A Postivestellensatz for Non-commutative Polynomials.

July 2, 2002Keywords: Positivestellensatz, Semialgebraic sets, ??

TO DO:

SCOTT

EXAMPLE(S) HERE which can be used later to see that in some cases ?must consider operators and not matrices. SCOTT will fill??

Write out Proof of Lemma giving absorbing etc for $\mathcal{N}_*, \mathcal{N}$ cases. then zcollins can type it

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MAYBE GLOBAL CHANGES ??:

MAYBE kill ℓ superscripts on $\mathcal{C}^\ell_\mathcal{P}$ much later.

Contents

Abstract

YAK YAK ??

1 Introduction

First we define NC polynomials and set notation for them. Then we describe a representaion for certain polynomials, then we state our non-commutative Positivestellenstaz which says which polynomials have this representation.

1.1 NC Polynomials and Special Classes

Let $\mathcal{F}_\}$ denote the free semi-group on the g generators $x = \{x_1, \dots, x_g\}$; in common language $\mathcal{F}_g$ is a set of words in $x_1, \cdots, x_g$. Given a word $w \in \mathcal{F}_g$,

$$w = x_{j_1} x_{j_2} \cdots x_{j_n}, \tag{1}$$ eq:wordw

a real Hilbert space H, and a tuple $X = (X_1, \dots, X_g)$ of operators on H, let

$$X^w = X_{j_1} X_{j_2} \cdots X_{j_n}. \tag{2}$$ eq:Xw

Our results apply to several classes of polynomials and we now introduce these classes.

BILL LOOK UP EXPOSITION IN hmcc.tex ?? [6/10 yeah. seems like we had it pretty polished there.]

1.1.1 Polynomials in Symmetric Entries, $\mathcal{N}$

Let $\mathcal{N}$ denote the polynomials, over the field of real numbers $\mathbb{R}$, in the noncommuting generators $x = \{x_1, \cdots, x_g\}$. $\mathcal{N}$ consists of real linear combinations of words w in x. Also $\mathcal{N}$ has an involution ^{T}, that is, given the word w from $\mathcal{F}_g$ viewed as an element of $\mathcal{N}$, define

$$w^T = x_{j_n} \cdots x_{j_2} x_{j_1}$$

and if $p = \sum p_w w \in \mathcal{N}$, define $p^T = \sum p_w w^T$. Here we emphasize that each p_w is a real number. We call a polynomial p in $\mathcal{N}$ symmetric provided $p^T = p$.

Often we shall be interested in evaluating a polynomial p in $\mathcal{N}$ on a tuple of matrices or operators that is, if $p = \sum p_w w$ is in $\mathcal{N}$ and $X = (X_1, \dots, X_g)$ is a tuple of real symmetric operators on a real Hilbert space define $p(X) = \sum p_w X^w$, where $X^w = X_{j_1} X_{j_2} \dots X_{j_n}$ as before. Thus, in the case of $\mathcal{N}$, we only allow substitution by real symmetric operators. Often the Hilbert space is simply $\mathbb{R}^\ell$, and so the operators X_j are real symmetric $\ell \times \ell$ matrices.

BILL -EXAMPLES ??

1.1.2 General Polynomials, $\mathcal{N}_*$

Let $\mathcal{N}_*$ denote the polynomials in the $2g$ non-commutative symbols $\{x_1, \ldots, x_g, x_1^T, \ldots, x_g^T\}$. $\mathcal{N}_*$ has an involution ^{T} defined on it on it, which behaves in the conventional way, for example,

$$\text{if } w = x_{j_1} x_{j_2} \cdots x_{j_n}, \text{ then } w^T = x_{j_n}^T x_{j_{n-1}}^T \cdots x_{j_1}^T$$
$$\text{if } w = x_{j_1} x_{j_2}^T x_{j_3} \cdots x_{j_n}, \text{ then } w^T = x_{j_n}^T x_{j_{n-1}}^T \cdots x_{j_2} x_{j_1}^T.$$

The involution on $\mathcal{N}_*$ can be conveniently thought of in another way. The polynomials in $\mathcal{N}_*$ as polynomials in the $2g$ non-commuting variables $\{x_1, \ldots, x_g, x_{g+1}, \ldots, x_{2g}\}$ and identify x_{g+j} with x_j^T. In this way the involution on $\mathcal{N}_*$ is the same as that on $\mathcal{N}$ in $2g$, rather than g, variables.

Often we shall be interested in evaluating a polynomial p in $\mathcal{N}$ on a tuple of matrices or operators. To define the substitution for $\mathcal{N}_*$, given a word $w \in \mathcal{F}_{2g}$, and the tuple X consisting of square matrices with entries from $\mathbb{R}$, substitute X_j for x_j and X_j^T for x_{j+g} for $1 \leq j \leq g$ and extend to all of $\mathcal{N}_*$ by linearity. For instance, if $g = 2$, for

$$p = x_1 x_2^T x_1 + 3x_2^T x_1 x_1^T, \text{ we have } p(X) = X_1 X_2^T X_1 + 3X_2^T X_1 X_1^T.$$

Here X_j^T is the transpose of X_j.

We can also make the analogous substitution if $X = (X_1, \ldots, X_g)$ are operators on a real Hilbert space.

BILL -EXAMPLES ?? Jeff says fwd and bkwd shift ops on ℓ^2.??

1.1.3 Hereditary Polynomials, $\mathcal{N}_*\mathcal{N}$

One type of polynomial we consider is a subset of $\mathcal{N}_*$ called the hereditary polynomials, denoted by $\mathcal{N}_*\mathcal{N}$. A polynomial $p \in \mathcal{N}_*$ is **hereditary** if the transposes, if any, always appear on the left. Note that the product of two hereditary polynomials need not be an hereditary polynomial.

In the hereditary case, given the word w, the definition of the word w^T is induced from the definition of ^{T} on $\mathcal{N}_*$. Note hereditary words have the form

$$X^{v^T w} = (X^v)^T X^w. \tag{3}$$ eq:hword1

If $p = \sum p_{v,w} v^T w$ is in $\mathcal{N}_*\mathcal{N}$, let p^T denote $p^T = \sum p_{v,w}(v^T w)^T$, which the reader notes is also a hereditary polynomial. Also given a Hilbert space H and a tuple $X = (X_1, \dots, X_g)$ of operators on H, the definition of $p(X)$ is induced from that on $\mathcal{N}_*$.

EXAMPLE(S) HERE which can be used later to see that in some cases ?must consider operators and not matrices. SCOTT will fill??

1.1.4 Matrix Valued Polynomials, $M_{l\times l}(\mathcal{N}_*)$ etc

We wish also to consider matrix-valued NC polynomials or, equivalently, NC polynomials with matrix coefficients. Let $M_{a\times b}$ denote the $a \times b$ matrices with entries from $\mathbb{R}$.

A $M_{a\times b}$-valued hereditary polynomial is a polynomial of the form $p = \sum p_{v,w} \otimes v^T w$, where the sum is finite and each $p_{v,w} \in M_{a\times b}$. Given our tuple X, the substitution rule is now

$$p(X) = \sum p_{v,w} \otimes X^{v^T w}$$

and the involution becomes $p^T = \sum p_{v,w}^T \otimes (v^T w)^T$, where $p_{v,w}^T$ is of course the transpose of the matrix $p_{v,w}$. The definitions for $\mathcal{N}$ and $\mathcal{N}_*$ are similar. A matrix-valued NC polynomial p is **symmetric** provided $p^T = p$, which is equivalent to the coefficient matrices satisfying $p_{v,w}^T = p_{w^T,v^T}$. These classes of **symmetric matrix-valued polynomials** are denoted $M_{l\times l}(\mathcal{N})$, $M_{l\times l}(\mathcal{N}_*)$, or $M_{l\times l}(\mathcal{N}_*\mathcal{N})$. We use $M_{\infty\times\infty}$ to denote $\cup_{\ell>0} M_{l\times l}$; it is not a closed set.

BILL PUT IN EXAMPLE ??

