in Subsection 8.1.1. In Subsection 8.1.2, we formulate the Hankel matrix $\mathbb{H}_{TFM}$ associated with the system described by the NCFM model, and show that one can identify $\mathbb{H}_{TFM}$ with the Hankel matrix $\mathbb{H}_{TR}$ of the recognizable system. We then establish the minimality condition(s) for NCFM model in connection with the rank of the Hankel matrix $\mathbb{H}_{TFM}$. The method discussed in Subsection 8.1.2 (i.e., construction of a shift realization by use of the Hankel matrix) can be modified to produce a minimal NCGR realization of a given transfer function; we shall give a brief discussion on this issue in Subsection 8.1.3.

8.1.1 Recognizable System, Σ^R

Let us consider the system described by the following equations:

$$\Sigma^R = \begin{cases} x(g_k w, \lambda) &= F_k x(w,\lambda) + G u(g_k w, \lambda), \quad x(\lambda,\lambda) = G u(\lambda,\lambda) \\ y(w,\lambda) &= H x(w,\lambda). \end{cases} \tag{8.1}$$

[2] In practice, one may be able to measure only finitely many of the experimental data associated with the input/output map. The *partial realization problem* is to construct a system realization $\{A, B, C, D\}$ which is consistent with such partial information. We do not discuss this type of problem here.

Application of the Z-transform (see Section 6.3 for the definition of the Z-transform) to the state equation in (8.1) yields,

$$\sum_{(w,\lambda)\in\mathcal{T}_f} x(g_k w,\lambda)z^w = \sum_{(w,\lambda)\in\mathcal{T}_f} F_k x(w,\lambda)z^w + \sum_{(w,\lambda)\in\mathcal{T}_f} G w(g_k w,\lambda)z^w,$$

where $\mathcal{T}_f = (\mathcal{F}_d \times \{\lambda\})$. Multiplying both sides of the above equation by z_k and sum over $k = 1,\dots,d$ to get

$$\sum_{k=1}^{d}\sum_{(w,\lambda)\in\mathcal{T}_f} x(g_k w,\lambda)z^{g_k w} = \sum_{k=1}^{d}\sum_{(w,\lambda)\in\mathcal{T}_f} z_k F_k x(w,\lambda)z^w + \sum_{k=1}^{d}\sum_{(w,\lambda)\in\mathcal{T}_f} G w(g_k w,\lambda)z^{g_k w},$$

which is equivalent to

$$x^{\wedge\mathcal{T}_f}(z,0) - x(\lambda,\lambda) = \left(\sum_{k=1}^{\infty} z_k F_k\right) x^{\wedge\mathcal{T}_f}(z,0) + G u^{\wedge\mathcal{T}_f}(z,0) - G u(\lambda,\lambda),$$

Recall that $x(\lambda,\lambda) = Gu(\lambda,\lambda)$, and hence we have

$$x^{\wedge\mathcal{T}_f}(z,0) = \left(I - \sum_{k=1}^{\infty} z_k F_k\right)^{-1} G u^{\wedge\mathcal{T}_f}(z,0),$$

and the output equation becomes

$$y^{\wedge\mathcal{T}_f}(z,0) = H\left(I - \sum_{k=1}^{\infty} z_k F_k\right)^{-1} G \cdot u^{\wedge\mathcal{T}_f}(z,0) \triangleq T_{\Sigma^R}(z)\cdot u^{\wedge\mathcal{T}_f}(z,0). \tag{8.2}$$

Note that

$$\begin{aligned} T_{\Sigma^R}(z) &= H\left(I - \sum_{k=1}^{d} F_k z_k\right)^{-1} G = H\sum_{j=0}^{\infty}\left(\sum_{k=1}^{d} F_k z_k\right)^{j} G \\ &= \sum_{v\in\mathcal{F}_d} HF^v G z^v \triangleq \sum_{v\in\mathcal{F}_d} T_v^R z^v. \end{aligned} \tag{8.3}$$

Evidently, this is of the form of the recognizable series (see Definition 26 on page 95). Thus, we shall call $T_{\Sigma^R}(z)$ a transfer function of the *recognizable system* Σ^R which is described by the linear equations (8.1). Note that the set of system matrices $\{H, F_1,\dots,F_d, G\}$ is said to be a recognizable-system realization of $T_{\Sigma^R}(z)$ if $T_v^R = HF^v G$ for all $v \in \mathcal{F}_d$, where $F^v = F_{i_n}F_{i_{n-1}}\cdots F_{i_1}$ if $v = g_{i_n}g_{i_{n-1}}\cdots g_{i_1}$.

Suppose now that we are given a transfer function $T_{\Sigma^R}(z)$, then the Hankel operator associated with $T_{\Sigma^R}(z)$ is represented by the matrix with rows indexed by $v \in \mathcal{F}_d$ and columns indexed by $w \in \mathcal{F}_d$ as

$$\mathbb{H}_{T^R} = \begin{bmatrix} HG & HF_1G & \cdots & HF_dG & HF_1^2G & HF_1F_2G & \cdots \\ HF_1G & HF_1^2G & \cdots & HF_1F_dG & HF_1^3G & HF_1^2F_2G & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ HF_dG & HF_dF_1G & \cdots & HF_d^2G & HF_dF_1^2G & HF_dF_1F_2G & \cdots \\ HF_1^2G & HF_1^3G & \cdots & HF_1^2F_dG & HF_1^4G & HF_1^3F_2G & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \end{bmatrix} \tag{8.4}$$

where the matrix entries of $\mathbb{H}_{T^R}$ are completely determined from the coefficients T_v^R $(v \in \mathcal{F}_d)$ as

$$[\mathbb{H}_{T^R}]_{v,w} = T_{vw}^R = HF^{vw}G.$$

By Theorem 5.1 and Theorem 5.2, the rank of this Hankel matrix is always finite, say $\operatorname{rank}(\mathbb{H}_{T^R}) = n < \infty$, since it is constructed from the transfer function of the recognizable system. Conversely, if $\operatorname{rank}(\mathbb{H}_{T^R}) = n < \infty$, then Theorem 5.2 implies that $T_{\Sigma^R}(z)$ is rational, and hence it is recognizable by Theorem 5.1. In other words, there exists a minimal realization $\{H, F_1, \ldots, F_d, G\}$ with the size of $F_i = n$.

Now one could observe from (8.4) that in fact the Hankel matrix $\mathbb{H}_{T^R}$ can be easily factored as

$$\mathbb{H}_{T^R} = [HF^v][F^wG] \triangleq \mathcal{O}^R \cdot \mathcal{C}^R,$$

where $\mathcal{O}^R$ and $\mathcal{C}^R$ are the observability and the controllability matrices of the recognizable system Σ^R. Then the following Theorem shows that the recognizable system constructed from the minimal realization discussed above is controllable and observable.

Theorem 8.1. *A recognizable-system realization $\{H, F_1, \ldots, F_d, G\}$ is minimal if and only if the corresponding recognizable system is both controllable and observable.*

Proof. A necessary part is quite straightforward. We need to show that, if the system is uncontrollable and/or unobservable, then a system realization $\{H, F_1, \ldots, F_d, G\}$ is not minimal.

Suppose first that the system is uncontrollable. Then one can decompose the state space $\mathcal{H}$ into a controllable subspace X^C and an uncontrollable subspace $\mathcal{H} \setminus X^C$. In other words, there exists an admissible transformation $S \in \mathcal{S}$ such that

$$SF_jS^{-1} = \begin{bmatrix} \widetilde{F}_j & * \\ 0 & * \end{bmatrix}, \quad SG = \begin{bmatrix} \widetilde{G} \\ 0 \end{bmatrix}, \tag{8.5}$$

where $\widetilde{F}_j \in \mathcal{L}(X^C, \bigoplus_1^d X^C)$ and $\widetilde{G} \in \mathcal{L}(\mathcal{U}, \bigoplus_1^d X^C)$.

It follows that the compression of the system realization to the controllable subspace

$$\{H, F_1, \dots, F_d, G\}|_{X^C} = \{\widetilde{H}, \widetilde{F}_1, \dots \widetilde{F}_d, \widetilde{G}\}$$

has the same transfer function as $\{H, F_1, \dots, F_d, G\}$ since

$$\begin{aligned} T_v^S = HF^vG &= HF_{i_n} \cdots F_{i_1} G \\ &= HS^{-1} \begin{bmatrix} \widetilde{F}_{i_n} & * \\ 0 & * \end{bmatrix} \cdots \begin{bmatrix} \widetilde{F}_{i_1} & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \widetilde{G} \\ 0 \end{bmatrix} \\ &= \widetilde{H}\widetilde{F}_{i_n} \cdots \widetilde{F}_{i_1}\widetilde{G} = \widetilde{H}\widetilde{F}^v\widetilde{G} = \widetilde{T}_v^S, \quad \text{where } HS^{-1} = \begin{bmatrix} \widetilde{H} & * \end{bmatrix}. \end{aligned}$$

Therefore, the system realization is not minimal since it is obvious from (8.5) that the size of F_j is greater than or equal to the size of $\widetilde{F}_j$.

By using the similar argument as shown above, one should be able to show that, if the system is unobservable, then its realization is not minimal. Thus it follows immediately that if the system realization is minimal, then the system is controllable and observable.

For sufficiency, let us assume that the system is both controllable and observable but its realization $\{H, F_1, \dots, F_d, G\}$ is not minimal. More precisely, suppose that $\{H, F_1, \dots, F_d, G\}$ is a realization of the controllable and observable system with state-space dimension n. Since the system is assumed not to be minimal, there is another realization, say $\{H', F'_1, \dots, F'_d, G'\}$, of smaller state-space dimension, say $n' < n$. Then there exists an admissible transformation $S \in \mathcal{S}$ such that $\{H'S^{-1}, SF'_1S^{-1}, \dots, SF'_dS^{-1}, SG\}$ has a Kalman-like decomposition structure (i.e., decomposed into a controllable/observable-uncontrollable/unobservable structure), and the corresponding state space is also decomposed into controllable/observable $X^C \cup X^O$ and uncontrollable/unobservable $\mathcal{H} \setminus (X^R \cup X^O)$ subspaces.

Pick states in the controllable/observable subspace of dimension, say $\tilde{n} \leq n'$, and form a new system where the compression of its realization to the controllable/observable subspace $\{H'S^{-1}, SF'_1S^{-1}, \dots, SF'_dS^{-1}, SG\}|_{X^C \cup X^O}$ has the same transfer function as $\{H, F_1, \dots, F_d, G\}$. It follows that two realizations: $\{H, F_1, \dots, F_d, G\}$ and $\{H'S^{-1}, SF'_1S^{-1}, \dots, SF'_dS^{-1}, SG\}$, are similar to each other (by Similarity Theorem 7.8) and hence, it implies that the dimension of the state-space must be preserved, i.e. $n = \tilde{n}$. On the other hand, we have $\tilde{n} \leq n' < n$. Therefore, this implies that $n < n$ which leads to contradiction. Thus, if the system is controllable and observable, then its realization is minimal. This completes the proof. ■

Remark 26. In [GKL88], the authors formulated an abstract notion of a *node* and examined the issues of controllability, observability, minimality, and state-space similarity in this general axiomatic setting. ▲

8.1.2 Minimal Realization for NCFM Models, Σ^{FM}

We begin this Subsection with a brief review on a linear system described by the NCFM model, and then formulate the Hankel operator $\mathbb{H}_{\mathcal{T}^{FM}}$ which is a linear map from input sequence in the past to the output sequence in the future. We identify this Hankel operator $\mathbb{H}_{\mathcal{T}^{FM}}$ with the Hankel operator associated with the recognizable system $\mathbb{H}_{\mathcal{T}^R}$ as shown in the previous Subsection. We finally develop the minimality theorem for the NCFM model in connection with the rank condition of $\mathbb{H}_{\mathcal{T}^{FM}}$.

Let us first recall that the Future-time i/s/o linear system described by the NCFM model is

$$\Sigma^{FM} = \begin{cases} x(g_1 w, \lambda) & = \underline{A}_1 x(w,\lambda) + \underline{B}_1 u(w,\lambda) \\ \quad \vdots & \qquad \vdots \\ x(g_d w, \lambda) & = \underline{A}_d x(w,\lambda) + \underline{B}_d u(w,\lambda) \\ y(w,\lambda) & = Cx(w,\lambda) + Du(w,\lambda), \end{cases} \tag{8.6}$$

with initial condition

$$x(\lambda,\lambda) = \sum_{v:|v|=n} \underline{A}^v x(\lambda, v) + \sum_{k=1}^{d} \sum_{v:|v|<n} \underline{A}^v \underline{B}_k u(\lambda, v g_k),$$

where $\underline{A}^v = \underline{A}_{i_n} \underline{A}_{i_{n-1}} \cdots \underline{A}_{i_1}$ if $v = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}$.

