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On the Solutions of Analytic Equations

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Section 1

Let k be a field of characteristic zero with a non-trivial valuation. We do not assume k to be complete. By convergent series, we mean a power series with coefficients in k which has a non-zero radius of convergence with respect to the given valuation.

Consider an arbitrary system of analytic equations

$$f(x,y)=0, \tag{1.1}$$

where

$$f(x,y)=(f_1(x,y),\dots,f_m(x,y))$$

are convergent series in the variables

$$x=(x_1,\dots,x_n),$$

$$y=(y_1,\dots,y_N).$$

Here m, n, N are arbitrary non-negative integers. We ask for solutions of (1.1) in which y_ν are convergent series in x. Our main result is the following basic fact:

Theorem (1.2). *Suppose that*

$$\bar{y}(x)=(\bar{y}_1(x),\dots,\bar{y}_N(x)) \qquad \bar{y}_\nu(x)\in k[[x]]$$

are formal power series without constant term which solve (1.1), *i.e., such that* $f(x,\bar{y}(x))=0$. *Let* c *be an integer. There exists a convergent series solution*

$$y(x)=(y_1(x),\dots,y_N(x))$$

of (1.1) *such that*

$$y(x)\equiv\bar{y}(x) \qquad (\text{modulo } \mathfrak{m}^c).$$

Here $\mathfrak{m}$ denotes the maximal ideal of the ring $k[[x]]$ of formal power series in x. Thus the congruence condition just means that the coefficients of monomials of degree $<c$ agree in $y_\nu(x)$ and $\bar{y}_\nu(x)$. Another way of stating the result is to say that the analytic solutions are dense in the space of formal solutions with its $\mathfrak{m}$-adic metric.

There is an algebraic analogue of this theorem which is a powerful tool in algebraic geometry. It says that if $f(x,y)=0$ is a system of *poly-*

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nomial equations with formal series solution $\bar{y}(x)$, then a series solution $y(x)$ can be found such that the $y_\nu(x)$ are algebraically dependent on $x_1, \ldots, x_n$. In this analogue k is an arbitrary field. There are also more precise formulations of (1.2), in which it is assumed only that $\bar{y}(x)$ be an approximate solution of (1.1), i.e., that the series $f_i(x, \bar{y}(x))$ have no non-zero coefficient of low degree[1]. However, the extension of our proof to this case presents some technical difficulties, and our results are not yet complete. Because of this, we will restrict ourselves here to the assertion in the form (1.2).

The proof is based on the Weierstrass preparation theorem. Before beginning it, we will indicate a few variants and applications. When we feel that there is no danger of confusion, we will use the symbol $\gg$ to stand for a large integer, quantified in a way which should be clear from the context. We use curly brackets to denote the ring of convergent series, viz. $k\{x\}$ is the ring of convergent series in $x_1, \ldots, x_n$. It is a noetherian ring ([6], (45.5)).

First of all, we note that if the formal solution $\bar{y}(x)$ is "already analytic" along some analytic set, then we can find $y(x)$ with the same value there:

Theorem (1.3). *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_N$ be proper ideals of $k\{x\}$. Suppose $\bar{y}(x)$ is a formal solution of* (1.1) *of the form*

$$\bar{y}_\nu(x) = u_\nu(x) + \bar{v}_\nu(x)$$

where $u_\nu(x)$ is a convergent series and where $\bar{v}_\nu(x)$ is a formal series congruent zero modulo $\hat{\mathfrak{a}}_\nu = \mathfrak{a}_\nu \cdot k[[x]]$. Then a solution $y(x)$ as in (1.2) *can be found such that moreover*

$$y_\nu(x) \equiv \bar{y}_\nu(x) \qquad (\operatorname{mod} \hat{\mathfrak{a}}_\nu).$$

Proof. The series $u_\nu(x)$ has no constant term. By the linear substitution $y = u(x) + v$ into $f(x, y)$ we reduce to the case $u(x) = 0$, i.e., that $\bar{y}_\nu(x) \equiv 0 \ (\operatorname{mod} \hat{\mathfrak{a}}_\nu)$. Let $\{a_{\nu j}\}$ be a finite set of generators for $\mathfrak{a}_\nu$. Then there are formal series $\bar{z}_{\nu j}(x)$ without constant term and elements $c_{\nu j} \in k$ such that

$$\bar{y}_\nu(x) = \sum_j (\bar{z}_{\nu j}(x) + c_{\nu j})\, a_{\nu j}.$$

Now apply (1.2) to the larger system of equations

$$f_i(x, y) = 0 \qquad i = 1, \ldots, m,$$

$$y_\nu - \sum_j (z_{\nu j} + c_{\nu j})\, a_{\nu j} = 0 \qquad \nu = 1, \ldots, N$$

in the unknowns $\{y_\nu, z_{\nu j}\}$.

[1] In the case $n=1$, $k=\mathbf{R}$, such a result is already in K. Spallek, Differenzierbare Kurven auf analytischen Mengen, Math. Annalen 177, 54–66 (1968).