If p is an $M_{b\times c}$-valued polynomial, q is a $M_{b\times a}$-valued polynomial, and r is an $M_{c\times d}$-valued polynomial, then $q^T pr$ is a $M_{a\times d}$-valued polynomial.

There are no consistency requirements in the $\mathcal{N}$ and $\mathcal{N}_*$ cases other than q and r should be $\mathcal{N}$ or $\mathcal{N}_*$ polynomials respectively. However, in the hereditary case q and r should both be transpose free. A special case of particular importance is when p is $M_{b\times b}$-valued, q is $M_{b\times a}$-valued and $r = q^T$.

1.2 Decomposition as Weighted Sums of Squares

Fix a collection of symmetric matrix-valued polynomials $\mathcal{P}$ from either $M_{\infty\times\infty}(\mathcal{N})$, $M_{\infty\times\infty}(\mathcal{N}_*)$, or $M_{\infty\times\infty}(\mathcal{N}_*\mathcal{N})$.

Let $\mathcal{C}^\ell_{\mathcal{P}}$ denote positive linear combinations of $s^T ps$ and $r^T r$ where $p \in \mathcal{P}$ and the sizes of s and r are such that the products make sense and result in $\ell \times \ell$ matrix-valued polynomials. Hence, elements of $\mathcal{C}^\ell_{\mathcal{P}}$ are, $M_{l\times l}$-valued NC polynomials of the form

$$q = \sum_1^N s_j^T p_j s_j + \sum_1^M r_k^T r_k, \tag{4}$$

eq:decomp

for some integers M and N, polynomials $p_j \in \mathcal{P}$ and polynomials s_j and r_k, where say p_j is $M_{\ell_j\times\ell_j}$-valued, s_j is $M_{\ell_j\times\ell}$-valued and r_j is $M_{1,\ell}$-valued. We emphasize that, while $\mathcal{P}$ may be an infinite set of polynomials, the decomposition above only requires a finite subset of them. Let $\mathcal{C}_{\mathcal{P}}$ denote the union of $\mathcal{C}^\ell_{\mathcal{P}}$ over ℓ. Note that the polynomials in $\mathcal{C}_{\mathcal{P}}$ are symmetric.

1.3 Domain of Positivity

The **positivity domain of** $\mathcal{P}$, denoted $\mathcal{D}_{\mathcal{P}}$, is the collection of tuples X of operators on a real separable Hilbert space H such that $p(X)$ is positive (semi-definite) for all $p \in \mathcal{P}$. Note that $\mathcal{D}_{\mathcal{P}}$ is really a graded object with the grading given as follows. For each $\nu = 1, 2, \ldots, \infty$, fix a nest of Hilbert spaces H_ν of dimension ν inside of H, then

$$\mathcal{D}_{\mathcal{P}} \supset \cup_\nu \{X = (X_1, \ldots, X_g) : X_j \in \mathcal{B}(H_\nu),\ p(X) \succeq 0 \text{ for every } p \in \mathcal{P}\},$$

where $\mathcal{B}(H)$ is the collection of (bounded) operators on H and $p(X) \succeq 0$ means $p(X)$ is PSD.

A positivity domain $\mathcal{D}_{\mathcal{P}}$ is called **convex** provided that, if X and Y are both operator tuples on the same Hilbert space H and both X and Y lie in $\mathcal{D}_{\mathcal{P}}$, then convex combinations $c_1 X + c_2 Y$ belong to $\mathcal{D}_{\mathcal{P}}$. Here real numbers $c_1, c_2 \geq 0$ satisfy $c_1 + c_2 = 1$. A **positivity domain is bounded** provided there is a constant $C > 0$ such that if $X \in \mathcal{D}_{\mathcal{P}}$, then $\|X_j\| \leq C$ for each $j = 1, 2, \ldots, g$.

We now define a special set of polynomials, and state our first lemma. Henceforth set $b_j := C^2 - x_j^T x_j$ and let b denote the set of polynomials $b := \{b_1, \cdots, b_g\}$.

thm:bd2 **Lemma 1.1** *For any j between 1 and g*

$$\tilde{b}_j := C^2 - x_j x_j^T \text{ belongs to } \mathcal{D}_{b_j}.$$

Proof.

$$\begin{pmatrix} 1 & -x \end{pmatrix} \begin{pmatrix} 1 & x \\ x^T & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -x^T \end{pmatrix} = 1 - xx^T.$$

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1.4 A NC Positivestellensatz

We shall be working only with bounded positivity domains, in which case we can, with no loss of generality, assume that b is appended to $\mathcal{P}$. More precisely, we take the convention:

- In the hereditary case and in the $\mathcal{N}_*$ case , for each j we adjoin the polynomial $C^2 - x_j^T x_j$ to $\mathcal{P}$ and obtain a bigger set $\tilde{\mathcal{P}}$.
- In the $\mathcal{N}$ case, for each j we adjoin the polynomial $C^2 - x_j^2$ to $\mathcal{P}$ and obtain a bigger set $\tilde{\mathcal{P}}$.

thm:main **Theorem 1.2** *Suppose $\mathcal{P}$ is a ??finite -SCOTT ?? DELETE?? list of symmetric matrix polynomials in $M_{\infty\times\infty}(\mathcal{N}_*)$, or respectively $M_{\infty\times\infty}(\mathcal{N})$, or respectively $M_{\infty\times\infty}(\mathcal{N}_*\mathcal{N})$. Suppose $\mathcal{D}_\mathcal{P}$ is bounded and q is a symmetric $M_{l\times l}$-valued NC polynomial (in either $M_{l\times l}(\mathcal{N})$, $M_{l\times l}(\mathcal{N}_*)$, or $M_{l\times l}(\mathcal{N}_*\mathcal{N})$ depending upon $\mathcal{P}$). If $q(X)$ is strictly positive definite on $\mathcal{D}_\mathcal{P}$, that is, if $q(X) \succ 0$ whenever $X \in \mathcal{D}_\mathcal{P}$, then $q \in \mathcal{C}^\ell_{\tilde{\mathcal{P}}}$.*

If in addition $\mathcal{D}_\mathcal{P}$ is convex, we need only verify $q(X) \succ 0$ for those $X \in \mathcal{D}_\mathcal{P}$ which are defined on a Hilbert space of dimension at most $\ell\sum_0^d (2g)^n$. From this test on finite dimensional matrices we obtain $q \in \mathcal{C}^\ell_{\tilde{\mathcal{P}}}$.

The proof has two parts which dictates the organization of the rest of the paper. The first is a Hahn-Banach result which separates $\mathcal{C}^\ell_\mathcal{P}$ from any polynomial q outside it with a linear functional λ. The second represents such linear functionals λ using a matrix tuple X. That q is outside $\mathcal{C}^\ell_\mathcal{P}$ forces $q(X)$ not to be PSD. Before launching into all of this we have a section which presents properties of convex positivity domains, since it is a pleasent topic, and we prove the last assertion of the Theorem thm:main 1.2 in this section, see Proposition thm:finiteI 2.3.

1.4.1 Complex Hilbert Spaces - ?? FIX LATER

Every Hilbert space over the complex numbers is automatically a real Hilbert space. Indeed the results we have stated hold for complex coefficient symmetric NC polynomials provided we test them on matrices with complex entries.

Here we replace ^{T} with the Hermetian adjoint *.

Key defs are : ?? FILLIN??

The main theorem for complex coefficient polynomials is

thm:mainComp **Theorem 1.3** *Suppose $\mathcal{P}$ is a finite list of self-adjoint matrix polynomials in $M_{\infty\times\infty}$. Suppose $\mathcal{D}_{\mathcal{P}}$?? is bounded and q is a symmetric $M_{l\times l}$??-valued NC polynomial (in either $M_{l\times l}(\mathcal{N}c)$, $M_{l\times l}(\mathcal{N}c_*)$, or $M_{l\times l}(\mathcal{N}c_*\mathcal{N}c)$ depending upon $\mathcal{P}$). If $q(X)$ is strictly positive definite on $\mathcal{D}_{\mathcal{P}}$, that is, if $q(X) \succ 0$ whenever $X \in \mathcal{D}_{\mathcal{P}}$, then $q \in \mathcal{C}_{\tilde{\mathcal{P}}}$??.*

1.4.2 Related Results— NEEDS TUNING

We are aware of several variations of Theorem thm:main 1.2 with proofs much like the one given here. In fact, the proof here borrows heavily from Putinar and Vasilescu [PV0??], although the results there are for the commutative $\mathcal{N}$ case, and much of the significance of their work is for the case of undbounded positivity regions.