Now for $k = 1, \ldots, d$, multiplication of the state equations in (8.6) by $z_k z^w$ yields,

$$x(g_k w, \lambda) z^{g_k w} = (z_k \underline{A}_k) x(w,\lambda) z^w + (z_k \underline{B}_k) u(w,\lambda) z^w.$$

Summing these over $k = 1, \ldots, d$ and all words $(w,\lambda) \in \mathcal{T}_f$ gives,

$$x^{\wedge \mathcal{T}_f}(z,0) - x(\lambda,\lambda) = \sum_{k=1}^{d} z_k \underline{A}_k x^{\wedge \mathcal{T}_f}(z,0) + \sum_{k=1}^{d} z_k \underline{B}_k u^{\wedge \mathcal{T}_f}(z,0),$$

and solving for $x^{\wedge \mathcal{T}_f}$,

$$x^{\wedge \mathcal{T}_f}(z,0) = \left(I - \sum_{k=1}^{d} z_k \underline{A}_k \right)^{-1} x(\lambda,\lambda) + \left(I - \sum_{k=1}^{d} z_k \underline{A}_k \right)^{-1} \sum_{k=1}^{d} z_k \underline{B}_k u^{\wedge \mathcal{T}_f}(z,0). \tag{8.7}$$

Application of the Z-transform to the output equation in (8.6) gives

$$y^{\wedge \mathcal{T}_f}(z,0) = Cx^{\wedge \mathcal{T}_f}(z,0) + Du^{\wedge \mathcal{T}_f}(z,0)$$

$$= C\left(I - \sum_{k=1}^{d} z_k \underline{A}_k\right)^{-1} x(\lambda,\lambda) + \left[C\left(I - \sum_{k=1}^{d} z_k \underline{A}_k\right)^{-1} \sum_{k=1}^{d} z_k \underline{B}_k + D\right] \cdot u^{\wedge \mathcal{T}_f}(z,0)$$

$$\triangleq C\left(I - \sum_{k=1}^{d} z_k \underline{A}_k\right)^{-1} x(\lambda,\lambda) + T_{\Sigma^{FM}}(z) \cdot u^{\wedge \mathcal{T}_f}(z,0), \tag{8.8}$$

where $T_{\Sigma^{FM}}(z)$ is called the *noncommutative Fornasini-Marchesini (*NCFM*) transfer function* for the system (8.6).

Now let $\{T_v\}_{v\in\mathcal{F}_d}$ denote a sequence of operators along a free semigroup $\mathcal{F}_d$ mapping from the input space $\mathcal{U}$ into the output space $\mathcal{Y}$ (i.e., $T_v \in \mathcal{L}(\mathcal{U},\mathcal{Y})$) defined by

$$T_\lambda = D \quad \text{and} \quad T_{wg_k} = C\underline{A}^w \underline{B}_k \quad \text{for all } w \in \mathcal{F}_d, k \in \mathcal{I}_d, \tag{8.9}$$

where $\underline{A}^w = \underline{A}_{i_n}\underline{A}_{i_{n-1}}\cdots\underline{A}_{i_1}$ if $w = g_{i_n}g_{i_{n-1}}\cdots g_{i_1}$.

Definition 40 (Realization). A set of system matrices $\{\underline{A}_1,\ldots,\underline{A}_d,\underline{B}_1,\ldots,\underline{B}_d,C,D\}$ is said to be a *realization* of the sequence $\{T_v\}_{v\in\mathcal{F}_d}$ if (8.9) holds.

Note that one could represent the NCFM transfer function $T_{\Sigma^{FM}}$ as a formal power series of the sequence $\{T_v\}_{v\in\mathcal{F}_d}$ with coefficients T_v defined in (8.9) as follows:

$$\begin{aligned} T_{\Sigma^{FM}}(z) &= C\left(I - \sum_{k=1}^{d} z_k \underline{A}_k\right)^{-1} \sum_{k=1}^{d} z_k \underline{B}_k + D \\ &= C\left[\sum_{j=0}^{\infty}\left(\sum_{k=1}^{d} z_k \underline{A}_k\right)^{j}\right] \sum_{k=1}^{d} z_k \underline{B}_k + D \\ &= \sum_{w\in\mathcal{F}_d}\sum_{k=1}^{\infty}(C\underline{A}^w\underline{B}_k) z^{wg_k} + D := \sum_{v\in\mathcal{F}_d} T_v z^v, \end{aligned} \tag{8.10}$$

Thus, for given a set of system matrices $\{\underline{A}_1,\ldots,\underline{A}_d,\underline{B}_1,\ldots,\underline{B}_d,C,D\}$, the coefficients T_v can be easily computed via the formula for the transfer function as in (8.10), and hence the set of system matrices $\{\underline{A}_1,\ldots,\underline{A}_d,\underline{B}_1,\ldots,\underline{B}_d,C,D\}$ is a realization of $T_{\Sigma^{FM}}(z)$ if and only if $D = T_\lambda$ and $C\underline{A}^w\underline{B}_k = T_{wg_k}$ for all $w \in \mathcal{F}_d, k \in \mathcal{I}_d$.

Our interest is in the reversing this computation, and in particular we want to establish conditions on T_v that guarantee existence of the corresponding linear system equations (8.6). In other words, given a formal power series $T(z) = \sum_{v\in\mathcal{F}_d} T_v z^v$, find the set of all minimal realizations $\{\underline{A}_1,\ldots,\underline{A}_d,\underline{B}_1,\ldots,\underline{B}_d,C,D\}$ satisfying $T_\lambda = D$ and $T_{wg_k} = C\underline{A}^w\underline{B}_k$ for all $w \in \mathcal{F}_d$ and $k = 1,\ldots,d$, where the notion of the minimal realization is defined as follows:

Definition 41 (Minimal Realization). The NCFM realization $\{\underline{A}_1,\ldots,\underline{A}_d,\underline{B}_1,\ldots,\underline{B}_d,C,D\}$ of a formal power series $T(z) = \sum_{v\in\mathcal{F}} T_v z^v$ is said to be *minimal* if given other realization, say

$\{\underline{A}'_1, \ldots, \underline{A}'_d, \underline{B}'_1, \ldots, \underline{B}'_d, C', D'\}$ of $T(z)$, then the size of the square matrices $\underline{A}_j$ is less than or equal to the size of $\underline{A}'_j$.

Now we are ready to establish the minimality theorem for the NCFM models which is an analogue to the recognizable case.

Theorem 8.2. *A* NCFM *realization* $\{\underline{A}_1, \ldots, \underline{A}_d, \underline{B}_1, \ldots, \underline{B}_d, C, D\}$ *is minimal if and only if the i/s/o linear system described by the* NCFM *model is both controllable and observable.*

Proof. Use the similar argument as the proof of Theorem 8.1. ■

Note that the concept of minimal realization makes sense only when (8.10) holds with the dimension of the state-space is finite, say $\dim \mathcal{H} < \infty$. Before we move on, let us consider the system equations (8.6) for a moment. It is of interest to solve recursively for the output sequence of the system (8.6), i.e.

$$\begin{bmatrix} y(\lambda,\lambda) \\ y(g_1,\lambda) \\ \vdots \\ y(g_d,\lambda) \\ y(g_1g_1,\lambda) \\ y(g_1g_2,\lambda) \\ \vdots \end{bmatrix} = \begin{bmatrix} C \\ C\underline{A}_1 \\ \vdots \\ C\underline{A}_d \\ C\underline{A}_1\underline{A}_1 \\ C\underline{A}_1\underline{A}_2 \\ \vdots \end{bmatrix} x(\lambda,\lambda) + \begin{bmatrix} D & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ C\underline{B}_1 & D & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \cdots \\ C\underline{B}_d & 0 & 0 & \cdots & D & 0 & 0 & \cdots \\ C\underline{A}_1\underline{B}_1 & C\underline{B}_1 & 0 & \cdots & 0 & D & 0 & \cdots \\ C\underline{A}_1\underline{B}_2 & 0 & C\underline{B}_1 & \cdots & 0 & 0 & D & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u(\lambda,\lambda) \\ u(g_1,\lambda) \\ \vdots \\ u(g_d,\lambda) \\ u(g_1g_1,\lambda) \\ u(g_1g_2,\lambda) \\ \vdots \end{bmatrix}$$

Apparently, the first block column matrix of the above expression is the NCFM *observability matrix* $\underline{\mathcal{O}}$, which is a linear map of the state-space $\mathcal{H}$ to the square summable sequence of the outputs, i.e.

$$\underline{\mathcal{O}} : \mathcal{H} \mapsto \ell^2((\mathcal{F}_d \times \{\lambda\}), \mathcal{Y}).$$

Thus, one can write the output sequence $\{y(w,\lambda)\}_{w\in\mathcal{F}_d}$ as

$$\mathrm{col}_{w\in\mathcal{F}_d}[y(w,\lambda)] = \underline{\mathcal{O}}x(\lambda,\lambda) + \mathbb{T}_{T^{FM}} \cdot \mathrm{col}_{w\in\mathcal{F}_d}[u(w,\lambda)], \tag{8.11}$$

where $\mathbb{T}_{T^{FM}}$ is the NCFM *Toeplitz operator*, which has a matrix representation generated by a symbol T_v $(v \in \mathcal{F}_d)$ according to the formula

$$[\mathbb{T}_{T^{FM}}]_{v,w} = \begin{cases} 0 & \text{if } |w| > |v|, \\ T_{vw^{-1}} & \text{if } |w| \le |v|, \end{cases} \tag{8.12}$$

where $[\mathbb{T}_{T^{FM}}]_{v,w}$ denotes the entry of $\mathbb{T}_{T^{FM}}$ in row v and column w. As usual, we interpret $T_{vw^{-1}}$ to be zero if vw^{-1} is undefined. Obviously, such a matrix $\mathbb{T}_{T^{FM}}$ has rows and columns indexed by $\mathcal{F}_d$.

Suppose now that the state at a time in the remote past is initialized to be zero (i.e., $x(\lambda, v) = 0$ for all $v \in \mathcal{F}_d$ such that $|v| = n$ for n sufficiently large) and the system is fed by an input sequence $\{u(w,v)\}$ supported only in the past, i.e.

$$u(w,v) = \begin{cases} u(\lambda, v) & \text{if } (w,v) \in \mathcal{T}_p, \\ 0 & \text{if } (w,v) \in \mathcal{T}_f. \end{cases} \tag{8.13}$$

Thus, the output sequence in (8.11) collapses to

$$\mathrm{col}_{w\in\mathcal{F}_d}[y(w,\lambda)] = \underline{\mathcal{O}}x(\lambda,\lambda), \text{ where } x(\lambda,\lambda) = \sum_{k=1}^{d} \sum_{v:|v|<n} \underline{A}^{v^\top} \underline{B}_k u(\lambda, vg_k).$$

Note that one can write $x(\lambda,\lambda)$ explicitly in terms of the input sequence as

$$\begin{aligned} x(\lambda,\lambda) &= \mathrm{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [\underline{A}^w \underline{B}_k] \cdot \mathrm{col}_{\substack{v\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, vg_k)] \\ &= \underline{\mathcal{C}} \cdot \mathrm{col}_{\substack{v\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, vg_k)], \end{aligned}$$

where $\underline{\mathcal{C}}$ is the NCFM *controllability matrix*, which is a linear map of the square summable sequence of the inputs to the state-space $\mathcal{H}$, i.e.

$$\underline{\mathcal{C}} : \ell^2((\{\lambda\} \times \mathcal{F}_d \setminus \{\lambda\}), \mathcal{U}) \mapsto \mathcal{H}.$$

Thus, we have

$$\mathrm{col}_{w\in\mathcal{F}_d}[y(w,\lambda)] = \underline{\mathcal{O}} \cdot \underline{\mathcal{C}} \cdot \mathrm{col}_{\substack{v\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, vg_k)] \triangleq \mathbb{H}_{TFM} \cdot \mathrm{col}_{\substack{v\in\mathcal{F}_d \\ k=1,\dots,d}} [u(\lambda, vg_k)] \tag{8.14}$$

where $\mathbb{H}_{TFM}$ is the NCFM *Hankel operator*

$$\mathbb{H}_{TFM} : \ell^2((\{\lambda\} \times \mathcal{F}_d \setminus \{\lambda\}), \mathcal{U}) \mapsto \ell^2((\mathcal{F}_d \times \{\lambda\}), \mathcal{Y}),$$

having a matrix representation with rows indexed by $v \in \mathcal{F}_d$ and columns indexed by $wg_k \in \mathcal{F}_d \cdot g_k$ where $w \in \mathcal{F}_d$ and $k \in \mathcal{I}_d$, i.e.

$$\mathbb{H}_{TFM} = \mathrm{col}_{v\in\mathcal{F}_d}[C\underline{A}^v] \cdot \mathrm{row}_{\substack{w\in\mathcal{F}_d \\ k=1,\dots,d}} [\underline{A}^w \underline{B}_k]$$

$$= \begin{bmatrix} C\underline{B}_1 & \cdots & C\underline{B}_d & C\underline{A}_1\underline{B}_1 & \cdots & C\underline{A}_1\underline{B}_d & C\underline{A}_2\underline{B}_1 & \cdots \\ C\underline{A}_1\underline{B}_1 & \cdots & C\underline{A}_1\underline{B}_d & C\underline{A}_1^2\underline{B}_1 & \cdots & C\underline{A}_1^2\underline{B}_d & C\underline{A}_1\underline{A}_2\underline{B}_1 & \cdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \cdots \\ C\underline{A}_d\underline{B}_1 & \cdots & C\underline{A}_d\underline{B}_d & C\underline{A}_d\underline{A}_1\underline{B}_1 & \cdots & C\underline{A}_d\underline{A}_1\underline{B}_d & C\underline{A}_d\underline{A}_2\underline{B}_1 & \cdots \\ C\underline{A}_1^2\underline{B}_1 & \cdots & C\underline{A}_1^2\underline{B}_d & C\underline{A}_1^3\underline{B}_1 & \cdots & C\underline{A}_1^3\underline{B}_d & C\underline{A}_1^2\underline{A}_2\underline{B}_1 & \cdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \cdots \end{bmatrix} \quad (8.15)$$