It is natural to ask the following question: Suppose that in (1.2) some variables x_i do not appear in certain of the formal series $\bar{y}_\nu(x)$. Then can one find $y_\nu(x)$ with the same property? This would be useful for algebraization of classical inductive procedures in algebraic geometry, involving generic projections and discriminant loci. However, our proof does not seem to give such a result.

From (1.3) one deduces immediately

Theorem (1.4). *Let I be an ideal of $k\{x\}$. Suppose that a solution $\bar{y}(x)$ of (1.1) is given for which $\bar{y}_\nu(x)$ are in the I-adic completion of $k\{x\}$. Let c be an integer. There is a solution $y(x)$ of (1.1) such that*

$$y(x) \equiv \bar{y}(x) \qquad (\mathrm{mod}\ I^c).$$

For, since the $\bar{y}_\nu(x)$ are in the I-adic completion of $k\{x\}$, they are congruent to analytic series modulo arbitrary powers of I. Hence we can set $\mathfrak{a}_\nu = I^c$ for all ν in (1.3).

There are a number of applications to existence locally of maps between analytic spaces. As examples, consider the extension and lifting problems. We will state them first in a geometric way in the classical case $k = \mathbf{C}$.

Let X, Y be complex analytic spaces, and let $x \in X$. By *formal map* from X to Y at x, we mean

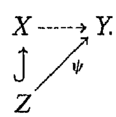
(i) a choice of point $y \in Y$,

(ii) a local $\mathbf{C}$-homomorphism $\mathcal{O}_{Y,y} \to \hat{\mathcal{O}}_{X,x}$

where $\mathcal{O}_{Y,y}$ (resp. $\mathcal{O}_{X,x}$) denotes the local ring of y in Y (resp. of x in X), and where ^ denotes completion. Clearly, a map ϕ: $X \to Y$ induces a formal map at x.

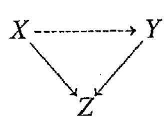
Theorem (1.5). *Let c be an integer.*

(i) *(extension problem). Let Z be an analytic subspace of X containing x. Let ψ: $Z \to Y$ be a map. Consider the problem of extending ψ to X locally at x:*



If ψ extends to a formal map $\bar{\phi}$ from X to Y at x, then there is an extension ϕ locally, which agrees with $\bar{\phi}$ (modulo $\mathfrak{m}_{X,x}^c$).

(ii) *(lifting problem). Let the solid arrows*



be given maps of analytic spaces, and consider the problem of finding a map $\phi: X \to Y$ locally at x which makes the diagram commute. If a formal lifting $\bar{\phi}$ exists, then there is a map ϕ locally, which agrees with $\bar{\phi}$ (modulo $\hat{\mathfrak{m}}^c_{X,x}$).

One could generalize these statements in various ways by requiring that the map preserve extra structure, such as a stratification.

It is clear that the above are assertions about the local rings in question. We will state them that way in the abstract case. Let us call *analytic local ring* A a non-zero local k-algebra isomorphic to a finite algebra over a ring $k\{z\}$ of convergent series in a suitable set of variables z. By *map* $\phi: B \to A$ of analytic local rings, we mean a local k-homomorphism, i.e., one carrying the maximal ideal $\mathfrak{m}_B$ of B to $\mathfrak{m}_A$. Let k' be the residue field of A, which will be a finite extension of k, separable since $\operatorname{char} k = 0$. The ring $k\{z\}$ is henselian ([6], (45.5)). Therefore, so is A. Thus the subfield $k \subset A$ extends uniquely to a coefficient field $k' \subset A$. This will simplify our considerations a little.

We recall the following elementary facts: The valuation of k extends to k', and the ring $k'\{z\}$ of convergent series with coefficients in k' is canonically isomorphic to $k' \otimes_k k\{z\}$. Given any finite set x of elements in the maximal ideal $\mathfrak{m}_A$, the map $k[x] \to A$ extends uniquely to a homomorphism $k\{x\} \to A$, hence to a k'-homomorphism $k'\{x\} \to A$, which will be surjective if x is a set of generators of $\mathfrak{m}_A$. Let

$$\phi: B \to A$$

be a map, and let x, y be generators for $\mathfrak{m}_A$, $\mathfrak{m}_B$ respectively. Because of uniqueness of extension of the coefficient fields, the coefficient field $k'' \subset B$ maps into the coefficient field $k' \subset A$. Let $y_\nu(x) \in k'\{x\}$ be a representative for $\phi(y_\nu)$. Then $y_\nu(x)$ has no constant term, hence substitution for y_ν yields a map $k''\{y\} \to k'\{x\}$, and it makes the following square commute:

$$\begin{array}{ccc} k''\{y\} & \longrightarrow & k'\{x\} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\phi} & A. \end{array}$$

Theorem (1.5a) *(algebraic form). Let c be an integer.*

(i) *Let A, B be analytic local rings, A° a non-zero quotient of A, and $u^\circ: B \to A^\circ$ a map. Let $\bar{u}: B \to \hat{A}$ be a local k-homomorphism lifting $\hat{u}^\circ: B \to \hat{A}^\circ$. There exists a map $u: B \to A$ lifting u°, such that $\tilde{u} \equiv \bar{u}$* (modulo $\mathfrak{m}^c_{\hat{A}}$).