Agler, in his seminal work on Schur class functions on the polydisc, begins with the collection $\mathcal{P} = \{1 - x_j^* x_j : j = 1, 2, \ldots, g\}$ (here * is the complex transpose, rather than just transpose) and shows that a matrix-valued analytic function W, on the g-fold polydisc, such that $W(X)$ is a contraction for each tuple of commuting strict contractions $X = (X_1, \ldots, X_g)$ can be written as $I - W(z)W(w)^* = \sum H_j(z)(1 - z_j\overline{w_j})H_j(w)^*$. Agler and McCarthy AM01 [?] prove a generalization involving a finite collection of scalar polynomials in place of $1 - x_j^* x_j$.

Our noncommutative Positivestellensatz result is also related to a result of Blecher and Paulsen which treats a contractive, rather than positive, version of (1.2) which, when translated as best I can, treats a situation more special than hereditary. For instance, the Blecher and Paulsen result specialized to the polydisc says, if W is matrix-valued and analytic in a neighborhood of the polydisc, then $W(X)$ is a contraction for each tuple X of commuting strict contractions, if and only if W can approximated (in a suitable sense) by functions of the form $C_0D_1C_1D_2C_2 \ldots D_nC_n$, where the C_j are (constant) contraction matrices, the D_j are diagonal matrices whose diagonal entries are contractive analytic functions on the polydisc.

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BILL. This could be omitted? Their argument involves rep theorem for operator algebras which in the end is really a HB separation argument. However, the actually statement is a bit afield from ours.

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When all of the x_j commute but they do not necssarily commute with their transpose x_k Theorem 1.2 should go through too, however we have not checked it carefully.

Also related to the NC Positivestellensatz is polynomials which are actaully sums of squares, that is, they have the representation ?? [**??**]

- For x_j which are unitary McCullough proved ?? SCOTT ?? []??
- A noncommutative polynomial q in $\mathcal{N}_*$ which is positive "everywhere", that is on $\mathcal{C}_{\mathcal{P}}$ where $\mathcal{P}$ consisists only of the polynomial $p = 1$, is a sum of squares. Helton [**?**]

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2 Convex Positivity Domains

Now we specialize our noncommutative Positivestellenstaz to convex domains and find that the structure is very rigid. We shall prove the finite dimensionality assertion of Theorem thm:main 1.2. To aid with the proof we introduce several properties of positivity domains and several natural notions of convexity.

2.1 Properties of Positivity Domains

Proposition 2.1 *Given a set $\mathcal{P}$ of polynomials in $M_{l\times l}(\mathcal{N}_*)$, or in $M_{l\times l}(\mathcal{N})$, or in $M_{l\times l}(\mathcal{N}_*\mathcal{N})$. The positivity domain $\mathcal{D}_\mathcal{P}$ has the properties that it is closed with respect to the operations:*

1. **Reducing** *Suppose X is a tuple of operators on Hilbert space H which lies in $\mathcal{D}_\mathcal{P}$ and K is a subspace of H which is invariant under every operator X_j and X_j^T in the tuple X, and suppose V is an isometry from K into H. Then the tuple of operators V^TXV on K is in $\mathcal{D}_\mathcal{P}$. In particular, $\mathcal{D}_\mathcal{P}$ is closed under conjugation U^TXU by a unitary operator U.*

2. **Direct Sums** *Suppose X is a tuple of operators on Hilbert space H, and suppose Y is a tuple of operators on Hilbert space K. If both X and Y are in $\mathcal{D}_\mathcal{P}$, then the direct sum $X\oplus Y$, which is a tuple of operators on $H\oplus K$, is in $\mathcal{D}_\mathcal{P}$.*

Proof This is an immediate consequence of the fact that

$$p(X\oplus Y) = p(X)\otimes p(Y).$$

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2.2 Convexity

A positivity domain $\mathcal{D}_{\mathcal{P}}$ is **closed w.r.t compression** means: suppose X is a tuple of operators on Hilbert space H which lies in $\mathcal{D}_{\mathcal{P}}$ and K is a Hilbert space and suppose V is an isometry from K into H. Then the tuple of operators $V^T XV$ on K is in $\mathcal{D}_{\mathcal{P}}$. In particular, $\mathcal{D}_{\mathcal{P}}$ is closed under conjugation $U^T XU$ by a unitary operator.

Compression is a stronger property than reducing, in that K need not be an invariant subspace and probably it is not enjoyed by all positivity domains. However, we now see that convex positivity domains are closed w.r.t compression.

thm:compress **Lemma 2.2** *Every convex reducing domain is closed w.r.t compression. In particular, every convex positivity domain is closed w.r.t compression.*

Conversely, a domain $\mathcal{D}$ which is closed w.r.t. direct sums and compression is convex.

Proof

Suppose X is a tuple form $\mathcal{B}(H)$ suppose K is subspace of H_1. Calculate partition X w.r.t K to obtain $X = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$. We want to show that the projection $V := (I \ \ 0)$ produces $A = V^T XY$ in $\mathcal{D}_{\mathcal{P}}$.

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A & B \\ B^T & D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which belongs to $\mathcal{D}_{\mathcal{P}}$, because it's convex and because $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is unitary. Since K reduces $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ we get that $A \in \mathcal{D}_{\mathcal{P}}$.

Proof of the converse. Given real numbers a_1, a_2 satisfying $a_1^2 + a_2^2 = 1$ construct an isometry $V : H \oplus H \to H$ by

$$V := \begin{pmatrix} a_1^T & a_2^T \end{pmatrix}.$$

If $X, Y \in \mathcal{D}$, then their direct sum $X \oplus Y$ is in $\mathcal{D}$, so

$$a_1^2 X + a_2^2 Y = a_1^T X a_1 + a_2^T Y a_2 = V^T (X \oplus Y) V$$

is in $\mathcal{D}$. ••

2.3 Proof of Finite Dimensionality

For a convex positivity domain $\mathcal{D}$ the Theorem thm:main 1.2 gives a bound on the dimension of Hilbert space H needed in the Positivestellensatz. Now we prove this fact which we state again in a form suited to our proof.

thm:finiteI **Proposition 2.3** *Let $\mathcal{P}$ be a collection of symmetric polynomials from $M_{l\times l}(\mathcal{N}_*))$, respectively $M_{l\times l}(\mathcal{N}))$, respectively $M_{l\times l}(\mathcal{N}_*\mathcal{N})$ and suppose $\mathcal{D}_\mathcal{P}$ is a bounded convex positivity domain. If q is a symmetric $\ell \times \ell$ matrix-valued $M_{l\times l}(\mathcal{N}_*))$ polynomial of degree d and if $q \notin \mathcal{C}_\mathcal{P}$, then there exists a Hilbert space H of dimension at most $\ell \sum_0^d (2g)^n$, a non-zero vector $\gamma \in H$, and a tuple $X = (X_1, \ldots, X_g)$ of operators on H such that $X \in \mathcal{D}_\mathcal{P}$, but $< q(X)\gamma, \gamma > \leq 0$.*

From the Positivestellensatz, there exists a tuple $Z = (Z_1, \ldots, Z_g)$ acting on a Hilbert space H and a non-zero vector $\gamma = \oplus \gamma_j \in \oplus_1^\ell H$ such that $< q(Z)\gamma, \gamma > \leq 0$. Here $q \in M_{l\times l}(\mathcal{N}_*)$ and has degree d. Let

$$K_d = \text{span}\{Z^w \gamma_j : \ w \text{ is a word of length at most } d, \ j = 1, \cdots, \ell\} \subset H.$$

Then K has dimension at most equal to the total number of words w of length at most d times ℓ. Since there are $2g$ generators for the words in $\mathcal{N}_*$, we get K_d has dimension at most $\ell \sum_0^d (2g)^n$.