Note that the matrix entries of $\mathbb{H}_{T^{FM}}$ are determined from the coefficients T_v $(v \in \mathcal{F}_d)$ as

$$[\mathbb{H}_{T^{FM}}]_{v,wg_k} = T_{vwg_k} = C\underline{A}^{vw}\underline{B}_k. \quad (8.16)$$

Next we shall identify this Hankel operator $\mathbb{H}_{T^{FM}}$ with the Hankel operator associated with the recognizable system $\mathbb{H}_{T^R}$ in (8.4). To this end, let us write a formal power series representation of the transfer function $T_{\Sigma^{FM}}(z)$ in (8.10) explicitly as

$$T_{\Sigma^{FM}}(z) = T_\lambda + T_{g_1}z^{g_1} + \cdots + T_{g_d}z^{g_d} + T_{g_1g_1}z^{g_1g_1} + T_{g_1g_2}z^{g_1g_2} + \cdots \quad (8.17)$$

Define $T_{\Sigma^R}(z)$ as

$$T_{\Sigma^R}(z) = \begin{bmatrix} T^1_{\Sigma^R}(z) & \cdots & T^d_{\Sigma^R}(z) \end{bmatrix} \triangleq \begin{bmatrix} T_{\Sigma^{FM}}(z)z_1^{-1} & \cdots & T_{\Sigma^{FM}}(z)z_d^{-1} \end{bmatrix}$$

i.e., for each $k = 1, \ldots, d$,

$$\begin{aligned} T^k_{\Sigma^R}(z) &= T_{\Sigma^{FM}}(z)z_k^{-1} \\ &= T_{g_k} + T_{g_1g_k}z^{g_1} + \cdots + T_{g_dg_k}z^{g_d} + T_{g_1g_1g_k}z^{g_1g_1} + \cdots \\ &= \sum_{v \in \mathcal{F}_d} T_{vg_k}z^v. \end{aligned}$$

Thus,

$$\begin{aligned} T_{\Sigma^R}(z) &= \sum_{v \in \mathcal{F}_d} \begin{bmatrix} T_{vg_1} & \cdots T_{vg_d} \end{bmatrix} z^v \\ &\triangleq \sum_{v \in \mathcal{F}_d} T^R_v z^v \quad \text{where } T^R_v = \begin{bmatrix} T_{vg_1} & \cdots & T_{vg_d} \end{bmatrix}. \end{aligned}$$

We then recover $T_{\Sigma^{FM}}(z)$ from $T_{\Sigma^R}(z)$ and T_λ via

$$T_{\Sigma^{FM}}(z) = T_\lambda + T^1_{\Sigma^R}(z) \cdot z_1 + \cdots + T^d_{\Sigma^R}(z) \cdot z_d. \quad (8.18)$$

Reall that the Hankel operator associated with $T_{\Sigma^R}(z)$ is a matrix $\mathbb{H}_{T^R}$ with rows and columns indexed by $\mathcal{F}_d$, where each entry of $\mathbb{H}_{T^R}$ is determined by $[\mathbb{H}_{T^R}]_{v,w} = T^R_{vw}$. It is easy to see that the Hankel matrix $\mathbb{H}_{T^R}$ is identical to the Hankel matrix $\mathbb{H}_{T^{FM}}$ of the NCFM model since

$$[\mathbb{H}_{T^R}]_{v,w} = T^R_{vw} = \begin{bmatrix} T_{vwg_1} & \cdots & T_{vwg_d} \end{bmatrix} = \begin{bmatrix} [\mathbb{H}_{T^{FM}}]_{v,wg_1} & \cdots & [\mathbb{H}_{T^{FM}}]_{v,wg_d} \end{bmatrix}. \tag{8.19}$$

Then the identification between $\mathbb{H}_{T^R}$ and $\mathbb{H}_{T^{FM}}$ is now clear, and hence the recognizable transfer function $T_{\Sigma^R}(z)$ is realizable if and only if the NCFM transfer function $T_{\Sigma^{FM}}(z)$ is.

We have already seen that if $T_{\Sigma^{FM}}(z)$ has a NCFM realization, then the associated Hankel operator $\mathbb{H}_{T^{FM}}$ has factorization $\mathbb{H}_{T^{FM}} = \underline{\mathcal{O}} \cdot \underline{\mathcal{C}}$ where

$$\begin{aligned} &\underline{\mathcal{C}} : \ell^2((\{\lambda\} \times \mathcal{F}_d \setminus \{\lambda\}), \mathcal{U}) \mapsto \mathcal{H}, \\ \text{and} \qquad &\underline{\mathcal{O}} : \mathcal{H} \mapsto \ell^2((\mathcal{F}_d \times \{\lambda\}), \mathcal{Y}), \end{aligned}$$

and hence $\operatorname{rank}(\mathbb{H}_{T^{FM}}) \leq \dim \mathcal{H} < \infty$. In addition, if such a realization is minimal, say $\dim \mathcal{H} = n$, then by Minimality Theorem 8.2 the system is controllable and observable. In other words, the NCFM controllability matrix $\underline{\mathcal{C}}$ is surjective and the NCFM observability matrix $\underline{\mathcal{O}}$ is injective, and this implies that $\operatorname{rank}(\mathbb{H}_{T^{FM}}) = \dim \mathcal{H} = n$.

Conversely, if $\operatorname{rank}(\mathbb{H}_{T^{FM}}) = n < \infty$, then the identification (8.19) implies that $\operatorname{rank}(\mathbb{H}_{T^R})$ is also equal to n. Then there exists a minimal recognizable realization $\{H, F_1, \ldots, F_d, G\}$ such that $T^R_v = HF^vG$ for all $v \in \mathcal{F}_d$ with the dimension of the state-space $\mathcal{H} = n$. To realize the original system equations (8.6), G necessarily has the form (cf. (8.10))

$$G = \begin{bmatrix} G_1 & \cdots & G_d \end{bmatrix}.$$

Thus the formal power series (8.18) can be represented as

$$\begin{aligned} T_{\Sigma^{FM}}(z) &= T_\lambda + \sum_{w \in \mathcal{F}_d} HF^w G_1 z^w \cdot z_1 + \cdots + \sum_{w \in \mathcal{F}_d} HF^w G_d z^w \cdot z_d \\ &= T_\lambda + \sum_{w \in \mathcal{F}_d} \sum_{k=1}^{d} HF^w G_k z^{wg_k} \end{aligned} \tag{8.20}$$

Then we set $\underline{A}_k = F_k, \underline{B}_k = G_k, C = H, D = T_\lambda$, and hence the minimal realization problem of the NCFM model is solved. Furthermore, since $\mathbb{H}_{T^{FM}} = \underline{\mathcal{O}} \cdot \underline{\mathcal{C}}$ and $\operatorname{rank}(\mathbb{H}_{T^{FM}}) = \dim \mathcal{H} = n$ which is the smallest one among other representations, one is able to construct a minimal realization $\{\underline{A}_1, \ldots, \underline{A}_d, \underline{B}_1, \ldots, \underline{B}_d, C, D\}$ as a system Σ^{FM} described by the NCFM model. By Minimality Theorem 8.2, this also implies that the system Σ^{FM} is controllable and observable ($\underline{\mathcal{O}}$ is injective and $\underline{\mathcal{C}}$ is surjective).

The above discussion is summarized in the following Theorem.

Theorem 8.3. *Let $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$ be a formal power series with coefficients $T_v \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then $T(z)$ has a realization $\{\underline{A}_1, \ldots, \underline{A}_d, \underline{B}_1, \ldots, \underline{B}_d, C, D\}$ as a* NCFM *model (i.e., $T = T_{\Sigma^{FM}}$) with state-space $\mathcal{H}$ if and only if the associated Hankel operator $\mathbb{H}_T$ has finite rank. In this case, the minimal possible dimension of the state-space $\mathcal{H}$ is equal to the rank of $\mathbb{H}_T$.*

Alternatively, one may construct the state-space $\mathcal{H}$, and the system matrices

$$\{\underline{A}_1, \ldots, \underline{A}_d, \underline{B}_1, \ldots, \underline{B}_d, C, D\}$$

directly from the experimental data (i.e., from the impulse response and the corresponding Hankel operator) rather than finding a recognizable-system realization $\{H, F_1, \ldots, F_d, G\}$; this procedure amounts to an adaptation of the procedure given by M. Fliess [Fli74] for the recognizable series. To do so, let us define the state-space $\mathcal{H}$ as

$$\mathcal{H} = \ell_{fin}((\{\lambda\} \times \mathcal{F}_d \setminus \{\lambda\}), \mathcal{U}) / \ker \mathbb{H}_T,$$

where $\mathbb{H}_T$ is constructed from coefficients T_v of the given formal power series $\sum_{v \in \mathcal{F}_d} T_v z^v$ as in (8.15), and define the operators $\underline{A}_j : \mathcal{H} \mapsto \bigoplus_1^d \mathcal{H}$, $\underline{B}_j : \mathcal{U} \mapsto \bigoplus_1^d \mathcal{H}$ and $C : \mathcal{H} \mapsto \mathcal{Y}$ by:

$$\begin{aligned}
\underline{A}_j &: [\{u(\lambda, v)\}_{v \in \mathcal{F}_d \setminus \{\lambda\}}]_{\mathcal{H}} \mapsto [\{u(\lambda, g_j^{-1} v)\}_{v \in g_j \mathcal{F}_d}]_{\mathcal{H}} \\
\underline{B}_j &: u \mapsto [\{\delta_{\tilde{w}, g_j} u\}_{\tilde{w} \in \mathcal{F}_d \setminus \{\lambda\}}]_{\mathcal{H}} \\
C &: [\{u(\lambda, v)\}_{v \in \mathcal{F}_d \setminus \{\lambda\}}]_{\mathcal{H}} \mapsto [\mathbb{H}_T \cdot \{u(\lambda, v)\}_{v \in \mathcal{F}_d \setminus \{\lambda\}}]_{\lambda},
\end{aligned}$$

where $[\,\cdot\,]_{\mathcal{H}}$ indicates the equivalence class modulo the kernel of $\mathbb{H}_T$, and $\delta_{v,w}$ is the noncommutative Kronecker delta defined by

$$\delta_{v,w} = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{otherwise.} \end{cases}$$

First we have to verify that these operators are well-defined. To show that $\underline{A}_j$ is well-defined, by linearity it suffices to show:

$$\{u(\lambda, v)\}_{v \in \mathcal{F}_d \setminus \{\lambda\}} \in \ker \mathbb{H}_T \Rightarrow \{u(\lambda, g_j^{-1} v)\}_{v \in g_j \mathcal{F}_d} \in \ker \mathbb{H}_T$$

For any fixed $w \in \mathcal{F}_d$, we compute

$$\begin{aligned}
[\mathbb{H}_T \cdot \{u(\lambda, g_j^{-1} v)\}_{v \in g_j \mathcal{F}_d}]_w &= \sum_{v \in g_j \mathcal{F}_d} [\mathbb{H}_T]_{w,v} \cdot u(\lambda, g_j^{-1} v) \\
&= \sum_{v' \in \mathcal{F}_d} [\mathbb{H}_T]_{w, g_j v'} \cdot u(\lambda, v')
\end{aligned}$$

$$= \sum_{v' \in \mathcal{F}_d} [\mathbb{H}_T]_{wg_j, v'} \cdot u(\lambda, v') = [\mathbb{H}_T \cdot \{u(\lambda, v')\}_{v' \in \mathcal{F}_d}]_{wg_j}.$$

Since we assume that $\{u(\lambda, v)\}_{v \in \mathcal{F}_d \setminus \{\lambda\}} \in \ker \mathbb{H}_T$, it follows that $[\mathbb{H}_T \cdot \{u(\lambda, v')\}_{v' \in \mathcal{F}_d}]_{wg_j} = 0$. Thus, we have $\{u(\lambda, g_j^{-1} v)\}_{v \in g_j \mathcal{F}_d} \in \ker \mathbb{H}_T$ as required. Note that the well-definedness of $\underline{B}_j$ is not an issue here since domain of $\underline{B}_j$ is not a space of equivalence class.

For the operator C, let $\{u(\lambda, v)\}_{v \in \mathcal{F}_d \setminus \{\lambda\}} \in \ker \mathbb{H}_T$. This implies that $\mathbb{H}_T \cdot u(\lambda, v)_{v \in \mathcal{F}_d \setminus \{\lambda\}} = 0$, and in paricular, $0 = [\mathbb{H}_T \cdot u(\lambda, v)_{v \in \mathcal{F}_d \setminus \{\lambda\}}]_\lambda := C \cdot \{u(\lambda, v)\}_{v \in \mathcal{F}_d \setminus \{\lambda\}}$ Hence, C is also well-defined.