(ii) *Let*

$$\begin{array}{ccc} A & & B \\ & {}_{v}\nwarrow \quad \nearrow_{w} & \\ & C & \end{array}$$

be a diagram of analytic local rings, and let $\bar{u}$: $B \to \hat{A}$ *be a local k-homomorphism such that* $\tilde{v} = \bar{u}\, w$. *There is a map* u: $B \to A$ *with* $v = u\, w$ *and such that* $\tilde{u} \equiv \bar{u}$ (modulo $\mathfrak{m}_{\hat{A}}^c$).

Here $\tilde{u}$: $B \to \hat{A}$ denotes the map induced by u, and the congruence condition means that the maps $B \to \hat{A}/\mathfrak{m}_{\hat{A}}^c = A/\mathfrak{m}_A^c$ obtained from $\bar{u}$ and u are equal.

Proof. We use the following notation:

$$A = k'\{x\}/(f_1, \ldots, f_a),$$

$$B = k''\{y\}/(g_1, \ldots, g_b),$$

$$C = k'''\{z\}/(h_1, \ldots, h_c),$$

$$A^\circ = k'\{x\}/(\phi_1, \ldots, \phi_d).$$

(i) Let $y_\nu^\circ(x) \in k'\{x\}$ represent the image $u^\circ(y_\nu)$ of y_ν in A°, and let $\bar{y}_\nu(x) \in k'[[x]]$ represent $\bar{u}(y_\nu) \in \hat{A}$. The fact that $\bar{u}$ extends u° means that

$$y^\circ(x) \equiv \bar{y}(x) \qquad (\operatorname{mod}(\phi)),$$

i.e., that there are elements $\bar{\alpha}_{\nu\mu}(x) \in k'[[x]]$ with

$$y_\nu^\circ(x) - \bar{y}_\nu(x) = \sum_\mu \bar{\alpha}_{\nu\mu}(x)\, \phi_\mu .$$

The fact that $\bar{u}$ is a homomorphism means that

$$g_j(\bar{y}(x)) \equiv 0 \qquad (\operatorname{mod}(f)),$$

i.e., that there are elements $\bar{\beta}_{ji}(x) \in k'[[x]]$ with

$$g_j(\bar{y}(x)) = \sum_i \bar{\beta}_{ji}(x)\, f_i .$$

Thus we may replace k by k' and apply (1.2) to the system of equations

$$y_\nu^\circ(x) - y_\nu - \sum_\mu \alpha_{\nu\mu}\, \phi_\mu = 0,$$

$$g_j(y) - \sum_i \beta_{ji}\, f_i = 0$$

in the unknowns $\{y_\nu, \alpha_{\nu\mu}, \beta_{ji}\}$.

(ii) Let $z'_\mu(x) \in k'\{x\}$ represent $v(z_\mu)$, let $z''_\mu(y) \in k''\{y\}$ represent $w(z_\mu)$, and let $\bar{y}_\nu(x) \in k'[[x]]$ represent $\bar{u}(y_\nu)$. Then the relevant equations are

$$z''_\mu(\bar{y}_\nu(x)) - z'_\mu(x) = \sum_i \bar{\alpha}_{\mu i}(x)\, f_i,$$

$$g_j(\bar{y}_\nu(x)) = \sum_i \bar{\beta}_{ji}(x)\, f_i$$

for certain $\bar{\alpha}_{\mu i}(x)$, $\bar{\beta}_{ji}(x) \in k'[[x]]$, whence we are again reduced to (1.2) with k replaced by k'.

The following answers a question raised by Grauert:

Corollary (1.6). *With the notation of* (1.5)(i) *or* (ii), *suppose that in addition* $\bar{u}$ *induces an isomorphism* $\hat{\bar{u}}: \hat{B} \xrightarrow{\sim} \hat{A}$. *Then* u *is an isomorphism, provided* $c \geqq 2$. *In particular, suppose two analytic spaces* X, Y *are formally isomorphic at points* x, y, *i.e., that* $\hat{\mathcal{O}}_{X,x} \approx \hat{\mathcal{O}}_{Y,y}$. *Then there are neighborhoods of* x *in* X *and* y *in* Y *which are isomorphic.*

The last assertion was proved by Hironaka and Rossi [4], extending a method of Grauert [2], in case x is an isolated singular point of X. Much more is true in that case; namely it suffices that the truncations of the local rings modulo a sufficiently high power of the maximal ideal be isomorphic, and that their dimensions be equal (cf. Hironaka [3] for more general assertions). In the general situation of (1.6), it is easy to give counterexamples to this.

To prove (1.6), note first that a map $u: B \to A$ of analytic local rings is an isomorphism if the induced map of completions $\hat{u}: \hat{B} \to \hat{A}$ is. For then the image of $\mathfrak{m}_{\hat{B}}$ generates $\mathfrak{m}_{\hat{A}}$, and it follows easily that $u(\mathfrak{m}_B)$ generates $\mathfrak{m}_A$. Moreover, the residue fields of B, A are equal, say to k'. Let y be a finite set of generators of $\mathfrak{m}_B$. Then $u(y)$ generates $\mathfrak{m}_A$. Hence $k'\{y\} \to A$ is surjective, whence u is surjective. In particular, A is a finite B-module. Therefore $\hat{A} \approx \hat{B} \otimes_B A$. Thus since $\hat{B}$ is faithfully flat over B, the fact that $\hat{B} \otimes_B u$ is an isomorphism implies that u is one.