If P is the projection of H onto K, then $< q(PZP)\gamma, \gamma > = < q(Z)\gamma, \gamma > \leq 0$. On the other hand, convexity implies $PZP \in \mathcal{D}_\mathcal{P}$, since by Lemma thm:compress 2.2 the domain $\mathcal{D}_\mathcal{P}$ is closed under compression. This proves the theorem by taking $X := PZP$. ••

A tighter bound on dimension is immediate for $\mathcal{N}$ and $\mathcal{N}_*\mathcal{N}$, since there are fewer words. The bound is $\ell \sum_0^d (g)^n$. CHECK?? Jeff??

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ORIGINAL PROOF - DELETE ??

From the Positivestellensatz, there exists a tuple $Z = (Z_1, \ldots, X_g)$ acting on a Hilbert space H and a non-zero vector $\gamma = \oplus\gamma_j \in \oplus_1^\ell H$ such that $< q(X)\gamma, \gamma >$. Here $q \in M_{l\times l}(\mathcal{N}_*)$ and has degree d. Let

$$\mathcal{M} = \text{span}\{X^w\gamma_j : w \text{ is a word of length at most } di,\ j = 1, \cdots, \ell\} \subset H.$$

Then $\mathcal{M}$ has dimension at most $\ell \sum_0^d (2g)^n$ and if P is the projection of H onto $\mathcal{M}$, then $< q(PXP)\gamma, \gamma >=< p(X)\gamma, \gamma >\leq 0$. On the other hand, $PXP \in \mathcal{D}_\mathcal{P}$ by the heavy convex hypothesis.

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3 Separating $\mathcal{C}_{\mathcal{P}}$ from Outsiders with a Linear Functional

sec:sepCP

This section gives a Hahn-Banach result for sets of the type $\mathcal{C}^{\ell}_{\mathcal{P}}$. It is based on a general theorem called the Krein extension theorem, which applies to linear functionals which are non-negative on cones meeting certain properties. In this section we first show that the cone $\mathcal{C}^{\ell}_{\mathcal{P}}$ has the needed properties. Then we build our separating linear functional.

3.1 Properties of $\mathcal{C}_{\mathcal{P}}$

We shall be working with bounded sets $\mathcal{D}_{\mathcal{P}}$ and we first focus on the structure related to boundedness. Recall $b_j = C^2 - x_j^T x_j$ and b denotes the set of polynomials $b = \{b_1, \cdots, b_g\}$; likewise $\tilde{b}$ corresponds to $\tilde{b}_j = C^2 - x_j x_j^T$.

lem:incalC **Lemma 3.1** *If $w \in \mathcal{F}_g$ is a word of length n,*

1. *in $\mathcal{N}_*$, then $C^{2n} - w^T w \in \mathcal{C}^{\ell}_{\{b,\tilde{b}\}}$.*
2. *in $\mathcal{N}$, then $C^{2n} - w^T w \in \mathcal{C}^{\ell}_{b}$.*
3. *in $\mathcal{N}_*\mathcal{N}$, then $C^{2n} - w^T w \in \mathcal{C}^{\ell}_{b}$.*

Proof. First consider the case of hereditary words w. We use $(C^2 - x_j^T x_j) \subset \mathcal{C}^{\ell}_{b}$ and argue by induction. Accordingly suppose the result is true for the word v of length n and consider $w = x_{j_0} v$. We have,

$$(C^{2n+2} - w^T w) = C^2 (C^{2n} - v^T v) + v^T (C^2 - x_{j_0}^T x_{j_0}) v. \qquad (5)$$ eq:CbdHerd

This implies

$$(C^{2n+2} - w^T w) \subset C^2 (C^{2n} - v^T v) + \mathcal{C}^{\ell}_{b}$$

which yields the induction step for going from hereditary word v to hereditary word w.

When w is a word in x_j, x_k^T which is not necessarily hereditary we proceed as before but also consider $w = x_{j_0}^T v$, and obtain

$$(C^{2n+2} - w^T w) = C^2(C^{2n} - v^T v) + v^T(C^2 - x_{j_0} x_{j_0}^T)v. \tag{6}$$

eq:CbdHerdT

Combine (5) and (6) to obtain the induction step for going from words v in $\mathcal{N}_*$ to words w in $\mathcal{N}_*$.

For words in symmetric variables, that is in $\mathcal{N}$ the same argument prevails. ●●

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lem:strict **Lemma 3.2** *Suppose $\lambda : M_{l\times l}(\mathcal{N}_*) \mapsto \mathbb{R}$ is linear.*

1. *If λ is $non-negative$ on $\mathcal{C}_b^\ell$, and if $\lambda(I) = 0$, then $\lambda = 0$ on $M_{l\times l}(\mathcal{N}_*\mathcal{N})$.*

2. *If the variables x_j are all symmetric and if λ is $non-negative$ on $\mathcal{C}_b^\ell$, together with $\lambda(I) = 0$, then $\lambda = 0$ on $M_{l\times l}(\mathcal{N})$.*

3. *If λ is $non-negative$ on $\mathcal{C}_{\{b,\tilde{b}\}}^\ell$, and if $\lambda(I) = 0$, then $\lambda = 0$ on $M_{l\times l}(\mathcal{N}_*)$.*

Proof.

Using Lemma lem:incalC 3.1 and the definition of $\mathcal{C}_b^\ell$, for a word $w \in \mathcal{F}_g$ of length n,

$$I \otimes (C^{2n} - w^T w) = \sum_k e_k (C^{2n} - w^T w) e_k^T \in \mathcal{C}_b^\ell.$$

Here $e_1, \ldots, e_\ell$ is the standard basis for $\mathbb{R}^\ell$ and e_k^T is viewed as the $1 \times \ell$ matrix-valued constant polynomial. Since λ is non-negativeon $\mathcal{C}_b^\ell$, we have $\lambda(I \otimes (C^{2n} - w^T w)) \succeq 0$. As $\lambda(I) = 0$, we have $-\lambda(I \otimes w^T w) \geq 0$, which since $w^T w \in \mathcal{C}_b^\ell$ implies that $\lambda(I \otimes w^T w) = 0$.

Given a unit vector $h_1 \in \mathbb{R}^\ell$ by choosing an orthonormal basis $\{h_1, h_2, \ldots, h_\ell\}$ for $\mathbb{R}^\ell$, writing $I = \sum h_j h_j^T$, and considering these as constant polynomials, it is evident that $\lambda(h_1 h_1^T) = 0$, since $h_j h_j^T \in \mathcal{C}_b^\ell$ for each j. By scaling, the unit vector hypothesis on h_1 may be dropped.

If $h \in \mathbb{R}^\ell$, then considering he_1^T as the $M_{l\times l}$-valued constant polynomial (here e_1 is first standard basis vector in $\mathbb{R}^\ell$) shows

$$hh^T \otimes (C^{2n} - w^T w) = he_1^T (I \otimes (C^{2n} - w^T w)) e_1 h^T$$

is in $\mathcal{C}_b^\ell$. Hence, as before, $\lambda(hh^T \otimes w^T w) = 0$.

Now we do a critical calculation. First we do it for hereditary words $v^T w$, since this calculation specializes easily to the other cases. Suppose $g, h \in \mathbb{R}^\ell$ and $v^T w$ is an hereditary word. For t real, let

$$r = (h^T \otimes v) + t(g^T \otimes w),$$

so that r is a $M_{1\times\ell}$-valued polynomial. Since $r^T r$ is in $\mathcal{C}_b^\ell$,

$$0 \leq \lambda(r^T r) = \lambda(hh^T \otimes v^T v) + t\lambda(hg^T \otimes v^T w + gh^T \otimes w^T v) + t^2 \lambda(gg^T \otimes w^T w)$$

for all real t. Thus,

$$\lambda(hg^T \otimes v^T w + gh^T \otimes w^T v) = 0. \qquad (7) \quad \boxed{\texttt{eq:lamda0}}$$

The proof of our lemma for the hereditary case is complete, since every symmetric $M_{l\times l}$-valued hereditary polynomial is a linear combination of polynomials of the form $hg^T \otimes v^T w + gh^T \otimes w^T v$. So $\lambda = 0$ on $M_{l\times l}(\mathcal{N}_* \mathcal{N})$.