It is quite straightforward to show that the operators defined above with $D = T_\lambda$ give a minimal NCFM realization for $T_\Sigma(z)$, i.e.

$$D = T_\lambda, \qquad C \underline{A}^w \underline{B}_k = T_{wg_k} \quad \text{for all } w \in \mathcal{F}_d, k = 1, \ldots, d.$$

To do so, suppose we are given $u \in \mathcal{U}$ and compute $C \underline{A}_{i_1} \underline{B}_k u$ as follows:

$$\underline{A}_{i_1} \underline{B}_k \cdot u = \underline{A}_{i_1} \cdot \{\delta_{\tilde{w}, g_k} u\}_{\tilde{w} \in \mathcal{F}_d \setminus \{\lambda\}} = \{\delta_{\tilde{w}, g_{i_1} g_k} u\}_{\tilde{w} \in \mathcal{F}_d \setminus \{\lambda\}}.$$

$$\text{Thus,} \qquad \begin{aligned} C \underline{A}_{i_1} \underline{B}_k \cdot u &= [\mathbb{H}_T \cdot \{\delta_{\tilde{w}, g_{i_1} g_k} u\}_{\tilde{w} \in \mathcal{F}_d \setminus \{\lambda\}}]_\lambda \\ &= \sum_{\tilde{w} \in \mathcal{F}_d \setminus \{\lambda\}} [\mathbb{H}_T]_{\lambda, \tilde{w}} \cdot \delta_{\tilde{w}, g_{i_1} g_k} u \\ &= [\mathbb{H}_T]_{\lambda, g_{i_1} g_k} \cdot u = T_{g_{i_1} g_k} \cdot u. \end{aligned}$$

Since u is arbitrary, one can deduce that $C \underline{A}_{i_1} \underline{B}_k = T_{g_{i_1} g_k}$. By induction, one can verify that for any $v \in \mathcal{F}_d$, $C \underline{A}^v \underline{B}_k = T_{vg_k}$.

8.1.3 Minimal Realization for NCGR Models, Σ^{GR}

This Subsection is devoted to solving the minimal realization problem of the i/s/o linear system described by the NCGR model. The reader will see that such a problem can be solved (i.e., to get a minimal NCGR realization) directly by a modification of the construction via Hankel matrices.

Let us first recall that the Future-time i/s/o linear system described by the NCGR model is

$$\Sigma^{GR} = \begin{cases} x_1(g_1 w, \lambda) &= \sum_{k=1}^d A_{1,k} x_k(w, \lambda) + B_1 u(w, \lambda) \\ \vdots & \qquad \vdots \\ x_d(g_d w, \lambda) &= \sum_{k=1}^d A_{d,k} x_k(w, \lambda) + B_d u(w, \lambda) \\ y(w, \lambda) &= \sum_{k=1}^d C_k x_k(w, \lambda) + D u(w, \lambda). \end{cases} \tag{8.21}$$

with initial condition

$$
\begin{aligned}
x_{i_n}(\lambda,\lambda) &= \sum_{i_{n-1},\dots,i_1,k=1}^{d} A_{i_n,i_{n-1}}\cdots A_{i_2,i_1}A_{i_1,k}x_k(\lambda, g_{i_n}g_{i_{n-1}}\cdots g_{i_1}) \\
&\quad + \sum_{i_{n-1},\dots,i_1=1}^{d} A_{i_n,i_{n-1}}\cdots A_{i_2,i_1}B_{i_1}u(\lambda, g_{i_n}g_{i_{n-1}}\cdots g_{i_1}) \\
&\quad + \cdots + \sum_{i_{n-1}=1}^{d} A_{i_n,i_{n-1}}B_{i_{n-1}}u(\lambda, g_{i_n}g_{i_{n-1}}) + B_{i_n}u(\lambda, g_{i_n}).
\end{aligned}
$$

Suppose that the state at a time in the remote past is initialized to be zero and the system is fed by an input sequence $\{u(w,v)\}$ supported only in the past as in the NCFM case. Then the output sequence is

$$
\mathrm{col}_{w\in\mathcal{F}_d}[y(w,\lambda)] = \mathbb{H}_{T^{GR}}\cdot \mathrm{col}_{w\in\mathcal{F}_d\setminus\{\lambda\}}[u(\lambda,v)], \tag{8.22}
$$

where $\mathbb{H}_{T^{GR}}$ denotes the NCGR *Hankel operator* given by the matrix with rows indexed by $\mathcal{F}_d$ and columns indexed by $\mathcal{F}_d\setminus\{\lambda\}$

$$
\begin{bmatrix}
C_1B_1 & \cdots & C_dB_d & C_1A_{1,1}B_1 & \cdots & C_1A_{1,d}B_d & C_2A_{2,1}B_1 & \cdots \\
C_1A_{1,1}B_1 & \cdots & C_1A_{1,d}B_d & C_1A_{1,1}^2B_1 & \cdots & C_1A_{1,1}A_{1,d}B_d & C_1A_{1,2}A_{2,1}B_1 & \cdots \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \cdots \\
C_dA_{d,1}B_1 & \cdots & C_dA_{d,d}B_d & C_dA_{d,1}A_{1,1}B_1 & \cdots & C_dA_{d,1}A_{1,d}B_d & C_dA_{d,2}A_{2,1}B_1 & \cdots \\
C_1A_{1,1}^2B_1 & \cdots & C_1A_{1,1}A_{1,d}B_d & C_1A_{1,1}^3B_1 & \cdots & C_1A_{1,1}^2A_{1,d}B_d & C_1A_{1,1}A_{1,2}A_{2,1}B_1 & \cdots \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \cdots
\end{bmatrix}
$$

The matrix entry of $\mathbb{H}_{T^{GR}}$ can be determined by

$$
[\mathbb{H}_{T^{GR}}]_{g_{i_n}g_{i_{n-1}}\cdots g_{i_1},g_{j_m}g_{j_{m-1}}\cdots g_{j_1}g_k} = C_{i_n}A_{i_n,i_{n-1}}\cdots A_{i_2,i_1}A_{i_1,j_m}A_{j_m,j_{m-1}}\cdots A_{j_2,j_1}A_{j_1,k}B_k.
$$

Note that the transfer function of the NCGR model is

$$
T_{\Sigma^{GR}}(z) = D + C(I - Z_d(z)A)^{-1}Z_d(z)B,
$$

which can also be expressed in terms of NCFM system matrices $\underline{A}_j, \underline{B}_j$ as

$$
\begin{aligned}
T_{\Sigma^{GR}}(z) &= D + C\left[\sum_{j=0}^{\infty}\left(\sum_{k=1}^{d}\underline{A}_k z_k\right)^j\right]\cdot\sum_{k=1}^{d}\underline{B}_k z_k \\
&= D + \sum_{w\in\mathcal{F}_d}\sum_{k=1}^{d}\left(C\underline{A}^w\underline{B}_k\right)z^{wg_k},
\end{aligned} \tag{8.23}
$$

where $\underline{A}_j = P_j A$ and $\underline{B}_j = P_j B$. (See (6.57) on page 133). One could observe that this Hankel operator $\mathbb{H}_{TGR}$ is identical to the Hankel operator $\mathbb{H}_{TFM}$ if the NCGR system model in (8.21) is viewed as (embedded into) an NCFM system model, where we set $\underline{A}_k = P_k A$ and $\underline{B}_k = P_k B$. Therefore, the method discussed in Subsection 8.1.2 produces a minimal NCFM realization of a given formal power series; however, this NCFM realization may not have the special structure to come from a NCGR realization. We here present a adapted procedure for construction a minimal NCGR realization.

To do so, let us partition the Hankel matrix $\mathbb{H}_{TGR}$ into the block row decomposition as

$$\mathbb{H}_{TGR} = \begin{bmatrix} \mathbb{H}^1_{TGR} & \mathbb{H}^2_{TGR} & \cdots & \mathbb{H}^d_{TGR} \end{bmatrix},$$

where the column indices are partitioned up by writing $\mathcal{F}_d \setminus \{\lambda\}$ as the pairwise-disjoint union of $g_k \cdot \mathcal{F}_d$ over $k = 1, \ldots, d$. In this case, we have d Hankel matrices where for each fixed k, the rows of $\mathbb{H}^k_{TGR}$ are indexed by $v \in \mathcal{F}_d$, whereas the columns are indexed by $w = g_k w' \in g_k \cdot \mathcal{F}_d$ (i.e., words beginning with g_k on the left). More explicitly, the k-th Hankel matrix $\mathbb{H}^k_{TGR}$ is given by

$$\mathbb{H}^k_{TGR} = \begin{bmatrix} C_k B_k & C_k A_{k,1} B_1 & \cdots & C_k A_{k,d} B_d & C_k A_{k,1} A_{1,1} B_1 & \cdots \\ C_1 A_{1,k} B_k & C_1 A_{1,k} A_{k,1} B_1 & \cdots & C_1 A_{1,k} A_{k,d} B_d & C_1 A_{1,k} A_{k,1} A_{1,1} B_1 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \cdots \\ C_d A_{d,k} B_k & C_d A_{d,k} A_{k,1} B_1 & \cdots & C_d A_{d,k} A_{k,d} B_d & C_d A_{d,k} A_{k,1} A_{1,1} B_1 & \cdots \\ C_1 A_{1,1} A_{1,k} B_k & C_1 A_{1,1} A_{1,k} A_{k,1} B_1 & & \vdots & \vdots & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \end{bmatrix}$$

where its matrix entries are completely determined by the NCGR system matrices as follows:

$$[\mathbb{H}^k_{TGR}]_{v,g_k w} = \begin{cases} C_k B_k & \text{if } v = \lambda, w = \lambda, \\ C_k A_{k,j_m} \cdots A_{j_2,j_1} B_{j_1} & \text{if } v = \lambda, w = g_{j_m} g_{j_{m-1}} \cdots g_{j_1} \\ C_{i_n} A_{i_n,i_{n-1}} \cdots A_{i_1,k} B_k & \text{if } v = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}, w = \lambda \\ C_{i_n} A_{i_n,i_{n-1}} \cdots A_{i_1,k} A_{k,j_m} \cdots A_{j_2,j_1} B_{j_1} & \text{if } v = g_{i_n} g_{i_{n-1}} \cdots g_{i_1}, w = g_{j_m} g_{j_{m-1}} \cdots g_{j_1}. \end{cases}$$

It is worth noting that each entry of $\mathbb{H}^k_{TGR}$ can also be determined by coefficients T_v of a formal power series as

$$\text{for fixed } k, \qquad [\mathbb{H}^k_{TGR}]_{v,g_k w} = T_{v g_k w}. \tag{8.24}$$

It is easily to see that for each k, the Hankel matrix $\mathbb{H}^k_{TGR}$ can be factored as

$$\mathbb{H}^k_{TGR} = \begin{bmatrix} C_k \\ C_1 A_{1,k} \\ \vdots \\ C_d A_{d,k} \\ C_1 A_{1,1} A_{1,k} \\ \vdots \end{bmatrix} \begin{bmatrix} B_k & A_{k,1}B_1 & \cdots & A_{k,d}B_d & A_{k,1}A_{1,1}B_1 & \cdots \end{bmatrix}$$

and recall that the block column matrix and the block row matrix, respectively in the above expression are the observability matrix $\mathcal{O}^k$ and the controllability matrix $\mathcal{C}^k$ with respect to a letter g_k (see Chapter 7 for the discussion on controllability and observability issues). Thus we have

$$\mathbb{H}^k_{TGR} = \mathcal{O}^k \cdot \mathcal{C}^k. \tag{8.25}$$

In general, for each $k = 1, \dots, d$

$$\operatorname{rank}(\mathbb{H}^k_{TGR}) \leq \dim(\mathcal{H}_k) < \infty$$

but if the NCGR system is NCGR-controllable and NCGR-observable (i.e., $\mathcal{C}^k$ is surjective and $\mathcal{O}^k$ is injective for each $k = 1, \dots, d$) and the state-space dimension for each k is $\dim \mathcal{H}_k = n_k$, then $\operatorname{rank}(\mathbb{H}^k_{TGR}) = \dim \mathcal{H}_k = n_k$.

Conversely, let $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$ be any formal power series and construct the corresponding Hankel matrix $\mathbb{H}^k_T$ as in (8.24). We then define the state-space $\mathcal{H}_k$ by

$$\mathcal{H}_k = \ell_{fin}((\{\lambda\} \times g_k\mathcal{F}_d), \mathcal{U}) / \ker \mathbb{H}^k_T, \tag{8.26}$$

and define the operators $A_{i,j} : \mathcal{H}_j \mapsto \mathcal{H}_i$, $B_j : \mathcal{U} \mapsto \mathcal{H}_j$, and $C_i : \mathcal{H}_i \mapsto \mathcal{Y}$ by:

$$\begin{aligned} A_{i,j} &: [\{u(\lambda, v)\}_{v \in g_j\mathcal{F}_d}]_{\mathcal{H}_j} \mapsto [\{u(\lambda, g_i^{-1}v)\}_{v \in g_i\mathcal{F}_d}]_{\mathcal{H}_i} \\ B_j &: u \mapsto [\{\delta_{\tilde{w}, g_j} u\}_{\tilde{w} \in g_j\mathcal{F}_d}]_{\mathcal{H}_j} \\ C_i &: [\{u(\lambda, v)\}_{v \in g_i\mathcal{F}_d}]_{\mathcal{H}_i} \mapsto [\mathbb{H}^i_T \cdot \{u(\lambda, v)\}_{v \in g_i\mathcal{F}_d}]_\lambda. \end{aligned}$$

As in the NCFM case, we have to verify that the operators $A_{i,j}$ and C_i are well-defined. Note that domain of B_i is not a space of equivalence class and hence the well-definedness is not an issue here.