Now to show u an isomorphism, we may replace the rings by their completions. We have maps

$$\hat{A} \xrightarrow{\hat{\bar{u}}^{-1}} \hat{B} \xrightarrow{\hat{u}} \hat{A},$$

and it suffices to show $\hat{u}\hat{\bar{u}}^{-1} = \hat{\phi}$ an isomorphism. By construction, $\hat{\phi} \equiv \text{identity} \pmod{\mathfrak{m}_{\hat{A}}^2}$. Therefore $\hat{\phi}(\mathfrak{m}_{\hat{A}})$ generates $\mathfrak{m}_{\hat{A}}$, by the Nakayama lemma. Hence the argument used above on u, with $k'[[y]]$ replacing $k'\{y\}$, shows that $\hat{\phi}$ is surjective. Now to show that a surjective map $\hat{\phi}: M \to N$ of finite $\hat{A}$-modules is bijective, it suffices to show that the lengths of $M/\mathfrak{m}^n M$ and of $N/\mathfrak{m}^n N$ are equal for all n. Since $\phi(\mathfrak{m}) = \mathfrak{m}$, this is trivial in our case.

Section 2. Proof of the Theorem

The proof of (1.2) is by induction on the number n of variables x_i. It is trivial if $n=0$. Hence we assume $n>0$ and that the theorem is true for $(n-1)$. We have arranged the argument around three lemmas, of which we will defer the proofs to the end.

The ring $k\{x, y\}$ of convergent series in x, y is a noetherian local ring ([6], (45.5)), and the formal series $\bar{y}(x)$ without constant term define by substitution a local k-homomorphism

$$\phi: k\{x, y\} \to k[[x]].$$

Since $\bar{y}(x)$ is a solution of (1.1), we have

$$f_i(x, y) \in \ker \phi = I.$$

Now it is clearly permissible to enlarge the system (1.1) by adding finitely many more equations $f_i(x, y) = 0$ so that the set $\{f_i\}$ generates the ideal I. Then since $k[[x]]$ is an integral domain, so is the ring

$$A = k\{x, y\}/(f) \approx \operatorname{im} \phi.$$

Let a be the Krull dimension of Spec A, and consider the minors of the jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial y_j}\right) \tag{2.1}$$

of rank $r = N + n - a$. Let $\{\delta_\nu\}$ be the set of determinants of these minors, and let Δ be the ideal of $k\{x, y\}$ generated by $\{\delta_\nu\}$. Then we claim that $\Delta \not\subset I$.

Note that the non-vanishing of some δ_ν is the classical condition that the projection map π: Spec $A = V \to$ Spec $k\{x\}$ be *smooth*. Thus the assertion is that the set of points of Spec A at which π is smooth in this sense is *not empty*. It is here that our assumption that k be of characteristic zero is used. But an extension of our proof to characteristic p would probably not be difficult, at least in case $[k : k^p] < \infty$.

We restate the assertion in the form of a lemma:

Lemma (2.2). *Let* $I = (f_1, \ldots, f_m)$ *be an ideal of* $k\{x, y\}$, *and let* $A = k\{x, y\}/I$. *Assume that all components of* Spec A *are of the same dimension* a, *and that the nilradical of* A *has support of dimension* $< a$, *so that* Spec A *is reduced in the neighborhood of its generic points. Assume finally that there is a formal solution* $\bar{y}(x) \subset k[[x]]$ *of* (1.1). *Let* Δ *be the ideal generated by the r-rowed determinants of the jacobian matrix* (2.1), *where* $r = N + n - a$. *Then*

$$\Delta \not\subset I.$$

Since $I = \ker \phi$, it follows from (2.2) that some determinant δ has the property

$$\delta(x, \bar{y}(x)) \neq 0 \tag{2.3}$$

where say

$$\delta = \det\left(\frac{\partial f_i}{\partial y_j}\right)_{i, j = 1, \ldots, r}. \tag{2.4}$$

Lemma (2.5). *Under the assumption of* (2.3), *there is an integer* C *with the following property: If* $y(x) = (y_1(x), \ldots, y_N(x))$ *are convergent series such that*

$$f_i(x, y(x)) = 0 \qquad \textit{for } i = 1, \ldots, r$$

and that

$$y(x) \equiv \bar{y}(x) \qquad (\mathrm{mod}\ \mathfrak{m}^{C}),$$

then $y(x)$ *is a solution of* (1.1), *i.e.*,

$$f_i(x, y(x)) = 0 \qquad \textit{for all } i = 1, \ldots, m.$$

In fact, it follows from standard considerations ([6], (39), (46)) that the locus of zeros of $\{f_1, \ldots, f_r\}$ in Spec $k\{x, y\}$ is of the form