The proof that $\lambda = 0$ on $M_{l\times l}(\mathcal{N}_*)$, follows from Lemma lem:incalC 3.1 as well as from (eq:lamda0 7 by taking w to be a word in $M_{l\times l}(\mathcal{N}_*)$ and v to be the empty word??. Use that Lemma lem:incalC 3.1 implies $r^t r \in \mathcal{C}_{\{b,\tilde{b}\}}\ell$. With v the empty word the equation becomes (eq:lamda0 7) becomes $\lambda(hg^T \otimes w + gh^T \otimes w^T) = 0$.

The symmetric case follows ?? immediately, from the $\mathcal{N}_*$ case by restricting to symmetric variables.?TRUE??

SCOTT?? define tensor product of words with matrices ?? empty word and its marelous properties. We used it above. LOOK AT OTHER NOTES??

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●●

The collection $M_{l\times l}(\mathcal{N}_*)$ of all symmetric $M_{l\times l}$-valued NC polynomials is a real vector space. The collection $\mathcal{C}_{\mathcal{P}}^{\ell}$ consisting of those $M_{l\times l}$ valued elements of $\mathcal{C}_{\mathcal{P}}$ is a cone in $M_{l\times l}(\mathcal{N}_*)$; i.e., $\mathcal{C}_{\mathcal{P}}^{\ell}$ is closed under sums and multiplication by non-negative scalars and the relation $s \geq t$ if and only if $s-t \in \mathcal{C}_{\mathcal{P}}$ on $M_{l\times l}(\mathcal{N}_*)$ makes $M_{l\times l}(\mathcal{N}_*)$ an ordered vector space. One proof of this for bounded $\mathcal{D}_{\mathcal{P}}$ can be gotten from Lemma lem:interior 3.3 below which treats a more refined situation.

———————————————

DELETE ?? OK??

If $\mathcal{M}$ is a (real) subspace of $M_{l\times l}(\mathcal{N}_*)$ and if $\lambda : \mathcal{M} \mapsto \mathbb{R}$ is linear, then λ is **positive** provided $\lambda(s) \geq 0$ for each $s \in \mathcal{C}_{\mathcal{P}}^{\ell} \cap \mathcal{M}$. Let I denote the $M_{l\times l}$-valued polynomial constantly equal to I, the $\ell \times \ell$ identity matrix.

?? BILL LOOk AT positive NOT A CONSISTENT DEF because positive depends on which $\mathcal{M}$. tune it ?? [6/10. The $\mathcal{M}$ could be moved to later SCOTT? WHERE LATER $\mathcal{M}$ causes expository problems . Needed to define λ positive for the lemma immediately below only on the whole space, but later needed for the subspace.]

END DELETE??

—————-

Let $|w|$ denote the length of the word $w \in \mathcal{F}_g$. An hereditary $M_{l\times l}$-valued polynomial s has degree at most d if $s = \sum_{|v|,|w|\leq d} s_{v,w} v^T w$. The degree of s is d if it has degree at most d, but not degree at most $d-1$. The definitions of the degree of NC polynomials in the classes $\mathcal{N}$ and $\mathcal{N}_*$ is similar. For instance, s in $\mathcal{N}$ or in $\mathcal{N}_*$ has degree at most d if $s = \sum_{|w|\leq d} s_w w$. The only difference being that, in the first case $w \in \mathcal{F}_g$, and in the second $w \in \mathcal{F}_{2g}$.

Let $\mathcal{C}_{\mathcal{P}|d}^{\ell}$ denote those elements of $\mathcal{C}_{\mathcal{P}}^{\ell}$ of degree at most d and let $M_{l\times l}(\mathcal{N}_*)_{|_d}$ denote the symmetric $M_{l\times l}$-valued polynomials of degree at most d. Note that $M_{l\times l}(\mathcal{N}_*)_{|_d}$ is a finite dimensional vector space and as such is a Banach space.

lem:interior **Lemma 3.3** *The set* $\mathcal{C}^{\ell}_{b|d} - I$ *is absorbing in* $M_{l\times l}(\mathcal{N}_*)_{|_d}$ *and*

$$M_{l\times l}(\mathcal{N}_*)_{|_d} = \mathcal{C}^{\ell}_{b|d} - \mathcal{C}^{\ell}_{b|d}$$

.

BILL WONDERS: $M_{l\times l}(\mathcal{N}_*)_{|_d}$ = Sos -SoS. which can be proved by taking the L^TDL decomposition of the Gramm matrix. Why the complication with b?

Proof. We do the hereditary case first, since the calculations involved include the calculations for the other cases. For the hereditary case, let $g, h \in \mathbb{R}^{\ell}$ and $v, w \in \mathcal{F}_g$ with $|w|, |v| \leq d$ be given and assume that $C \geq 1$ so that $C^{2d} \geq C^{2|w|}$ and $C^{2d} \geq C^{2|v|}$. Observe, an arbitrary hereditary polynomial (in $M_{l\times l}(\mathcal{N}_*\mathcal{N})$) is a linear combination of terms which can be expanded as

$$\begin{aligned} &gh^T \otimes w^T v \ + \ hg^T \otimes v^T w \\ &\qquad = (g^T \otimes w + h^T \otimes v)^T (g^T \otimes w + h^T \otimes v) - gg^T \otimes v^T v - hh^T \otimes w^T w \end{aligned}$$

and note that

$$\begin{aligned} &C^{2d}(g^T g + h^T h)I \ - \ gg^T \otimes v^T v - hh^T \otimes w^T w \\ &\quad = gg^T \otimes (C^{2d} \ - \ v^T v) + hh^T \otimes (C^{2d} - w^T w) + C^{2d}(g^T g I - gg^T) + C^{2d}(h^T h I - hh^T) \end{aligned}$$

is in $\mathcal{C}^{\ell}_{\mathcal{P}|_d}$. Since also

$$(g^T \otimes w + h^T \otimes v)^T (g^T \otimes w + h^T \otimes v) \tag{8}$$ eq:absorb1

is in $\mathcal{C}^{\ell}_{\mathcal{P}|_d}$, it follows that

$$C^{2d}(g^T g + h^T h)I - (gh^T \otimes w^T v + hg^T \otimes v^T w) \in \mathcal{C}^{\ell}_{\mathcal{P}|_d}.$$

Thus

$$gh^T \otimes w^T v + hg^T \otimes v^T w = C^{2d}(g^T g + h^T h)I - (\text{ a poly in } \mathcal{C}^{\ell}_{\mathcal{P}|_d}) \subset \mathcal{C}^{\ell}_{\mathcal{P}|_d} - \mathcal{C}^{\ell}_{\mathcal{P}|_d}$$

which proves one assertion of the theorem. Further, (8) implies

$$t(-gh^T \otimes w^T v - hg^T \otimes v^T w) \in \mathcal{C}^{\ell}_{\mathcal{P}|_d} - I,$$

where $t = (C^{2d}(g^T g + h^T h))^{-1} > 0$. Thus, as every member of $M_{l\times l}(\mathcal{N}_*)_{|_d}$ is a finite linear combination of terms of the form $-gh^T \otimes w^T v - hg^T \otimes v^T w \in M_{l\times l}(\mathcal{N}_*)_{|_d}$, the set $\mathcal{C}^{\ell}_{\mathcal{P}|_d} - I$ is absorbing. ●●

SCOTT ?? DO $\mathcal{N}$ and $\mathcal{N}_*$. These are easier??