To verify that $A_{i,j}$ is well-defined, let us assume that $\{u(\lambda, v)\}_{v \in g_j\mathcal{F}_d} \in \ker \mathbb{H}^j_T$. Thus, we have

$$0 = [\mathbb{H}_T^j \cdot \{u(\lambda, v)\}_{v \in g_j \mathcal{F}_d}]_w = \sum_{v \in g_j \mathcal{F}_d} [\mathbb{H}_T^j]_{w,v} \cdot u(\lambda, v)$$
$$= \sum_{v' \in \mathcal{F}_d} [\mathbb{H}_T^j]_{w, g_j v'} \cdot u(\lambda, g_j v') \tag{8.27}$$

The above expression is true for all $w \in \mathcal{F}_d$. In particular, when w has the form $w = w' g_i$ for some $w' \in \mathcal{F}_d$. Thus, it follows from (8.27) that

$$\begin{aligned} 0 &= \sum_{v' \in \mathcal{F}_d} [\mathbb{H}_T^j]_{w' g_i, g_j v'} \cdot u(\lambda, g_j v') \\ &= \sum_{v' \in \mathcal{F}_d} [\mathbb{H}_T^i]_{w', g_i g_j v'} \cdot u(\lambda, g_j v') \\ &= \sum_{v \in g_i \mathcal{F}_d} [\mathbb{H}_T^i]_{w', v} u(\lambda, g_i^{-1} v) \quad \text{(here we set } v = g_i g_j v') \\ &= [\mathbb{H}_T^i \cdot \{u(\lambda, g_i^{-1} v)\}_{v \in g_i \mathcal{F}_d}]_{w'} \end{aligned}$$

This implies that $\{u(\lambda, g_i^{-1} v)\}_{v \in g_i \mathcal{F}_d} \in \ker \mathbb{H}_T^i$. For well-definedness of C_i is trivial from the definition.

Thus, these operators together with $D = T_\lambda$ perform a minimal NCGR realization for $T_\Sigma(z)$ as in the NCFM case. We should also note that $\dim \mathcal{H} = \operatorname{rank}(\mathbb{H}_{T^{GR}})$, say $= n_k < \infty$ for each k, by definition in (8.26).

The above discussion is summarized in the following Theorem.

Theorem 8.4. *Let $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$ be a formal power series with coefficients $T_v \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then $T(z)$ has a realization $\{[A_{i,j}]_{i,j=1}^d, [B_j]_{j=1}^d, [C_i]_{i=1}^d, D\}$ as a* NCGR *model (i.e., $T = T_{\Sigma^{GR}}$) with state-space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d$ if and only if for each $k = 1, \ldots, d$, the associated Hankel operators $\mathbb{H}_T^k$ has finite rank. In this case, the minimal possible dimension of the state-space $\mathcal{H}_k$ in any* NCGR *realization for $T(z)$ is equal to* $\operatorname{rank}(\mathbb{H}_T^k)$.

Remark 27. C. Beck [Bec01] gave some partial results concerning minimal NCGR realizations for a given formal power series by making connections between series of the form $T_{\Sigma^{GR}}$ and recognizable series T_{Σ^R}. ▲

8.2 Stability

This Section concerns the stability issue which plays an important role in the designing a stabilizing linear controller. The system we are dealing with here is an i/s/o linear system described by the Future-time NCGR model (6.4). If the system is fed by zero input sequence, we shall call such a system as an *unforced system*. First we introduce some notions of stability and then define the so-called *noncommutative d-variable Schur class* which is analogous to the Schur class in the commutative case. We end this Section with the development of the Lyapunov theory for stability test.

Definition 42. A system is said to have *finite ℓ^2-gain* if for every input sequence $\{u(w,\lambda)\} \in \ell^2((\mathcal{F}_d \times \{\lambda\}), \mathcal{U})$, the system generates an output sequence $\{y(w,\lambda)\}$ such that

$$\|y(w,\lambda)\|^2_{\ell^2((\mathcal{F}_d \times \{\lambda\}, \mathcal{Y})} < \infty$$

when the initial and boundary conditions are bounded, i.e.

$$\|x(\lambda,\lambda)\|^2 + \sum_{k=1}^{d} \sum_{w: w \neq g_k \tilde{w}} \|x_k(w,\lambda)\|^2 < \infty.$$

Definition 43. An unforced system is said to be *exponentially stable* provided that for zero boundary condition, there exist $K < \infty$ and $r < 1$ so that $\sum_{w:|w|=n} \|x(w,\lambda)\|^2 \leq K r^n \|x(\lambda,\lambda)\|^2$ for all n.

Definition 44. An unforced system is said to be *asymptotically stable* provided that for zero boundary condition, $\lim\limits_{n\to\infty} \sum\limits_{w:|w|=n} \|x(w,\lambda)\|^2 = 0$.

Recall that for any word w of length m given by $w = g_{j_m} g_{j_{m-1}} \cdots g_{j_1}$, the general solution $x_j(g_j w, \lambda)$ for the system (6.4) with zero boundary conditions and zero input sequence is given by (see (6.47) on page 130):

$$x_j(g_j w,\lambda) = A_{j,j_m} A_{j_m,j_{m-1}} \cdots A_{j_3,j_2} A_{j_2,j_1} \sum_{i_n=1}^{d} A_{j_1,i_n} x_{i_n}(\lambda,\lambda),$$

or in terms of the NCFM system matrices

$$x(gw,\lambda) = \sum_{k=1}^{d} \underline{A}^{g_k w} x(\lambda,\lambda).$$

Thus the stability criteria in Definition 43 and Definition 44, respectively are equivalent to:

- An unforced system is said to be *exponentially stable* provided that for zero boundary condition, there exist $K < \infty$ and $r < 1$ so that $\sum_{w:|w|=n} \|\underline{A}^w x(\lambda,\lambda)\|^2 \leq K r^n \|x(\lambda,\lambda)\|^2$ for all n.

- An unforced system is said to be *asymptotically stable* provided that for zero boundary condition,

$$\lim_{n\to\infty}\sum_{w:|w|=n}\|\underline{A}^w x(\lambda,\lambda)\|^2 = 0.$$

8.2.1 Noncommutative d-variable Schur Class

Let us first recall that the Future-time NCGR model is described by

$$\Sigma_f^{GR} = \begin{cases} x_k(g_k w,\lambda) & = \sum_{j=1}^d A_{k,j}x_j(w,\lambda) + B_k u(w,\lambda), \quad \text{for } k=1,\dots,d \\ y(w,\lambda) & = \sum_{j=1}^d C_j x_j(w,\lambda) + Du(w,\lambda), \end{cases} \tag{8.28}$$

together with the connecting operator

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{1,1} & \cdots & A_{1,d} & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{d,1} & \cdots & A_{d,d} & B_d \\ C_1 & \cdots & C_d & D \end{bmatrix} : \begin{bmatrix} \bigoplus_{i=1}^d \mathcal{H}_i \\ \mathcal{U} \end{bmatrix} \mapsto \begin{bmatrix} \bigoplus_{i=1}^d \mathcal{H}_i \\ \mathcal{Y} \end{bmatrix}.$$

Suppose now that the connecting operator U is contractive; i.e., $\|U\xi\|^2 \le \|\xi\|^2$ for all $\xi \in \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix}$. Then from the system (8.28), we have the dissipative inequality over all trajectories of the system (cf. (6.39) on page 127)

$$\|x(gw,\lambda)\|^2 - \|x(w,\lambda)\|^2 \le \|u(w,\lambda)\|^2 - \|y(w,\lambda)\|^2 \tag{8.29}$$

It is of interest to take the summation of both sides of (8.29) over all words w of length at most n and this yields

$$\sum_{w:|w|\le n}\left(\|x(gw,\lambda)\|^2 - \|x(w,\lambda)\|^2\right) \le \sum_{w:|w|\le n}\left(\|u(w,\lambda)\|^2 - \|y(w,\lambda)\|^2\right),$$

which is equivalent to

$$\begin{aligned} \sum_{w:|w|=n}\|x(gw,\lambda)\|^2 - \|x(\lambda,\lambda)\|^2 - \sum_{\substack{w:1\le|w|\le n \\ w\ne g_k\tilde{w}}}\sum_{k=1}^d \|x_k(w,\lambda)\|^2 \\ \le \sum_{w:|w|\le n}\left(\|u(w,\lambda)\|^2 - \|y(w,\lambda)\|^2\right) \end{aligned} \tag{8.30}$$

If now we assume that $\{u(w,\lambda)\} \in \ell^2((\mathcal{F}_d \times \{\lambda\},\mathcal{U})$ and that n be arbitrarily large, one can see that

$$
\begin{aligned}
&\|y(w,\lambda)\|^2_{\ell^2((\mathcal{F}_d\times\{\lambda\},\mathcal{Y})}\\
&\leq \|u(w,\lambda)\|^2_{\ell^2((\mathcal{F}_d\times\{\lambda\},\mathcal{U})} + \|x(\lambda,\lambda)\|^2 - \lim_{n\to\infty}\sum_{w:|w|=n}\|x(gw,\lambda)\|^2 + \lim_{n\to\infty}\sum_{k=1}^{d}\sum_{\substack{w:1\leq|w|\leq n\\ w\neq g_k\tilde{w}}}\|x_k(w,\lambda)\|^2\\
&\leq \|u(w,\lambda)\|^2_{\ell^2((\mathcal{F}_d\times\{\lambda\}),\mathcal{U})} + \|x(\lambda,\lambda)\|^2 + \lim_{n\to\infty}\sum_{k=1}^{d}\sum_{\substack{w:1\leq|w|\leq n\\ w\neq g_k\tilde{w}}}\|x_k(w,\lambda)\|^2 \qquad (8.31)
\end{aligned}
$$

If we assume in addition that the initial and boundary conditions are bounded in the sense of Definition 42, then the system has finite ℓ_2-gain. Now if we suppose also that the boundary terms are all zero, then (8.31) collapses to

$$
\|y(w,\lambda)\|^2_{\ell^2((\mathcal{F}_d\times\{\lambda\},\mathcal{Y})} \leq \|u(w,\lambda)\|^2_{\ell^2((\mathcal{F}_d\times\{\lambda\},\mathcal{U})} + \|x(\lambda,\lambda)\|^2 \qquad (8.32)
$$

This implies that for any contractive operator U if the system is asymptotically stable, then it does also have finite ℓ_2-gain.

Let us now consider the dissipative system (i.e., a system associated with a contraction U) given in (8.31). It is of interest when the initial states $x_k(\lambda,\lambda)=0$ and the boundary conditions $x_k(w,\lambda)=0$ unless $w=g_k\tilde{w}$ for all $k\in\mathcal{I}_d$. Then (8.31) collapses to

$$
\|y(w,\lambda)\|^2_{\ell^2((\mathcal{F}_d\times\{\lambda\},\mathcal{Y})} \leq \|u(w,\lambda)\|^2_{\ell^2((\mathcal{F}_d\times\{\lambda\},\mathcal{U})}. \qquad (8.33)
$$

After taking the Z-transform (see Eq. (6.55) on page 133), it yields

$$
\|y^{\wedge\mathcal{T}_f}(z,0)\|^2_{L^2((\mathcal{F}_d\times\{\lambda\},\mathcal{Y})} = \|T_{\Sigma_f}(z)\cdot u^{\wedge\mathcal{T}_f}(z,0)\|^2_{L^2((\mathcal{F}_d\times\{\lambda\},\mathcal{U})} \leq \|u^{\wedge\mathcal{T}_f}(z,0)\|^2_{L^2((\mathcal{F}_d\times\{\lambda\},\mathcal{U})} \quad (8.34)
$$

In analogy with the classical case, we denote $\mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y})$ the *noncommutative, d-variable Schur class*:

$$
\begin{aligned}
\mathcal{S}_{nc,d}(\mathcal{U},\mathcal{Y}) = \Big\{&T(z)=\sum_{w\in\mathcal{F}_d}T_w z^w \text{ such that}\\
&M_T\colon L^2((\mathcal{F}_d\times\{\lambda\},\mathcal{U})\mapsto L^2((\mathcal{F}_d\times\{\lambda\},\mathcal{Y}) \text{ analytic and } \|M_T\|_{Op}\leq 1\Big\} \qquad (8.35)
\end{aligned}
$$

Thus (8.34) implies that $T_{\Sigma_f}\in\mathcal{S}_{nc,d}$.

8.2.2 Lyapunov Theory

This Subsection presents the development of the so-called *Lyapunov Theory* for an i/s/o linear system with evolution along the elements of a free semigroup which is represented by the Future-time NCGR model. Let us begin by letting $V : \mathcal{H} \to \mathbb{R}^+$ be a *quadratic Lyapunov function candidate* defined by

$$V(x) = \langle x, Px \rangle \quad \text{for all } x \in \mathcal{H} \tag{8.36}$$

where P is a given positive definite matrix. Note that the quadratic Lyapunov function V is always positive as long as $x \neq 0$. Let w be a word of length n, i.e. $|w| = n$. Then

$$\begin{aligned} \triangle V(x) &\triangleq V(x(gw,\lambda)) - V(x(w,\lambda)) \\ &= \langle x(gw,\lambda), Px(gw,\lambda)\rangle - \langle x(w,\lambda), Px(w,\lambda)\rangle \\ &= \langle Ax(w,\lambda), PAx(w,\lambda)\rangle - \langle x(w,\lambda), Px(w,\lambda)\rangle \\ &= \langle x(w,\lambda), (A^\top PA - P)x(w,\lambda)\rangle \end{aligned} \tag{8.37}$$

Since $V(x)$ was chosen to be positive definite, we require, for asymptotic stability, that $\triangle V(x)$ be negative definite. Therefore, we demand that

$$\triangle V(x) = -\langle x(w,\lambda), Qx(w,\lambda)\rangle, \tag{8.38}$$

where,

$$Q \triangleq P - A^\top PA > 0 \tag{8.39}$$

The equation (8.39) is called *Lyapunov equation.*

By taking summation of all possible words of length $|w| = 0$ to $|w| = N$ on the right hand side of (8.37), we get

$$\begin{aligned} \sum_{w:|w|=0}^{N} \langle x(w,\lambda), Qx(w,\lambda)\rangle &= \sum_{w:|w|=0}^{N} [V(x(w,\lambda)) - V(x(gw,\lambda))] \\ &= V(x(\lambda,\lambda)) - \sum_{w:|w|=N+1} V(x(w,\lambda)) \\ &\leq V(x(\lambda,\lambda)) < \infty \end{aligned} \tag{8.40}$$

Since the sequence of partial sums of the associated quadratic functions is bounded above by

$V(x(\lambda,\lambda))$ and is increasing due to the positivity assumption on Q in (8.39), this implies that

$$\lim_{N\to\infty}\sum_{w:|w|=0}^{N}\langle x(w,\lambda), Qx(w,\lambda)\rangle \qquad \text{converges.}$$

Indeed $\lim_{N\to\infty}\sum_{w:|w|=0}^{N}\langle x(w,\lambda), Qx(w,\lambda)\rangle = 0$ by the n^{th}-term test. Since Q is (strictly) positive definite by assumption (8.39), we have

$$\langle x(w,\lambda), Qx(w,\lambda)\rangle \geq m\|x(w,\lambda)\|^2, \quad \text{where } m = \inf_{x:\|x\|=1}\langle x(w,\lambda), Qx(w,\lambda)\rangle > 0$$

and therefore, this implies

$$\lim_{n\to\infty}\sum_{w:|w|=n}\|x(w,\lambda)\|^2 = 0,$$

i.e., the system is asymptotically stable. Hence, we have established the following Lemma.