$$X = V \cup X'$$

where X' is a closed set not containing V. In terms of ideals, we can write

$$(f_1, \ldots, f_r) = I \cap K$$

where K is an ideal whose locus is X', and where $K \not\subset I$. Let $h(x, y)$ be a convergent series vanishing on X' but not on V. Then $h(x, \bar{y}(x)) \neq 0$. Hence some monomial has non-zero coefficient in the formal series $h(x, \bar{y}(x))$. If $y(x) \equiv \bar{y}(x)$ modulo sufficiently high powers of $\mathfrak{m}$, then that monomial will have the same coefficient in $h(x, y(x))$, whence

$$h(x, y(x)) \neq 0.$$

Let $\mathfrak{p}$ be the kernel of the map

$$k\{x, y\} \to k\{x\}$$

defined by the substitution

$$g(x, y) \mapsto g(x, y(x)).$$

This kernel is a prime ideal, and by assumption $\mathfrak{p} \supset (f_1, \ldots, f_r)$. But since $h(x, y(x)) \neq 0$, we have $\mathfrak{p} \not\supset K$. Hence $\mathfrak{p} \supset I$. This proves Lemma (2.5).

Thus to prove (1.2), it suffices to treat the case that $m = r$ and that the jacobian (2.4) satisfies (2.3).

Suppose we find convergent series $y^\circ(x) = (y_1^\circ(x), \ldots, y_N^\circ(x))$ with

$$y^\circ(x) \equiv \bar{y}(x) \qquad (\mathrm{mod}\ \mathfrak{m}^{\gg}), \tag{2.6}$$

and such that

$$\delta^2(x, y^\circ(x)) \quad \text{divides} \quad f_i(x, y^\circ(x)) \qquad i = 1, \ldots, r \tag{2.7}$$

as series in x. Since

$$\delta(x, y^\circ(x)) \equiv \delta(x, \bar{y}(x)),$$

$$f_i(x, y^\circ(x)) \equiv f_i(x, \bar{y}(x)) = 0 \qquad (\mathrm{mod}\ \mathfrak{m}^{\gg})$$

it will follow that actually

$$f_i(x, y^\circ(x)) \equiv 0 \qquad (\mathrm{mod}\ \delta^2(x, y^\circ(x)) \cdot \mathfrak{m}^{\gg}).$$

The following lemma will then complete the proof:

Lemma (2.8). *Let $f_1, \ldots, f_r$ be convergent series in x, y, let J be the square matrix*

$$J = \left(\frac{\partial f_i}{\partial y_j}\right)_{i,j=1,\ldots,r}$$

(so that $N \geqq r$), and let $\delta = \det J$. Suppose $y^\circ(x) = (y_1^\circ(x), \ldots, y_N^\circ(x))$ are convergent series in x without constant term, such that for $i = 1, \ldots, r$ we have

$$f_i(x, y^\circ(x)) \equiv 0 \qquad (\operatorname{mod} \delta^2(x, y^\circ(x)) \cdot \mathfrak{m}^c).$$

Then there exist convergent series $y(x) = (y_1(x), \ldots, y_N(x))$ with

$$y(x) \equiv y^\circ(x) \qquad (\operatorname{mod} \delta(x, y^\circ(x)) \cdot \mathfrak{m}^c)$$

such that

$$f_i(x, y(x)) = 0 \qquad \text{for } i = 1, \ldots, r.$$

This lemma is a standard tool. However, we know of no reference which treats our case and so we will repeat a proof at the end of the paper.

Thus it suffices to find convergent series $y^\circ(x)$ satisfying (2.6), (2.7). Now since $f_i(x, \bar{y}(x)) = 0$, we have trivially

$$\delta^2(x, \bar{y}(x)) \mid f_i(x, \bar{y}(x))$$

for $i = 1, \ldots, r$. Moreover, by (2.3)

$$\delta^2(x, \bar{y}(x)) \neq 0.$$

We put $g(x, y) = \delta^2(x, y)$ and drop the symbol $^\circ$. Then the theorem results from the following lemma.

Lemma (2.9). *Assume Theorem (1.2) true in dimension $(n-1)$. Let $g(x, y)$, $f_i(x, y)$ $(i = 1, \ldots, m)$ be convergent series in x, y. Let $\bar{y}(x) = (\bar{y}_1(x), \ldots, \bar{y}_N(x))$ be formal series without constant term, such that*

$$g(x, \bar{y}(x)) \neq 0$$

and that

$$g(x, \bar{y}(x)) \mid f_i(x, \bar{y}(x)) \qquad \text{for } i = 1, \ldots, m.$$

Let c be an integer. There are convergent series $y(x) = (y_1(x), \ldots, y_N(x))$ with

$$y(x) \equiv \bar{y}(x) \qquad (\operatorname{mod} \mathfrak{m}^c)$$

such that

$$g(x, y(x)) \mid f_i(x, y(x)) \qquad \text{for } i = 1, \ldots, m.$$

Note that this lemma is actually an assertion of the same type as (1.2).