We shall be working with bounded sets $\mathcal{D}_{\mathcal{P}}$ and instead of writing $\tilde{\mathcal{P}}$ to indicate that we have included polynomials $b, \tilde{b}$, we shall henceforth use the convention $\mathcal{P} = \tilde{\mathcal{P}}$, that is $\mathcal{P}$ includes $\mathcal{P} = \tilde{\mathcal{P}}$.

lem:extend **Lemma 3.4** *Suppose the set $\mathcal{P}$ of polynomials contains b. If $\Lambda : M_{l\times l}(\mathcal{N}_*)_{|d} \mapsto \mathbb{R}$ is a linear functional which is non-negative on $\mathcal{C}^{\ell}_{\mathcal{P}|d}$, and not identically zero, then there exists a linear functional $\lambda : M_{l\times l}(\mathcal{N}_*) \mapsto \mathbb{R}$ extending Λ which is non-negative on $\mathcal{C}^{\ell}_{\mathcal{P}}$.*

Proof. ?? JEFF owns this proof ??

We first take up the hereditary case. The real vector space $M_{l\times l}(\mathcal{N}_*)$ with the relation $s \geq t$ provided $s - t \in \mathcal{C}^{\ell}_{\mathcal{P}}$ is an ordered vector space, in the sense found in Conway's text on functional analysis. ?? WHATS THAT SENSE?? Given $gh^T \otimes v^T w + hg^T \otimes w^T v$ we have

$$(g^T \otimes v - h^T \otimes w)^T (g^T \otimes v - h^T \otimes w) + (gh^T \otimes v^T w + hg^T \otimes w^T v) = gg^T \otimes v^T v + hh^T \otimes ww^T.$$

Hence,

$$(gh^T \otimes v^T w + hg^T \otimes w^T v) \leq gg^T \otimes v^T v + hh^T \otimes ww^T.$$

From Lemma lem:incalC 3.1 it now follows that

$$(gh^T \otimes v^T w + hg^T \otimes w^T v) \leq C^{2|v|} gg^T + C^{2|w|} hh^T,$$

where $|v|$ and $|w|$ are the lengths of the words v and w and $C > 0$ is our constant such that $C^2 - x_j^T x_j \in \mathcal{P}$ for each j. Since every element of $M_{l\times l}(\mathcal{N}_*)$ is a finite linear combination of those of the form $(gh^T \otimes v^T w + hg^T \otimes w^T v)$, for each $p \in M_{l\times l}(\mathcal{N}_*)$ there exists a number $t > 0$ such that $p \leq tI$.

———-

?? JEFF - will lookup Krein Ext

Hence I is an ??order unit?? in $M_{l\times l}(\mathcal{N}_*)$ and $M_{l\times l}(\mathcal{N}_*)_{|d}$ is ??cofinal?? in $M_{l\times l}(\mathcal{N}_*)$. The result now follows by a version of the Hahn-Banach

Theorem, often called the Krein Extension Theorem, see for instance Conway's text on functional analysis page 87 item 9.8.

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Now do other cases. Similar? ••

3.2 Separating a Polynomial from $\mathcal{C}_{\mathcal{P}}$

Now we give the main result of Section sec:sepCP 3.

thm:sep **Proposition 3.5** *If q is in $M_{l\times l}(\mathcal{N}_*), M_{l\times l}(\mathcal{N}), M_{l\times l}(\mathcal{N}_*\mathcal{N})$ but not in the corresponding $\mathcal{C}^{\ell}_{\mathcal{P}}$, then there is a non-zero linear functional λ*

$$\lambda : M_{l\times l}(\mathcal{N}_*) \mapsto \mathbb{R}, \ \textit{respectively}\ \lambda : M_{l\times l}(\mathcal{N}_*) \mapsto \mathbb{R},$$
$$\textit{respectively}\ \lambda : M_{l\times l}(\mathcal{N}_*) \mapsto \mathbb{R}$$

which is non-negative on $\mathcal{C}^{\ell}_{\mathcal{P}}$ and non-positive elsewhere.

Proof. We focus on the $M_{l\times l}(\mathcal{N}_*)$ case since the others are very similar. ??SCOTT - im bluffing OK?? Do you believe THIS ?? Suppose $q \notin \mathcal{C}^{\ell}_{\mathcal{P}}$ and has degree d. Let $\mathcal{B} = \{tq : t > 0\}$. Both $\mathcal{C}^{\ell}_{\mathcal{P}|_d}$ and $\mathcal{B}$ are convex, the intersection is empty, and $\mathcal{C}^{\ell}_{\mathcal{P}|_d} - I$ is absorbing by Lemma lem:interior 3.3. Hence, by a result in Rudin's Functional Analysis (Exercise 3 Chapter 3), there exists a non-zero (real) linear functional $\Lambda : M_{l\times l}(\mathcal{N}_*)_{|_d} \mapsto \mathbb{R}$ such that $\Lambda(\mathcal{C}^{\ell}_{\mathcal{P}|_d}) \cap \Lambda(\mathcal{B})$ contains at most one point. Observe that $\Lambda(\mathcal{C}^{\ell}_{\mathcal{P}|_d}) \neq \{0\}$ as otherwise Lemma lem:interior 3.3 implies $M_{l\times l}(\mathcal{N}_*)_{|_d}$ is contained in a closed hyperplane in $M_{l\times l}(\mathcal{N}_*)_{|_d}$. Because 0 is in $\mathcal{C}^{\ell}_{\mathcal{P}|_d}$ and in the closure of $\mathcal{B}$, the sets $\Lambda(\mathcal{B})$ and $\Lambda(\mathcal{C}^{\ell}_{\mathcal{P}|_d})$ each have 0 in their closure (here we have used that $\Lambda(\mathcal{B})$ and $\Lambda(\mathcal{C}^{\ell}_{\mathcal{P}|_d})$ are finite dimensional so that we can unambiguously speak of closures). The sets are also convex. Hence $\Lambda(\mathcal{C}^{\ell}_{\mathcal{P}|_d}) \cap \Lambda(\mathcal{B})$ is either $\{0\}$ or empty. In the first case, $\Lambda(q) = 0$ and, without loss of generality, $\Lambda(\mathcal{C}^{\ell}_{\mathcal{P}|_d}) \geq 0$. In the second case, it again may be assumed that $\Lambda(\mathcal{C}^{\ell}_{\mathcal{P}|_d}) \geq 0$ and $\Lambda(q) < 0$. Thus, there exists a linear functional $\Lambda : M_{l\times l}(\mathcal{N}_*)_{|_d} \mapsto \mathbb{R}$ which is non-negative on $\mathcal{C}^{\ell}_{\mathcal{P}|_d}$ such that $\Lambda(q) \leq 0$. By Lemma lem:extend 3.4 the functional Λ extends to a non-zero linear functional $\lambda : M_{l\times l}(\mathcal{N}_*) \mapsto \mathbb{R}$ which is non-negative on $\mathcal{C}^{\ell}_{\mathcal{P}}$.

4 Representing Linear Functionals

This section is devoted to a representation which will soon be applied to λ of the previous section.

lem:Hilbertspace **Proposition 4.1** *If $\lambda : M_{l\times l}(\mathcal{N}_*) \mapsto \mathbb{R}$ (resp $\lambda : M_{l\times l}(\mathcal{N}_*\mathcal{N}) :\mapsto \mathbb{R}$, resp. $\lambda : M_{l\times l}(\mathcal{N}) \mapsto \mathbb{R}$) is positive and not identically zero, then there exists a real Hilbert space H, a tuple $X = (X_1, \ldots, X_g)$ of operators (resp. operators, resp. symmetric operators) on H, and a non-zero vector $\gamma \in \oplus_1^\ell H$, the ℓ fold direct sum of H, such that $p(X) \succeq 0$ for all $p \in \mathcal{P}$ and for any symmetric $s \in M_{\ell\times\ell}(\mathcal{N}_*)$-valued polynomial (resp $s \in M_{\ell\times\ell}(\mathcal{N}_*\mathcal{N})$, resp $s \in M_{\ell\times\ell}(\mathcal{N})$),*

$$< s(X)\gamma, \gamma >= \lambda(s).$$

Proof for $M_{l\times l}(\mathcal{N}_*)$ Case. Given $1 \times \ell$ matrix-valued polynomials s, t with entries in $\mathcal{N}_*$, define

$$< s, t >= (1/2)\lambda(t^T s + s^T t) \tag{9}$$ eq:innerproduct

and verify that $< s, t >$ is indeed bilinear. It is positive semi-definite as $s^T s \in \mathcal{C}_\mathcal{P}$ and λ is positive. Let H be the Hilbert space formed by moding out $< \cdot, \cdot >$-null vectors and forming the completion. Note H may be infinite dimensional.