Lemma 8.5. *An i/s/o linear unforced system described by the* NCGR *model as in* (8.28) *is asymptotically stable if for some positive definite Q, there exists a positive definite solution P satisfying the Lyapunov equation* (8.39)*.*

Suppose now that the unforced system is asymptotically stable, i.e.

$$\lim_{n\to\infty}\sum_{w:|w|=n}\|x(w,\lambda)\|^2 = \lim_{n\to\infty}\sum_{w:|w|=n}\|\underline{A}^w x(\lambda,\lambda)\|^2 = 0 \tag{8.41}$$

For any positive definite Q, consider the equation for $M : M = Q + A^\top MA$. Solve recursively for M as follows:

$$\begin{aligned}
M &= Q + A^\top MA \\
&= Q + \sum_{i=1}^{d}\underline{A}_i^\top M\sum_{j=1}^{d}\underline{A}_j^\top \\
&= Q + \sum_{i=1}^{d}\underline{A}_i^\top\left[Q + \sum_{k=1}^{d}\underline{A}_k^\top M\sum_{\ell=1}^{d}\underline{A}_\ell^\top\right]\sum_{j=1}^{d}\underline{A}_j^\top \\
&= Q + \sum_{i=1}^{d}\underline{A}_i^\top Q\sum_{j=1}^{d}\underline{A}_j^\top + \sum_{i=1}^{d}\sum_{k=1}^{d}(\underline{A}_k\underline{A}_i)^\top M\sum_{\ell=1}^{d}\sum_{j=1}^{d}(\underline{A}_\ell\underline{A}_l) \\
\text{Or, } &= Q + \left[(\underline{A}^{g_1})^\top Q(\underline{A}^{g_1}) + (\underline{A}^{g_1})^\top Q(\underline{A}^{g_2}) + \cdots + (\underline{A}^{g_d})^\top Q(\underline{A}^{g_d})\right] \\
&\quad + \left[(\underline{A}^{g_1g_1})^\top Q(\underline{A}^{g_1g_1}) + (\underline{A}^{g_1g_1})^\top Q(\underline{A}^{g_1g_2}) + \cdots + (\underline{A}^{g_dg_d})^\top Q(\underline{A}^{g_dg_d})\right]
\end{aligned}$$

$$+ \sum_{i=1}^{d}\sum_{k=1}^{d}\sum_{m=1}^{d} (\underline{A}_m\underline{A}_k\underline{A}_i)^\top M \sum_{n=1}^{d}\sum_{\ell=1}^{d}\sum_{j=1}^{d} \left(\underline{A}_n\underline{A}_\ell\underline{A}_j\right)$$

$$\vdots$$

$$= Q + \sum_{w:|w|=1}^{N}\sum_{v:|v|=1}^{N} (\underline{A}^w)^\top Q\,(\underline{A}^v) + \sum_{w:|w|=N+1}\sum_{v:|v|=N+1} (\underline{A}^w)^\top M\,(\underline{A}^v) \tag{8.42}$$

Note that when N is large enough, the last term of (8.42) goes to zero due to the stability condition (8.41). Thus, (8.42) collapses to

$$M = Q + \sum_{w:|w|=1}^{N}\sum_{v:|v|=1}^{N} (\underline{A}^w)^\top Q\,(\underline{A}^v)$$

$$\text{Or, } M \geq \sum_{w:|w|=1}^{N}\sum_{v:|v|=1}^{N} (\underline{A}^w)^\top Q\,(\underline{A}^v) \qquad (\text{since } Q > 0) \tag{8.43}$$

Since the partial sums (8.43) of the associated quadratic functions are bounded above, the limit

$$\lim_{N\to\infty} \sum_{w:|w|=1}^{N}\sum_{v:|v|=1}^{N} (\underline{A}^w)^\top Q\,(\underline{A}^v) \qquad \text{exists.}$$

Hence, we shall take P to be

$$P = Q + \sum_{w:|w|=1}^{\infty}\sum_{v:|v|=1}^{\infty} (\underline{A}^w)^\top Q\,(\underline{A}^v) \tag{8.44}$$

Obviously, P is well-defined and positive definite since Q is. Moreover, such a P satisfies the Lyapunov equation (8.39), since

$$\begin{aligned}
A^\top PA - P &= \left(\sum_{i=1}^{d}\underline{A}_i^\top\right) P \left(\sum_{j=1}^{d}\underline{A}_j\right) - P \\
&= \sum_{i=1}^{d}\underline{A}_i^\top \left[Q + \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top Q \sum_{v:|v|=1}^{\infty} (\underline{A}^v)\right] \sum_{j=1}^{d}\underline{A}_j - P \\
&= \sum_{i=1}^{d}\underline{A}_i^\top Q \sum_{j=1}^{d}\underline{A}_j + \sum_{i=1}^{d}\sum_{w:|w|=1}^{\infty} (\underline{A}^w\underline{A}_i)^\top Q \sum_{j=1}^{d}\sum_{v:|v|=1}^{\infty} \left(\underline{A}^v\underline{A}_j\right) - P \\
&= \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top Q \sum_{v:|v|=1}^{\infty} (\underline{A}^v) - P
\end{aligned}$$

$$= -Q \tag{8.45}$$

Thus, from this analysis, we can conclude that if the system is asymptotically stable, then for some positive definite matrix Q, there exists a positive definite matrix P satisfying the Lyapunov equation. Moreover, such a P is unique. To see this, let us assume that P_1 is another solution of (8.39).

Then,

$$\begin{aligned}
P &= Q + \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top Q \sum_{v:|v|=1}^{\infty} (\underline{A}^v) \\
&= [P_1 - A^\top P_1 A] + \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top \left[P_1 - A^\top P_1 A\right] \sum_{v:|v|=1}^{\infty} (\underline{A}^v) \\
&= P_1 + \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top P_1 \sum_{v:|v|=1}^{\infty} (\underline{A}^v) - A^\top P_1 A - \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top A^\top P_1 A \sum_{v:|v|=1}^{\infty} (\underline{A}^v) \\
&= P_1 + \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top P_1 \sum_{v:|v|=1}^{\infty} (\underline{A}^v) - \sum_{w:|w|=0}^{\infty} (\underline{A}^w)^\top A^\top P_1 A \sum_{v:|v|=0}^{\infty} (\underline{A}^v) \\
&= P_1 + \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top P_1 \sum_{v:|v|=1}^{\infty} (\underline{A}^v) - \sum_{w:|w|=0}^{\infty} \sum_{i=1}^{d} (\underline{A}_i \underline{A}^w)^\top P_1 \sum_{v:|v|=0}^{\infty} \sum_{j=1}^{d} \left(\underline{A}_j \underline{A}^v\right) \\
&= P_1 + \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top P_1 \sum_{v:|v|=1}^{\infty} (\underline{A}^v) - \sum_{w:|w|=1}^{\infty} (\underline{A}^w)^\top P_1 \sum_{v:|v|=1}^{\infty} (\underline{A}^v) \quad = \; P_1
\end{aligned} \tag{8.46}$$

By this analysis combined with Lemma 8.5, we have proved the following Theorem.

Theorem 8.6 (Lyapunov Theorem for Noncommutative Roesser's System). *An i/s/o linear unforced system described by the* NCGR *model is asymptotically stable if and only if for some positive definite matrix Q, there exists a unique positive definite solution P of the Lyapunov equation* (8.39).

8.3 Conclusion

This Chapter establishes the realization theory for an i/s/o system described by the noncommutative d-D linear models in a connection with the so-called *recognizable system*. Our contribution here is to give the state-space interpretation of systems whose transfer function is expressed as a formal power series with noncommuting d-indeterminants, and to generalize the work of C. Beck in [Bec01] in systematic way. We also introduce various notions of stability and establish the Lyapunov theory in terms of state-space coordinate.

Chapter 9

Conclusion and Open Problems

In Part 1, we consider an i/s/o linear system whose the "time axis" is an integer lattice, $\mathbb{Z}^d, d > 1$. The thermal process, heat exchangers, gas absorption, and satellite photo analysis are typical examples of this type of system. We present here two well-known linear models which are commonly used in the multidimensional system literature, namely the Givone-Roesser (GR) and the Fornasini-Marchesini (FM) models. These models are not completely independent; in fact, one can identify one model with the other if certain assumptions are imposed on the FM model. The application of the d-variable Z-transform to the linear system equations yields the transfer function which is a rational function of d complex variables.

We then consider the so-called (output) feedback stabilizable problem for a given plant $P(\mathbf{z})$, where $\mathbf{z} = (z_1, \ldots, z_d)$, in the connection with the model matching problem and the interpolation problem. By assuming that $P(\mathbf{z})$ admits a double coprime factorization (DCF), one can convert the feedback stabilizable problem into a model matching problem via the Youla parameter, $Q(\mathbf{z})$. Let $F(\mathbf{z})$ denote the performance function, which is affine in $Q(\mathbf{z})$. Then, with the performance function $F(\mathbf{z})$ as the design parameter than $Q(\mathbf{z})$, one has an interpolation problem for $F(\mathbf{z})$. One then solves an interpolation problem to get $F(\mathbf{z})$, and then backsolves for $Q(\mathbf{z})$ and finally for $K(\mathbf{z})$, a desired controller. For the internal stability issue, if the performance function $F(\mathbf{z})$ is stable and satisfies the appropriate interpolation conditions, then $K(\mathbf{z})$ is internally stable. Incorporation of a tolerance level on $F(\mathbf{z})$ then leads to an NPIP type.

The procedure to solve the H^∞ control problem via the interpolation approach is summarized in Section 4.6. While this procedure does solve the problem, there are a number of remaining issues which may be directions for future research. In addition to Remark 12 (see page 74), let us mention:

1. While the existence of a DCF of a rational matrix-valued function in several complex variables in general appears not to have been proven at the moment, it is conjectured that such a DCF always does exist; the set of conditions in Proposition 4.6 is sufficient to

guarantee the existence of such a DCF.

2. The case where the third coupled interpolation condition (4.21) appears (i.e., the case where $q_u = s_v$ for some pair of indices (u, v)), and the case when interpolating along the zero variety of irreducible function with multiplicities k_u or ℓ_v are more than one remain mysterious.

3. The procedure in Section 4.6 is rather cumbersome and quite analogously to the situation for the classical 1D H^∞ problem in the early days of the development of the theory in the 1980s (see [Fra87]). What is missing is the formulation and solution of the interpolation problem in terms of state-space coordinates (or some other practical choice of coordinates which would streamline the computations). Despite the fact that much of the familiar Hardy space function theory for the classical case fails for $d > 1$ (see e.g. [Rud69]), the results here suggest that something should be possible.

4. At least to our knowledge, there is a lack of a reliable analysis on how to solve an infinite LMI or an LOI; some analysis of whether solutions of a sequence of approximating LMIs can be used to approximate a true solution of the infinite LMI would be helpful. Simple numerical experiments on small-size ID set done by the author suggest that one cannot expect to find the solution of the full infinite LMI by approximating with solutions of finite sub-LMIs.

5. It would be of interest to remove the assumption that T_2 and T_3 are invertible, and to handle the case of the general configuration of the standard H^∞ problem (see Fig. 4.1). In this connection, see Remark 13.

In Part 2, we examine linear systems with evolution along the elements of a free semigroup $\mathcal{F}_d$. In this case, the time-axis can be viewed as a homogeneous tree of order d with a root. We establish the system of linear equations in the form of the so-called noncommutative Givone-Roesser (NCGR) and noncommutative Fornasini-Marchesini (NCFM) models. As we have already seen, these models have similar mathematical structures as the GR and FM in Part 1. The application of the noncommutative d-variable Z-transform to the system of equations yields the transfer function which can be expressed as a formal power series in d noncommuting variables.