Proof. If $g(x, \bar{y}(x))$ is a unit, the lemma is trivial since $g(x, y(x))$ will be a unit whenever $y(x) \equiv \bar{y}(x)$ (mod $\mathfrak{m}$), and hence will divide anything. Thus we may assume that $g(x, \bar{y}(x))$ is not invertible. We adjust coordinates so that the formal Weierstrass preparation theorem ([6], (45.3)) applies to the series $g(x, \bar{y}(x))$ with respect to the variable x_n:

$$g(x, \bar{y}(x)) = \bar{a}(x_n) \cdot (\text{unit}) \tag{2.10}$$

where

$$\bar{a}(x_n) = x_n^r + \bar{a}_{r-1} x_n^{r-1} + \cdots + \bar{a}_1 x_n + \bar{a}_0, \tag{2.11}$$

the $\bar{a}_j$ being formal series in $k[[x_1, \ldots, x_{n-1}]]$ without constant term. Next, apply the division algorithm to the series $\bar{y}_\nu(x)$:

$$\bar{y}_\nu(x) = \bar{a}(x_n)\, \bar{z}_\nu(x) + \sum_{j=0}^{r-1} \bar{y}_{\nu j}\, x_n^j \tag{2.12}$$

where $\bar{y}_{\nu j}$ are series in $x_1, \ldots, x_{n-1}$. Write

$$\bar{y}_{\nu j} = \bar{y}'_{\nu j} + c_{\nu j} \qquad c_{\nu j} \in k \tag{2.13}$$

where $c_{\nu j}$ is the constant term of $\bar{y}_{\nu j}$. Since $\bar{y}_\nu(x)$ is without constant term, we have

$$c_{\nu 0} = 0 \qquad \text{for all } \nu = 1, \ldots, N. \tag{2.14}$$

Set

$$\bar{y}'_\nu(x) = \sum_{j=0}^{r-1} \bar{y}_{\nu j}\, x_n^j = \sum_{j=0}^{r-1} (\bar{y}'_{\nu j} + c_{\nu j})\, x_n^j.$$

Then $\bar{y}'_\nu(x)$ is a series in $x_1, \ldots, x_n$ without constant term. Hence substitution into $g(x, y)$, $f_i(x, y)$ is permitted. Since

$$\bar{y}'(x) \equiv \bar{y}(x) \qquad (\operatorname{mod} \bar{a}(x_n)),$$

we have

$$g(x, \bar{y}'(x)) \equiv g(x, \bar{y}(x)),$$
$$f_i(x, \bar{y}'(x)) \equiv f_i(x, \bar{y}(x)) \qquad (\operatorname{mod} \bar{a}(x_n))$$

by Taylor's formula. Hence the assumption that

$$g(x, \bar{y}(x)) \mid f_i(x, \bar{y}(x))$$

for all i, and (2.10), imply that

$$\bar{a}(x_n) \mid g(x, \bar{y}'(x)) \tag{2.15}$$
$$\bar{a}(x_n) \mid f_i(x, \bar{y}'(x)) \qquad i = 1, \ldots, m.$$

Now consider the polynomial

$$A(x_n) = x_n^r + A_{r-1} x_n^{r-1} + \cdots + A_1 x_n + A_0$$

with the variable coefficients A_j, and the polynomials

$$Y'_\nu = \sum_{j=0}^{r-1} (Y'_{\nu j} + c_{\nu j})\, x_n^j$$

with $Y'_{\nu j}$ variable, where $c_{\nu j}$ are the constants appearing in (2.13). Since (2.14) $c_{\nu 0}=0$, the substitution of Y' for y is permissible. Apply the division algorithm to divide by $A(x_n)$:

$$g(x, Y')=A(x_n)\,Q+\sum_{j=0}^{r-1} G_j\, x_n^j \tag{2.16}$$

$$f_i(x, Y')=A(x_n)\,Q'+\sum_{j=0}^{r-1} F_{ij}\, x_n^j$$

where Q, Q_i, G_i, F_{ij} are convergent series in the variables

$$x_1, \dots, x_n; \quad A_0, \dots, A_{r-1}; \quad Y'_{10}, \dots, Y'_{N\,r-1},$$

and where G_i, F_{ij} do not involve the variable x_n.

Consider the substitution of the power series $\bar{a}_j$, $\bar{y}'_{\nu j}$ in $k[[x_1, \dots, x_{n-1}]]$ for A_j, $Y'_{\nu j}$ into (2.16). It follows from uniqueness of division and (2.15) that

$$G_j(x, \{\bar{a}_j\}, \{\bar{y}'_{\nu j}\})=0,$$

$$F_{ij}(x, \{\bar{a}_j\}, \{\bar{y}'_{\nu j}\})=0$$

for all $j=0, \dots, r-1$; $i=1, \dots, m$. Thus we may apply theorem (1.2) in dimension $(n-1)$ to find convergent series

$$a_j,\, y'_{\nu j} \in k\{x_1, \dots, x_{n-1}\}$$

with

$$a_j \equiv \bar{a}_j,$$

$$y'_{\nu j} \equiv \bar{y}'_{\nu j} \qquad (\operatorname{mod}(x_1, \dots, x_{n-1})^{\beta})$$