Recall our constant $C > 0$ such that $C^2 - x_j^T x_j \in \mathcal{C}_\mathcal{P}$ and $C^2 - x_j x_j^T \in \mathcal{C}_\mathcal{P}$. Since

$$C^2 < s, s > - < x_j s, x_j s >= \lambda(s^T(C^2 - x_j^T x_j)s) \tag{10}$$ eq:repBd

and since $s^T(C^2 - x_j^T x_j)s \in ??\mathcal{C}??$, it follows that multiplication by x_j on $1 \times \ell$ matrix-valued $\mathcal{N}_*$ polynomials defines a bounded operator X_j on H. Likewise

$$C^2 < s, s > - < x_j^T s, x_j^T s >= \lambda(s^T(C^2 - x_j x_j^T)s)$$

implies multiplication by x_j^T on $1 \times \ell$ matrix-valued is bounded on H. Denote this multiplication operator by X_j and denote multiplication by

x_j^T by X_j^T. They are both bounded and they are adjoints of each other, since

$$< x_j s, s >= \lambda(s^T x_j^T s) = \lambda(s^T x_j^T s) =< s, x_j^T s > .$$

Now suppose $p \in \mathcal{C}_{\mathcal{P}}$ is $m \times m$ and symmetric. We shall be substituting X_j for x_j in $p(x)$ and this forces the substitution X_j^T for x_j^T in $p(x)$, since X_j^T is the adjoint of X_j^T. If r is an m vector where each entry is a $1 \times \ell$ matrix-valued polynomial, then

$$< p(X)r, r >= \sum_{a,b} < p_{a,b}(X)r_b, r_a >= \lambda(\sum_{a,b} r_a^T p_{a,b} r_b) = \lambda(r^T pr) \geq 0,$$

where r is also canonically identified with the $m \times \ell$ matrix-valued polynomial and where the inequality results from $r^T pr \in ??\mathcal{C}??$. Hence $p(X) \succeq 0$.

Let γ_j denote the (equivalence class of the) constant (basis) polynomial $e_j \mathbb{R}^\ell$ and note that $\sum \gamma_j \gamma_j^T = I$. Thus, in view of Lemma lem:strict 8.1, which says that $\lambda(I) \neq 0$, there is a j_0 such that $< \gamma_{j_0}, \gamma_{j_0} >= \lambda(\gamma_{j_0}\gamma_{j_0}^T) > 0$. Hence the vector $\gamma = \oplus \gamma_j$ is non-zero. Finally, if s is a symmetric $M_{l \times l}$-valued polynomial, then

$$< s(X)\gamma, \gamma >= \lambda(\sum \gamma_a^T s_{a,b} \gamma_b) = \lambda(s)$$

where the last equality takes into account that s is symmetric and that $\gamma_a^T s_{a,b} \gamma_b$ is the $\ell \times \ell$ matrix with $s_{a,b}$ in the (a, b) entry. This completes the proof for $M_{l \times l}(\mathcal{N}_*)$ polynomials. ••

Proof for $M_{l \times l}(\mathcal{N})$ Case .

The construction for the $M_{l \times l}(\mathcal{N})$ case is very similar to that for the $M_{l \times l}(\mathcal{N}_*)$ case. Given $1 \times \ell$ matrix-valued polynomials s, t with entries in $\mathcal{N}$, define

$$< s, t >= \lambda(s^T t) \quad (??USET = transpose??) \tag{11}$$ eq:innerproduct

and verify that $< s, t >$ is indeed bilinear. It is positive semi-definite as $s^2 \in \mathcal{C}_{\mathcal{P}}$ and λ is positive on $\mathcal{C}_{\mathcal{P}}$. Let H be the Hilbert space formed by moding out $< \cdot, \cdot >$-null vectors and forming the completion. Note it is infinite dimensional.

Recall our constant $C > 0$ such that $C^2 - x_j^2 \in \mathcal{C}_{\mathcal{P}}$. Since

$$C^2 < s, s > - < x_j s, x_j s >= \lambda(s(C^2 - x_j x_j)s)$$

and since $s(C^2 - x_j^2)s \in \mathcal{C}$, it follows that multiplication by x_j on $1 \times \ell$ matrix-valued polynomials defines a bounded operator on the Hilbert space H. Denote this multiplication operator by X_j and note it is symmetric, since

$$< x_j s, s >= \lambda(s x_j s) = \lambda(s x_j s) =< s, x_j s > .$$

From this point on the proof is exactly the same as it was in the $M_{l\times l}(\mathcal{N}_*)$ case. ●●

Proof for Hereditary $M_{l\times l}(\mathcal{N}_*\mathcal{N})$ Case .

Now we turn to the hereditary case. Given $1\times\ell$ matrix-valued polynomials s, t with entries in $\mathcal{N}$ (so s, t contain no transposes and consequently $s^T r$ and $r^T s$ are hereditary), define

$$< s, t >= \frac{1}{2}\lambda(t^T s + s^T t) \qquad (12)$$ eq:innerproduct

and verify that $< s, t >$ is indeed bilinear. It is positive semi-definite as $s^T s \in \mathcal{C}_{\mathcal{P}}$ and λ is positive. Let H be the Hilbert space formed by moding out $< \cdot, \cdot >$-null vectors and forming the completion. Note H may be infinite dimensional.

Recall our constant $C > 0$ such that $C^2 - x_j^T x_j \in \mathcal{C}_{\mathcal{P}}$. Estimate (10) eq:repBd gives us that that multiplication by x_j on $1 \times \ell$ matrix-valued polynomials defines a bounded operator on the Hilbert space H. Denote this multiplication operator by X_j.

In this case it is not true that X_j^* is multiplication by x_j^T, as $x_j^T s$ is not in $\mathcal{N}$. However, if P is an hereditary polynomial, $P = \sum P_{v,w} v^T w$,

$X = (X_1, \ldots, X_g)$ is a tuple, and $\gamma = \oplus \gamma_j$ and $\delta = \oplus \delta_j$ are vectors, then

$$< P(X)\gamma, \delta >= \sum P_{v,w} < X^w \gamma, X^v \delta >$$

so that it is not actually necessary to have an explicit representation for X_j^*.

Now suppose $p \in \mathcal{C}_{\mathcal{P}}$ is $m \times m$ and symmetric. If f is an m vector where each entry is a $1 \times \ell$ matrix-valued polynomial, then

$$< p(X)r, r >= \sum_{a,b} < p_{a,b}(X)r_b, r_a >= \lambda(\sum_{a,b} r_a^T p_{a,b} r_b) = \lambda(r^T pr) \geq 0,$$

where r is also canonically identified with the $m \times \ell$ matrix-valued polynomial and where the inequality results from $r^T pr \in \mathcal{C}_{\mathcal{P}}$. Hence $p(X) \succeq 0$. We emphasize that in the calculations above $r^T pr$ as well as all other polynomials occuring are hereditary.

From this point the proof is exactly as it was for the $M_{l \times l}(\mathcal{N}_*)$ case. ••

5 Proof of Theorem 1.2

Suppose is in $M_{l \times l}(\mathcal{N}_*)$ and $q \notin \mathcal{C}_{\mathcal{P}}$. By Proposition 3.5 there is a non-zero positive linear functional $\lambda : M_{l \times l}(\mathcal{N}_*) \mapsto \mathbb{R}$ which makes $\lambda(q) \leq 0$.