Motivation for the study of such formal power series comes from the Beck-Doyle paper [BD99] on the robust control for systems with structured uncertainty. We have seen that once we have an i/o map from input u to output y given by

$$y = LFT_u(U, \Delta)u = [D + C(I - \Delta A)^{-1}\Delta B]u,$$

we can replace the uncertainty operator Δ by the generalized system dynamics $Z(z)$ where z is a d-tuple of noncommuting indeterminants, say $z = (z_1, \ldots, z_d)$, $z_i z_j \neq z_j z_i$ unless $i = j$. Thus, we arrive at the transfer function of the NCFM system. If we assume in addition that Δ has the form

$$\Delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_d \end{bmatrix},$$

then we have the transfer function of the NCGR system. This gives a connection between the robust control theory and the noncommutative d-D system theory. The role of formal power series in analyzing linear time-invariant systems having time-varying structured uncertainties was introduced in the work of Doyle, Zhou, and Beck [Bec01, BD99, ZDG96] in a more formal, but less precise way.

In this Part, we study the "time-domain" properties of such systems such as reachability, controllability, observability, all of which are given in Chapter 7. In Chapter 8, we consider the minimality and the stability issues. The minimal realization problems for the NCFM and NCGR are also stated and solved in the connection with the recognizable system. Beck [Bec01] gave some partial results concerning minimal NCGR-realizations for a given formal power series; however, she did not provide the system of equations explicitly. We also establish the Lyapunov theory for the NCGR system.

The following is a list of remaining issues which may interest the reader and can be considered as topics for future research:

1. It is of interest to reformulate the results in this Part with the generalized system dynamics $Z(z)$ rather than the $Z_d(z)$ or $Z_r(z)$ (see Remark 19 on page 107), and this will give the general results in connection with the robust control as mentioned in [Bec01, BD99].

2. Suppose we are given a formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$. Then, one could replace noncommuting variable z by uncertainty operator δ, and the question would be, how can we construct a system realization $\{A, B, C, D\}$ from $T(\delta)$ so that $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is unitary. This can be stated precisely as follows:

   For a given formal power series $T(z) = \sum_{v \in \mathcal{F}_d} T_v z^v$, if $\|T(\delta)\| = \|\sum_{v \in \mathcal{F}_d} T_v \otimes \delta^v\| \leq 1$ for all $\delta = (\delta_1, \ldots, \delta_d)$, where for each j, $\delta_j \in \mathcal{L}(\ell^2)$ such that $\|\delta_j\| < 1$, then the conjecture is that there exists a *unitary operator* U:

$$U := \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{1,1} & \cdots & A_{1,d} & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{d,1} & \cdots & A_{d,d} & B_d \\ C_1 & \cdots & C_d & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_d \\ \mathcal{U} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_d \\ \mathcal{Y} \end{bmatrix} \tag{9.1}$$

so that $T(z) = D + C(I - Z_d(z)A)^{-1}Z_d(z)B$. If this conjecture is true, then the linear fractional transformation (LFT) model always exists for a given formal power series.

3. By using the result of the conjecture given above, one can prove the Bounded Real Lemma for the noncommutative d-D linear system which is stated as follows:

   Given a linear system which is NCGR-controllable/observable with the connecting operator U:
$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \bigoplus_{j=1}^{d} \mathcal{H}_j \\ \mathcal{U} \end{bmatrix} \mapsto \begin{bmatrix} \bigoplus_{j=1}^{d} \mathcal{H}_j \\ \mathcal{Y} \end{bmatrix}$$
   such that $T(z) = D + C(I - Z_d(z)A)^{-1}Z_d(z)B$ satisfies the norm constraint
$$\|T(\delta)\| \leq 1 \quad \text{whenever} \quad \left\| \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_d \end{bmatrix} \right\| \leq 1,$$
   then there is a positive definite matrix $X := \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_d \end{bmatrix} > 0$ so that
$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \tag{9.2}$$

4. Stability issue:

   (a) Stability for the NCGR systems defined in Section 8.2.

   (b) $(I - A\Delta)$ is invertible in $\mathcal{L}(\ell^2)$ for all $\Delta \in \mathbf{B}_\Delta$—$\Delta$'s are the time-varying structured uncertainties with bounded norm (see e.g. [Bec01, BD99, ZDG96]).

   (c) There is a structured similarity S so that SAS^{-1} is contractive—this is equivalent to: There exists a positive definite Y in $\mathcal{S}$ (i.e., commuting with the structured block-diagonal operators) so that $Y - A^*YA := Q$ is (strictly) positive definite.

   Shamma [Sha94, Sha91], and Feintuch-Markus [FM00] showed the equivalent between the statements (b) and (c). Our conjecture is that all statements above are equivalent.

Appendix A

Tensor Product

The tensor product notation gives a calculus which organizes everything and keeps everything relatively simple. Thus, it is more convenient for analysis purposes to represent the block-diagonal operators and the ℓ^2-space in the tensor product notation formally defined as follows:

Definition 45 (Algebraic Tensor Product [AM02]). Given two vector spaces E and F, the *algebraic tensor product*, denoted by $E \otimes F$, is the set of finite linear combinations

$$\left\{ \sum_{i=1}^{n} c_i e_i \otimes f_i \mid c_i \in \mathbb{C}, e_i \in E, \text{ and } f_i \in F \text{ with } n < \infty \right\} \tag{A.1}$$

modulo the equivalence relations

$$\begin{aligned} (e_1 + e_2) \otimes f &\sim (e_1 \otimes f) + (e_2 \otimes f) \\ e \otimes (f_1 + f_2) &\sim (e \otimes f_1) + (e \otimes f_2) \\ c(e \otimes f) &\sim (ce) \otimes f \\ c(e \otimes f) &\sim e \otimes (cf), \end{aligned}$$

A tensor is called an *elementary tensor* (or sometimes called a *pure tensor*) if it is of the form $e \otimes f$, for $e \in E$ and $f \in F$.

Note that an elementary tensor is a primitive, undefined object. Certain elements of $E \otimes F$ are associated with pairs of elements, the first from E and the second from F. It should also be noted that elements of the algebraic tensor product $E \otimes F$ do not have unique representations as finite linear combinations of elementary tensors. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. If $\mathcal{H}$ has an orthonormal basis $\{e_i \mid i \in I\}$ and $\mathcal{K}$ has an orthonormal basis $\{f_j \mid j \in J\}$, then $\{e_i \otimes f_j : (i, j) \in I \times J\}$ is an orthonormal basis of $E \otimes F$. It follows from this fact that if $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional spaces, then $\dim(\mathcal{H} \otimes \mathcal{K}) = \dim(\mathcal{H}) \dim(\mathcal{K})$. We also need the notion of an operator acting on $\mathcal{H} \otimes \mathcal{K}$ which is defined as follows: Let A and B be operators on $\mathcal{H}$ and $\mathcal{K}$, respectively (i.e., $A \in \mathcal{L}(\mathcal{H})$, and $B \in \mathcal{L}(\mathcal{K})$). We denote by $A \otimes B$ the operator defined on elementary tensors by

$$(A \otimes B) : (e \otimes f) \mapsto (Ae) \otimes (Bf) \tag{A.2}$$

and extended by linearity and continuity to all of $E \otimes F$.

Remark 28. In the finite dimensional case, one can make the identification

$$\mathcal{L}(E \otimes F) = \mathcal{L}(E) \otimes \mathcal{L}(F).$$

This still is roughly true, however, in the infinite dimensional case, but one has to be careful as to which topology is used to be close up finite linear combinations of elementary tensors. ▲

To demonstrate the application of the tensor product, let us consider the following concrete example which is adopted from [Ode79, page 76–78]. Let E and F denote the Euclidean vector spaces defined over the same field, say $E = \mathbb{R}^m$ and $F = \mathbb{R}^n$. Now suppose that $G = E \otimes F = \mathbb{R}^m \otimes \mathbb{R}^n$. Clearly, each element $g \in G$ can be represented in the form

$$g = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} e_i \otimes f_j,$$

where $e_i \in \mathbb{R}^m$, $f_i \in \mathbb{R}^n$, and the scalars $c_{i,j}$ are regarded as components of g relative to the basis $e_i \otimes f_j$. Once a basis is established, the operators of vector addition and scalar multiplication in G are particularly simple:

$$\begin{gathered} g_1 + g_2 = g_3 \Longrightarrow c^3_{i,j} = c^1_{i,j} + c^2_{i,j} \\ \alpha g_1 = g_2 \Longrightarrow c^2_{i,j} = \alpha c^1_{i,j} \\ \text{for } i = 1, \dots, m; j = 1, \dots, n. \end{gathered}$$

Since $\dim(E) = m$ and $\dim(F) = n$, this implies that $\dim(G) = mn$. To verify this, let us assume that $E = \mathbb{R}^3 = F$ together with the standard basis vectors, $\mathbf{1}_1 = (1, 0, 0)$, $\mathbf{1}_2 = (0, 1, 0)$, and $\mathbf{1}_3 = (0, 0, 1)$. Then the tensor product $G = E \otimes F$ has nine standard basis elements

$$\begin{matrix} \mathbf{1}_1 \otimes \mathbf{1}_1 & \mathbf{1}_1 \otimes \mathbf{1}_2 & \mathbf{1}_1 \otimes \mathbf{1}_3 \\ \mathbf{1}_2 \otimes \mathbf{1}_1 & \mathbf{1}_2 \otimes \mathbf{1}_2 & \mathbf{1}_2 \otimes \mathbf{1}_3 \\ \mathbf{1}_3 \otimes \mathbf{1}_1 & \mathbf{1}_3 \otimes \mathbf{1}_2 & \mathbf{1}_3 \otimes \mathbf{1}_3 \end{matrix}$$

If $e = (1, 2, -1)$ and $f = (-1, 4, -2)$, then $g = e \otimes f$ is given by

$$g = \sum_{i,j=1}^{3} c_{i,j} \mathbf{1}_i \mathbf{1}_j,$$

where

$$c = [c_{i,j}]_{i,j=1}^{3} = \begin{bmatrix} -1 & 4 & -2 \\ -2 & 8 & -4 \\ 1 & -4 & 2 \end{bmatrix}.$$

Similarly, let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be matrices of arbitrary sizes. Then their tensor product is defined as the matrix given by

$$A \otimes B = \begin{bmatrix} Ab_{1,1} & Ab_{1,2} & \cdots & Ab_{1,n} \\ Ab_{2,1} & Ab_{2,2} & \cdots & Ab_{2,n} \\ \vdots & \vdots & & \vdots \\ Ab_{m,1} & Ab_{m,2} & \cdots & Ab_{m,n} \end{bmatrix}.$$

For instance, let $B = I_{\ell^2}$, then

$$A \otimes B = A \otimes I_{\ell^2} = \text{diag}\{A, A, \dots\}.$$

Hence, one may write $\mathbf{A}$ as $A \otimes I_{\ell^2}$.

Therefore, by using the tensor product notation, the system Σ in (5.15) can be expressed as

$$\Sigma = \begin{cases} x &= (I_{\mathcal{H}} \otimes S)\left[(A \otimes I_{\ell^2})x + (B_1 \otimes I_{\ell^2})w + (B_2 \otimes I_{\ell^2})\right] u \\ z &= (C_1 \otimes I_{\ell^2})x + (D_1 \otimes I_{\ell^2})w + (D_2 \otimes I_{\ell^2})u \\ y &= (C_2 \otimes I_{\ell^2})x + (D_3 \otimes I_{\ell^2})w + (D_4 \otimes I_{\ell^2})u, \end{cases} \tag{A.3}$$

with $w = (\Delta \otimes I_{\ell^2})z$.

Or in the vector-matrix form:

$$\begin{bmatrix} x \\ w \\ y \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}} \otimes S & 0 & 0 \\ 0 & \Delta \otimes I_{\ell^2} & 0 \\ 0 & 0 & I_{\mathcal{Y} \otimes \ell^2} \end{bmatrix} \begin{bmatrix} A \otimes I_{\ell^2} & B_1 \otimes I_{\ell^2} & B_2 \otimes I_{\ell^2} \\ C_1 \otimes I_{\ell^2} & D_1 \otimes I_{\ell^2} & D_2 \otimes I_{\ell^2} \\ C_2 \otimes I_{\ell^2} & D_3 \otimes I_{\ell^2} & D_4 \otimes I_{\ell^2} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}.$$

By using the rows and columns permutation properly, one obtains the results as in Subsection 5.2.3 where each matrix entry $\widetilde{a}_{i,j}$ of an operator $\widetilde{A}$ is of the form $\widetilde{a}_{i,j} = a_{i,j} \otimes I_{\ell^2}$ if $A = [a_{i,j}]_{i,j=1}^{n}$ and $a_{i,j} \in \mathbb{C}$. The other operators $\widetilde{B}_i, \widetilde{C}_i, \widetilde{D}_j$ are defined in the similar way.