such that

$$G_j(x, \{a_j\}, \{y'_{\nu j}\})=0,$$

$$F_{ij}(x, \{a_j\}, \{y'_{\nu j}\})=0$$

for all relevant i, j. Then by formula (2.16), we have

$$a(x_n)\,|\,g(x, y'(x)),$$

$$a(x_n)\,|\,f_i(x, y'(x))$$

where

$$a(x_n)=x_n^r+a_{r-1}\,x_n^{r-1}+\cdots+a_1\,x_n+a_0,$$

$$y'_\nu(x)=\sum_{j=0}^{r-1}(y'_{\nu j}+c_{\nu j})\,x_n^j.$$

Let $\bar{z}_\nu(x)$ be as (2.12), and choose convergent series $z_\nu(x)$ such that

$$z(x) \equiv \bar{z}(x) \qquad (\operatorname{mod} \mathfrak{m}^{\beta}).$$

Put

$$y_\nu(x)=a(x_n)\,z_\nu(x)+y'_\nu(x)$$

so that

$$y(x) \equiv \bar{y}(x) \qquad (\operatorname{mod} \mathfrak{m}^{\beta}).$$

By Taylor's formula,

$$a(x_n) \mid g(x, y(x)) \tag{2.17}$$
$$a(x_n) \mid f_i(x, y(x)).$$

Now the exponent r defined by (2.11) can also be described as the order of $g(x, \bar{y}(x))$ in x_n, i.e., as the smallest integer s such that the monomial x_n^s has non-zero coefficient in $g(x, \bar{y}(x))$. If $y(x) \equiv \bar{y}(x)$ $(\mathrm{mod}\, \mathfrak{m}^{r+1})$, then $g(x, y(x)) \equiv g(x, \bar{y}(x))$ $(\mathrm{mod}\, \mathfrak{m}^{r+1})$. Hence the order of $g(x, y(x))$ in x_n is also r. Thus if we write by Weierstrass

$$g(x, y(x)) = b(x_n) \cdot (\text{unit})$$

where $b(x_n)$ is a monic polynomial in x_n with coefficients non-units in $k\{x_1, \dots, x_{n-1}\}$, then $\deg(b(x_n)) = r$. By (2.17), $a(x_n)$ divides $b(x_n)$. Since $\deg a = \deg b$, this implies that $a = b$. Therefore

$$g(x, y) = a(x_n) \cdot (\text{unit})$$

whence by (2.17)

$$g(x, y(x)) \mid f_i(x, y(x)) \qquad i = 1, \dots, m$$

as required. This completes the proof of Lemma (2.9).

Proof of Lemma (2.2). We first reduce the problem to the case of a complete valued field. Let $\tilde{k} \supset k$ be a complete field and let $\tilde{\Delta}$, $\tilde{I}$ denote the ideal generated in $\tilde{k}\{x, y\}$ by $\{\delta_v\}$, $\{f_i\}$ respectively. Clearly $\tilde{\Delta} \not\subset \tilde{I}$ implies $\Delta \not\subset I$. Now since A is of dimension a, the Weierstrass preparation theorem ([6], (45.3)) implies that A is a finite module over a ring of convergent series $k\{z\}$ in suitable variables $z_1, \dots, z_a$, which may be taken to be linear combinations of x, y ([6], (45.3)). One sees easily that

$$\tilde{A} \approx \tilde{k}\{z\} \otimes_{k\{z\}} A$$

where $\tilde{A} = \tilde{k}\{x, y\}/\tilde{I}$. The condition that A be reduced in the neighborhood of its general points is equivalent with the assertion that the extension A of $k\{z\}$ be generically separable. Hence this is preserved in the extension $\tilde{A}$ of $\tilde{k}\{z\}$. It remains to check that $\operatorname{Spec} \tilde{A}$ is equidimensional, and it suffices to prove this for $\hat{\tilde{A}}$. Now it follows from ([6], (45.5)) that the ring of convergent series $k\{x, y\}$ is pseudo-geometric ([6], (36), i.e., japanese), hence that $\hat{A}$ is equi-dimensional (this is a consequence of ([6], (36.4))). Thus we need only prove the analogous assertion for the extension of complete local rings $k[[x, y]] \hookrightarrow \tilde{k}[[x, y]]$, which is well known (it follows easily from ([1], (2.5))).

Now assume k complete. Let $V = \operatorname{Spec} A$. By smooth point of V we mean one at which V is nonsingular. This is equivalent with the non-vanishing of a suitable jacobian determinant (for the jacobian criterion in this context, see ([6], (46))), and hence is an open condition on V.

By assumption, every component of V contains smooth points. Define

$$V_0 = V,$$
$$V_i = V(f, x_1, \ldots, x_i) = \operatorname{Spec} A/(x_1, \ldots, x_i).$$

If we replace $x_1, \ldots, x_n$ by suitably generic linear combinations, it is clear that the following will hold for each $i = 1, \ldots, n-1$.

(a) The element x_i is not identically zero on any irreducible component of V_{i-1} except those lying over the closed point of $k\{x\}$.

(b) Each component of V_i not over the closed point contains points which are smooth on V.