By Proposition 4.1, there exists a Hilbert space H, a non-zero vector γ in $\oplus_1^\ell H$ and a tuple of operators $X = (X_1, \ldots, X_g)$ on H such that $p(X) \succeq 0$ for all $p \in \mathcal{P}$ and for each symmetric $M_{l \times l}$-valued polynomial s, we have $< s(X)\gamma, \gamma >= \lambda(s)$. In particular, substituting q for s gives, $< q(X)\gamma, \gamma >= \lambda(q) = \Lambda(q) \leq 0$. Since γ is non-zero, it follows that there is an $X \in \mathcal{D}_{\mathcal{P}}$ such that $q(X)$ is not strictly positive definite and this proves the contrapositive of Theorem 1.2.

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Bill SoS operator paper??

SCOTT SoS operator paper ??

Lotsa guys in Remark [6/10. Think all are accounted for excepting Agler and Mcarthy. But I just recall John saying something about a theorem like Jim's on the polydisc starting with a finite collection of scalar polynomials, rather than $1 - x_j x_j^T$.]

7 DUMP DUMP

7.1 ORIGINAL VERSION of Proof of Representation

Now we turn to the hereditary case. Given $1 \times \ell$ matrix-valued $\mathcal{N}$ polynomials s, t (so s, t contain no adjoints here ?? SCOTT appropriate here ?? and so that $s^T r$ and $r^T s$ are hereditary), define

$$< s, t >= \frac{1}{2}(\lambda(t^T s + s^T t) + i\lambda(-it^T s + is^T t)) \tag{13}$$

eq:innerproduct

and verify that $< s, t >$ is indeed sesquilinear. It is positive semi-definite as $s^T s \in \mathcal{C}_{\mathcal{P}}$ and λ is positive. Let H be the Hilbert space formed by moding out $< \cdot, \cdot >$-null vectors and forming the completion. Note it is infinite dimensional.

Recall our constant $C > 0$ such that $C^2 - x_j^T x_j \in \mathcal{C}_{\mathcal{P}}$.

Since

$$C^2 < s, s > - < x_j s, x_j s >= \lambda(s^T (C^2 - x_j^T x_j) s)$$

and since $s^T (C^2 - x_j^T x_j) s \in ??\mathcal{C}??$, it follows that multiplication by x_j on $1 \times \ell$ matrix-valued polynomials defines a bounded operator on the Hilbert space H. Denote this multiplication operator by X_j and denote multiplication by x_j^T by X_j^T. They are both bounded and they are adjoints of each other, since

$$< x_j s, s >= \lambda(s^T x_j^T s) = \lambda(s^T x_j^T s) =< s, x_j^T s > .$$

SCOTT bill added detail above and below. OK?? [6/10. Bill, In the $\mathcal{N}$ and $\mathcal{N}_*$ cases X_j^T is multiplication by x_j^T (and of course in the $\mathcal{N}$ case this is just the same as x_j). However, in the hereditary case, $x_j^T s$ is not an analytic (adjoint free) polynomial so that $< s, x_j^T s >$ is not defined. The difficulty is with say the definition of $< x_\ell^T s, x_j^T t >$ which would involve non-hereditary polynomials like $s^T x_\ell x_j^T t$. As a side note, this really gives non-self adjoint operator algebra as X_j^T is the compression of mu All else looks fine.]

Now suppose $p \in \mathcal{C}_{\mathcal{P}}$ is $m \times m$ and symmetric. We shall be substituting X_j for x_j in $p(x)$ and this forces the substitution X_j^T for x_j^T in $p(x)$, since X_j^T is the adjoint of X_j^T. If r is an m vector where each entry is a $1 \times \ell$ matrix-valued polynomial, then

$$< p(X)r, r >= \sum_{a,b} < p_{a,b}(X)r_b, r_a >= \lambda(\sum_{a,b} r_a^T p_{a,b} r_b) = \lambda(r^T pr) \geq 0,$$

where r is also canonically identified with the $m \times \ell$ matrix-valued polynomial and where the inequality results from $r^T pr \in ??\mathcal{C}??$. Hence $p(X) \succeq 0$.

Let γ_j denote the (class of) the constant polynomial x_j and note that $\sum \gamma_j \gamma_j^T = I$. Thus, in view of Lemma lem:strict 8.1, there is a j_0 such that $< \gamma_{j_0}, \gamma_{j_0} >= \lambda(\gamma_{j_0}\gamma_{j_0}^T) > 0$. Hence the vector $\gamma = \oplus \gamma_j$ is non-zero. Finally, if s is a symmetric $M_{l \times l}$-valued polynomial, then

$$< s(X)\gamma, \gamma >= \lambda(\sum \gamma_a^T s_{a,b} \gamma_b) = \lambda(s)$$

where the last equality takes into account that s is symmetric and that $\gamma_a^T s_{a,b} \gamma_b$ is the $\ell \times \ell$ matrix with $s_{a,b}$ in the (a, b) entry. This completes the proof for hereditary polynomials. ••

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8 DUMP – Proof of a Lemma ORIGINAL VERSION -DELETE??

lem:strict **Lemma 8.1** *Suppose* $\lambda : M_{l \times l}(\mathcal{N}_*) \mapsto \mathbb{R}$ *is linear. If* λ *is positive, and if* $\lambda(I) = 0$*, then* $\lambda = 0$.

The proof of Lemma lem:strict 8.1 depends upon the boundedness hypothesis on $\mathcal{P}$.

Proof. First consider the case of hereditary polynomials, $\mathcal{N}_*\mathcal{N}$. Using Lemma lem:incalC 3.1 and the definition of $\mathcal{C}_{\mathcal{P}}$, for a word $w \in \mathcal{F}_g$ of length n,

$$I \otimes (C^{2n} - w^T w) = \sum_k e_k (C^{2n} - w^T w) e_k^T \in \mathcal{C}_{\mathcal{P}}^{\ell}.$$

Here $e_1, \dots, e_\ell$ is the standard basis for $\mathbb{R}^\ell$ and e_k^T is viewed as the $1 \times \ell$ matrix-valued constant polynomial. Since λ is positive, $\lambda(I \otimes (C^{2n} - w^T w)) \geq 0$. As $\lambda(I) = 0$, we have $-\lambda(I \otimes w^T w) \geq 0$, which since $w^T w \in \mathcal{C}_{\mathcal{P}}^\ell$ implies that $\lambda(I \otimes w^T w) = 0$.

Given a unit vector $h_1 \in \mathbb{R}^\ell$ by choosing an orthonormal basis $\{h_1, h_2, \dots, h_\ell\}$ for $\mathbb{R}^\ell$, writing $I = \sum h_j h_j^T$, and considering these as constant polynomials, it is evident that $\lambda(h_1 h_1^T) = 0$, since $h_j h_j^T \in \mathcal{C}_{\mathcal{P}}^\ell$ for each j. By scaling, the unit vector hypothesis on h_1 may be dropped.

If $h \in \mathbb{R}^\ell$, then considering $h e_1^T$ as the $M_{l \times l}$-valued constant polynomial (here e_1 is first standard basis vector in $\mathbb{R}^\ell$) shows

$$hh^T \otimes (C^{2n} - w^T w) = h e_1^T (I \otimes (C^{2n} - w^T w)) e_1 h^T$$

is in ??$\mathcal{C}$??. Hence, as before, $\lambda(hh^T \otimes w^T w) = 0$.

Suppose $g, h \in \mathbb{R}^\ell$ and $v^T w$ is an hereditary word. For t real, let

$$r = (h^T \otimes v) + t(g^T \otimes w),$$

so that r is a $M_{1 \times \ell}$-valued polynomial. Since $r^T r$ is in $\mathcal{C}_{\mathcal{P}}^\ell$,

$$0 \leq \lambda(r^T r) = \lambda(hh^T \otimes v^T v) + t\lambda(hg^T \otimes v^T w + gh^T \otimes w^T v) + t^2 \lambda(gg^T \otimes w^T w)$$

for all real t. Thus, $\lambda(hg^T \otimes v^T w + gh^T \otimes w^T v) = 0$ and the proof for the hereditary case is complete since every symmetric $M_{l \times l}$-valued hereditary polynomial is a linear combination of polynomials of the form $hg^T \otimes v^T w + gh^T \otimes w^T v$.

??NOW DO THE OTHER CASES. Very similar.?? ••END ORIG