Consider a connecting operator U of the form

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \otimes \mathbb{C}^m \\ \mathcal{U} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{H} \otimes \mathbb{C}^n \\ \mathcal{Y} \end{bmatrix}$$

and suppose that we are given a homogeneous operator pencil

$$L(z) = \sum_{j=1}^{d} L_j z_j \text{ with } L_j \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m).$$

The associated system is

$$\Sigma \colon \begin{cases} x(w) &= (I_{\mathcal{H}} \otimes L(\sigma^*))Ax(w) + (I_{\mathcal{H}} \otimes L(\sigma^*))Bu(w) \\ y(w) &= Cx(w) + Du(w) \end{cases}$$

(where $\sigma^* = (\sigma_1^*, \ldots, \sigma_d^*)$ and $\sigma_j^* x(w) = x(g_j^{-1} w)$) with associated transfer function equal to the formal power series in d noncommuting variables $z = (z_1, \ldots, z_d)$ given by

$$T(z) = T_\Sigma(z) = D + C(I - (I_{\mathcal{H}} \otimes L(z))A)^{-1}(I_{\mathcal{H}} \otimes L(z))B \tag{A.4}$$

$$= \sum_{v \in \mathcal{F}_d} T_v z^v \tag{A.5}$$

where $T_v \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ (for $v \in \mathcal{F}_d$) are given by

$$T_\lambda = D, \qquad T_{vg_j} = CA^v B_j$$

where $A^v = A_{i_n} \cdots A_{i_1}$ if $v = g_{i_n} \cdots g_{i_1}$ and where

$$A_j := (I_{\mathcal{H}} \otimes L_j)A, \qquad B_j := (I_{\mathcal{H}} \otimes L_j)B.$$

The i/o operator associated with the system Σ and disturbance $\delta = (\delta_1, \ldots, \delta_d)$ is exactly

$$T(\delta) = D \otimes I_{\ell^2} + (C \otimes I_{\ell^2}) \left(I - \sum_{j=1}^{d} A_j \otimes \delta_j \right)^{-1} \left(\sum_{j=1}^{d} B_j \otimes \delta_j \right).$$

The theorem is:

Theorem A.1. *$T(\delta)$ can also be expressed as*

$$T(\delta) = \sum_{v \in \mathcal{F}_d} T_v \otimes \delta^v.$$

Proof. Use that $(A_j \otimes \delta_j) \cdot (A_k \otimes \delta_k) = A_j A_k \otimes \delta_j \delta_k$ to see that

$$\left(\sum_{j=1}^{d} A_j \otimes \delta_j \right)^n = \sum_{v\colon |v|=n} (A \otimes \delta)^v = \sum_{v\colon |v|=n} A^v \otimes \delta^v$$

and hence

$$\left(I - \sum_{j=1}^{d} A_j \otimes \delta_j \right)^{-1} = \sum_{n=0}^{\infty} \left(\sum_{j=1}^{d} A_j \otimes \delta_j \right)^n = \sum_{v \in \mathcal{F}_d} A^v \otimes \delta^v.$$

Use also that

$$(A^v \otimes \delta^v)(B_j \otimes \delta_j) = A^v B_j \otimes \delta^v \delta_j$$

to deduce that

$$T(\delta) = D \otimes I_{\ell^2} + (C \otimes I_{\ell^2}) \left(\sum_{v \in \mathcal{F}_d} A^v \otimes \delta^v \right) \left(\sum_{j=1}^{d} B_j \otimes \delta_j \right) = \sum_{v \in \mathcal{F}_d} T_v \otimes \delta^v$$

as desired. ■

For further discussion on the tensor product, the readers are referred to [AM02, Ma02, Ode79, Rya02].

Appendix B

MATLAB Source Code

For a detailed description of all LMI Control Toolbox functions, readers should refer to [GNLC95].

```
%FILE: LMI_experiment.m
%
%Suppose that for given n-1 data points
%{alpha_1,beta_1,omega_1},...,{alpha_n-1,beta_n-1,omega_n-1},
%there exist P and Q > 0 which satisfy the Bidisk interpolation condition
%OMEGA = P.ALPHA + Q.BETA where "." is the Schur product (entry times entry).
%We want to know that if we add one more point {alpha_n,beta_n,omega_n},
%can we find newP and newQ such that
%1. newP and newQ > 0, and
%2. submatrix (1:N-1,1:N-1) of newP (resp. newQ) is P (resp. Q)
%
clear all;
%
%given the data "alpha", "beta", and "omega"
%
alpha_row = input('given row vector of alpha = ');
beta_row = input('given row vector of beta = ');
omega_row = input('given row vector of omega = ');
%
%Construct the matrices from the given data where alpha and beta are diagonal
%
alpha = diag(alpha_row); beta = diag(beta_row);
N = length(omega_row);
```

```
for m=1:N
   for n=1:N
    omega(m,n) = 1-omega_row(m)*conj(omega_row(n));
    A(m,n) = 1-alpha_row(m)*conj(alpha_row(n));
    B(m,n) = 1-beta_row(m)*conj(beta_row(n));
   end;
end;
%
%Form the real matrix where A + Bi ~ [A -B; B A]
%
ALPHA = [real(alpha) -imag(alpha);imag(alpha) real(alpha)];
BETA = [real(beta) -imag(beta); imag(beta) real(beta)];
OMEGA = [real(omega) -imag(omega); imag(omega) real(omega)];
ALPHAcon = [real(alpha) imag(alpha);-imag(alpha) real(alpha)];
BETAcon = [real(beta) imag(beta); -imag(beta) real(beta)];
%
% Using LMI CONTROL TOOLBOX
%
setlmis([]);
%
% declare P = [Re(p) -Im(p); Im(p) Re(p)]
%
[P1,nP1,sP1] = lmivar(1,[N,1]);
[P2,nP2,sP2] = lmivar(3,skewdec(N,nP1));
[P,nP,sP] = lmivar(3,[sP1 -sP2; sP2 sP1]);
%
[Q1,nQ1,sQ1] = lmivar(1,[N,1]);
[Q2,nQ2,sQ2] = lmivar(3,skewdec(N,nQ1));
[Q,nQ,sQ] = lmivar(3,[sQ1 -sQ2; sQ2 sQ1]);
%
LMI_P = newlmi;
lmiterm([-LMI_P 1 1 P],1,1);                        % LMI #1: P
%
LMI_Q = newlmi;
lmiterm([-LMI_Q 1 1 Q],1,1);                        % LMI #2: Q
%
LMI_OMEGA_POS = newlmi;
```

```
lmiterm([-LMI_OMEGA_POS 1 1 0],OMEGA);
lmiterm([LMI_OMEGA_POS 1 1 P],1,1);
lmiterm([LMI_OMEGA_POS 1 1 P],-ALPHA,ALPHAcon);
lmiterm([LMI_OMEGA_POS 1 1 Q],1,1);
lmiterm([LMI_OMEGA_POS 1 1 Q],-BETA,BETAcon);
%
LMI_OMEGA_NEG = newlmi;
lmiterm([LMI_OMEGA_NEG 1 1 0],OMEGA);
lmiterm([-LMI_OMEGA_NEG 1 1 P],1,1);
lmiterm([-LMI_OMEGA_NEG 1 1 P],-ALPHA,ALPHAcon);
lmiterm([-LMI_OMEGA_NEG 1 1 Q],1,1);
lmiterm([-LMI_OMEGA_NEG 1 1 Q],-BETA,BETAcon);
%
lmisyst=getlmis;
%
[tmin,xfeas] = feasp(lmisyst);
Pf = dec2mat(lmisyst,xfeas,P);
Qf = dec2mat(lmisyst,xfeas,Q);
%
%Display the eigenvalues of Pf and Qf
%
eig_P = eig(Pf)
eig_Q = eig(Qf)
%
%Form the complex-valued matrices from the real matrices obtained by the
%LMI Toolbox
%
p = Pf(1:N,1:N) + i*Pf(N+1:2*N,1:N)
q = Qf(1:N,1:N) + i*Qf(N+1:2*N,1:N)
rhs = p.*A + q.*B;
fprintf('the value of  [P]*[alpha] + [Q]*[beta] is  \n'); rhs
fprintf('the data matrix (omega) is \n'); omega
%
%Suppose now that we add one more point, say {alpha_n,beta_n,omega_n}
%
alpha_n = input('given one more point of alpha = ');
beta_n = input('given one more point of beta = ');
```

```
omega_n = input('given one more point of omega = ');
%
new_alpha = [alpha_row alpha_n];  new_beta = [beta_row beta_n];
new_omega = [omega_row omega_n];
ext_alpha = diag(new_alpha);  ext_beta = diag(new_beta);
newN = N+1;
for m=1:newN
   for n=1:newN
    ext_omega(m,n) = 1-new_omega(m)*conj(new_omega(n));
    ext_A(m,n) = 1-new_alpha(m)*conj(new_alpha(n));
      ext_B(m,n) = 1-new_beta(m)*conj(new_beta(n));
   end;
end;
%
%Form the real matrix where A + Bi ~ [A -B; B A]
%
ext_ALPHA = [real(ext_alpha) -imag(ext_alpha);imag(ext_alpha) real(ext_alpha)];
ext_BETA = [real(ext_beta) -imag(ext_beta); imag(ext_beta) real(ext_beta)];
ext_OMEGA = [real(ext_omega) -imag(ext_omega); imag(ext_omega) real(ext_omega)];
ext_ALPHAcon = [real(ext_alpha) imag(ext_alpha);-imag(ext_alpha) real(ext_alpha)];
ext_BETAcon = [real(ext_beta) imag(ext_beta); -imag(ext_beta) real(ext_beta)];
%
%we need to ensure that the left-top corner block-matrices (1:N-1,1:N-1) of
%the new reconstruct matrices are the same as those in the matrices P and Q
%that we already computed from the first step.
%Therefore, we need to impose few conditions here
%
Re_P = Pf(1:N,1:N); Im_P = Pf(N+1:2*N,1:N);
firstblock_P = [Re_P zeros(N,1); zeros(1,N+1)];
secondblock_P = [-Im_P zeros(N,1); zeros(1,N+1)];
thirdblock_P = [Im_P zeros(N,1);zeros(1,N+1)];
forthblock_P = firstblock_P;
ext_P = [firstblock_P secondblock_P; thirdblock_P forthblock_P];
%
Re_Q = Qf(1:N,1:N); Im_Q = Qf(N+1:2*N,1:N);
firstblock_Q = [Re_Q zeros(N,1); zeros(1,N+1)];
secondblock_Q = [-Im_Q zeros(N,1); zeros(1,N+1)];
```

```
thirdblock_Q = [Im_Q zeros(N,1);zeros(1,N+1)];
forthblock_Q = firstblock_P;
ext_Q = [firstblock_Q secondblock_Q; thirdblock_Q forthblock_Q];
%
firstblock_M = [diag(ones(1,N)) zeros(N,1);zeros(1,N+1)];
secondblock_M = zeros(N+1,N+1);
thirdblock_M = secondblock_M;
forthblock_M = firstblock_M;
M = [firstblock_M secondblock_M; thirdblock_M forthblock_M];
%
% Using LMI CONTROL TOOLBOX
%
setlmis(lmisyst)
%
% declare P = [Re(p) -Im(p); Im(p) Re(p)]
%
[newP1,newnP1,newsP1] = lmivar(1,[newN,1]);
[newP2,newnP2,newsP2] = lmivar(3,skewdec(newN,newnP1));
[newP,newnP,newsP] = lmivar(3,[newsP1 -newsP2; newsP2 newsP1]);
%
[newQ1,newnQ1,newsQ1] = lmivar(1,[newN,1]);
[newQ2,newnQ2,newsQ2] = lmivar(3,skewdec(newN,newnQ1));
[newQ,newnQ,newsQ] = lmivar(3,[newsQ1 -newsQ2; newsQ2 newsQ1]);
%
LMI_newP = newlmi;
lmiterm([-LMI_newP 1 1 newP],1,1);                        % LMI #1: P
%
LMI_newQ = newlmi;
lmiterm([-LMI_newQ 1 1 newQ],1,1);                        % LMI #2: Q
%
LMI_OMEGA_POS = newlmi;
lmiterm([-LMI_OMEGA_POS 1 1 0],ext_OMEGA);
lmiterm([LMI_OMEGA_POS 1 1 newP],1,1);
lmiterm([LMI_OMEGA_POS 1 1 newP],-ext_ALPHA,ext_ALPHAcon);
lmiterm([LMI_OMEGA_POS 1 1 newQ],1,1);
lmiterm([LMI_OMEGA_POS 1 1 newQ],-ext_BETA,ext_BETAcon);
%
```

```
LMI_OMEGA_NEG = newlmi;
lmiterm([LMI_OMEGA_NEG 1 1 0],ext_OMEGA);
lmiterm([-LMI_OMEGA_NEG 1 1 newP],1,1);
lmiterm([-LMI_OMEGA_NEG 1 1 newP],-ext_ALPHA,ext_ALPHAcon);
lmiterm([-LMI_OMEGA_NEG 1 1 newQ],1,1);
lmiterm([-LMI_OMEGA_NEG 1 1 newQ],-ext_BETA,ext_BETAcon);
%
LMI_ext_P_NEG = newlmi;
lmiterm([LMI_ext_P_NEG 1 1 0],ext_P);
lmiterm([-LMI_ext_P_NEG 1 1 newP],M,M);
%
LMI_ext_P_POS = newlmi;
lmiterm([-LMI_ext_P_POS 1 1 0],ext_P);
lmiterm([LMI_ext_P_POS 1 1 newP],M,M);
%
newlmisyst=getlmis;
%
[Tmin,newxfeas] = feasp(newlmisyst);
newPf = dec2mat(newlmisyst,newxfeas,newP);
newQf = dec2mat(newlmisyst,newxfeas,newQ);
%
%Display the eigenvalues of Pf and Qf
%
eig(newPf)
eig(newQf)
l =  length(newPf);
%
%Form the complex-valued matrices from the real matrices obtained by the
%LMI Toolbox
%
new_p = newPf(1:l/2,1:l/2) + i*newPf(l/2 +1:l,1:l/2)
new_q = newQf(1:l/2,1:l/2) + i*newQf(l/2 +1:l,1:l/2)
rhs = new_p.*ext_A + new_q.*ext_B;
fprintf('the value of [ext_P]*[ext_alpha] + [ext_Q]*[ext_beta] is \n'); rhs
pause;
fprintf('the data matrix (ext_omega) is \n'); ext_omega
```

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Index

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