Since the map $V \to \operatorname{Spec} k\{x\}$ has a formal section $\operatorname{Spec} k[[x]] \to V$ given by $\bar{y}(x)$, so does the map

$$V_{n-1} \to \operatorname{Spec} k\{x_n\}.$$

Hence this map does not carry V_{n-1} to the closed point. Thus there is at least one component C of V_{n-1} not lying over the closed point. Since V is pure a-dimensional, it follows from (a) by dimension theory and induction that

$$\dim C = a - n + 1.$$

Denote by a script letter the local analytic space corresponding to the various spectra above. We will denote the local space corresponding to $\operatorname{Spec} k\{x\}$ by $\mathscr{E}$. Since C does not lie over the closed point of $\operatorname{Spec} k\{x_n\}$, the function x_n is not identically zero on $\mathscr{C}$. Therefore its level sets $x_n = \text{constant}$ are all of dimension $a-n$ [5]. These level sets are the fibres of the map $\varpi\colon \mathscr{V} \to \mathscr{E}$ above the x_n-axis. It follows that there are points arbitrarily near the origin on $\mathscr{V}$ at which the projection map ϖ has fibres of dimension $a-n$, and by (b) we may further assume the points smooth on $\mathscr{V}$. Now clearly the assertion of the lemma is equivalent with the condition that the map ϖ be smooth, in the sense of analytic spaces [5] at points arbitrarily near the origin. Thus it suffices to find such points near each of the points c above. We translate c to the origin and localize. Then what we have to prove is the following:

Lemma (2.18). *Let $I = (f_1, \ldots, f_m)$ be an ideal of $k\{x, y\}$. Assume that $A = k\{x, y\}/I$ is a regular ring of dimension a, and that the Krull dimension of $A/(x)$ is $a-n$. Then with the notation of* (2.2), *we have $\Delta \not\subset I$.*

Proof. Adjust coordinates so that $x_1, \ldots, x_n; y_1, \ldots, y_{a-n}$ generates an ideal primary to the maximal ideal in A, which is possible because of the assumption on $\dim A/(x)$. Then by the Weierstrass preparation theorem, A is a finite algebra over $k\{x; y_1, \ldots, y_{a-n}\}$. Since A is regular, and of dimension a, the field of fractions of A is a separable extension of that of $k\{x; y_1, \ldots, y_{a-n}\}$. This fact is equivalent with the non-vanishing of some $(N+n-a)$-rowed determinant of the jacobian matrix

$$J=\left(\frac{\partial f_i}{\partial y_j}\right) \qquad i=1, \ldots, m,\ j=a-n+1, \ldots, N$$

at the generic point of $\operatorname{Spec} A$, which proves the lemma.

Proof of Lemma (2.8). We let $f_i(x,y)=y_i-y_i^\circ(x)$ when $i=r+1, \ldots, N$. Then the new jacobian matrix has the form

$$\left(\frac{\partial f_i}{\partial y_j}\right)=\begin{pmatrix} J & * \\ 0 & I \end{pmatrix}$$

where I is the appropriate identity matrix. This matrix also has determinant δ. Hence the hypotheses of the lemma are preserved if we add the extra equations $f_i(x,y)=0$ for $i=r+1, \ldots, N$, and so we may as well assume $r=N$ from the start.

Put $f=(f_1, \ldots, f_N)$, $h=(h_1, \ldots, h_N)$, $y^\circ=y^\circ(x)$. Taylor's formula reads

$$f(x, y^\circ+h)=f(x, y^\circ)+J(x, y^\circ)\,h+P$$

where P is a vector of series in x and h all of whose terms are of degree $\geqq 2$ in $\{h_i\}$.

Set $J=J(x, y^\circ)$, $\delta=\delta(x, y^\circ)$. By Cramer's rule, there is a matrix M with entries series in $x_1, \ldots, x_n$ such that

$$MJ=JM=\delta I,$$

where I is the identity matrix.

We try to solve the equations

$$f(x, y^\circ+\delta u)=0$$

for a vector $u=(u_1, \ldots, u_N)$ of series in x. Write

$$f_i(x, y^\circ)=\delta^2\,\varepsilon_i(x)$$

with $\varepsilon_i(x)$ a series congruent zero modulo $\mathfrak{m}^C$. Substitution into Taylor's formula yields equations

$$0=\delta^2\,\varepsilon+J\,\delta u+\delta^2\,Q$$

where Q is a vector of series in x, u all of whose terms are of degree $\geqq 2$ in $\{u_i\}$. Hence we obtain

$$0=JM\,\delta\varepsilon+J\,\delta u+JM\,\delta Q,$$

whence it suffices to solve

$$0=M\,\varepsilon+u+M\,Q.$$

This is a system of analytic equations in $x_1, \ldots, x_n$; $u_1, \ldots, u_N$, and its jacobian with respect to the variables u has the value I at $u=0$. Hence the implicit function theorem implies the existence of a solution of u. Since $\varepsilon_i \equiv 0 \pmod{\mathfrak{m}^c}$, we have $u_i \equiv 0 \pmod{\mathfrak{m}^c}$ as well. Thus

$$y(x) = y^\circ(x) + \delta(x, y^\circ(x))\, u(x)$$

is the required solution of $f(x,y)=0$.

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