Reciprocal Linear Matrix Inequalities

Here is a class of functions some engineering friends of mine think include many interesting examples not covered by LMI.

R(x,y,z):= $\begin{pmatrix} x^{-1} & L_{12}(x,y,z) \\ L_{12}(x,y,z)^T & L_{22}(x,y,z) \end{pmatrix}$

Here L_{22} are $d \times d$.

Here L_{12} are $1 \times d$.

So R is $(d+1) \times (d+1)$.

Any number of variables is fair game,
not just 3 variables.

Q1. Let

$$C := \{(x,y,z) : \ R(x,y,z) > 0, x > 0\}$$

How general a class of sets are these?
Q2. Set $r(x,y,z) := det\ R(x,y,z)$.

Does it satisfy a simple modification of the RZ condition?
AnS: Bill has an MMa notebook which makes Q2 look true.

Here is a question.

Suppose C is a set with minimum degree defining polynomial p.

Suppose 0 is in C.

Def:

p meets the RZminus2 condition wrt 0 means:

on complex lines thru 0 all but maybe 2 zeroes are real.

CONJECTURE 1:

Suppose p meets the RZminus2 condition wrt 0

Then for any b in C,

p meets the RZminus2 condition wrt b.

Def:

4-Intercepts from b condition: every line L thru b

intersects bdryC in at most 4 points.

VV (below BAD NEWS) disputes this conjecture

CONJECTURE 2:

Suppose p meets the RZminus2 condition wrt 0

Then for every b in C the 4-intercepts from b

condition is satisfied.

Examplish:

this is an analog of the RZ situation.

We saw that any line L thru 0 intersects bdryC

at most 2 times.

Proof of ConJ2:

If C is convex,

then a line L intersects the bdry 2 times.

If C is convex except it

has a smooth concave bump B stricking into it

then pick b near B, so that some lines L go thru B and some do not.

At the transition the usual behavior would be

that L changes from d intersects to d-2

intersects with Z_p.

So this is consistent with 4-Intercepts from b condition;

indeed it is the canonical picture of what happens

with p satisfying RZmnus2.

If 4-Intercepts from b condition FAILS,

then we have 2 bumps B1 and B2, or we have bumps on bumps.

Anyway the above moving L thru transition now

maybe gives that there are 4 unreal zeroes.

Bill thinks

Conjecture 3

RZminus2 condition for a set C

means C is the difference of two convex sets.

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Namely,  K1 \contains K2 ,  K1 and  K2 are convex.

and

  C = K1 - K2=: { x : x is in K1 but nt in K2 }


This leads up to the big question.


  Q3. Given R(x,y,z)  the RLMI as above.

  Let \cD_R denote its positivity set { X,Y,Z : R(X,Y,Z) >0}.

  Can we write down a simple LMI

         (a) L1 whose positivity set is the convex hull of \cD_R?

          (b) L2 whose positivity set is the set K2, which one removes?

  In other words how does one get a hold of K1, and K2 directly?
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IDEA:
Given

$$\text{R(x,y,z)}:= \begin{pmatrix} x^{-1} & L_{12}(x,y,z) \\ L_{12}(x,y,z)^T & L_{22}(x,y,z) \end{pmatrix}$$

The two inequalities $R_A(X,Y,Z) > 0$ and $X^{-1} > A$, where

$$\mathrm{R}_A(x,y,z) := \begin{pmatrix} A & L_{12}(x,y,z) \\ L_{12}(x,y,z)^T & L_{22}(x,y,z) \end{pmatrix}$$

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with a very insightful choice of A
might help. That is, split the problem in two
somehow.

1 Some Answers

Bill did an MMa notebook.

2 From VV

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GOOD NEWS:

 Assume that p defines a smooth irreducible curve.

 Then p admits a self adjoint determinantal representation with

at most 2 negative eigenvalues in J iff C is enclosed by at least

[d/2]-1 ovals (d=deg p).
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Of course in this case RZminus2 condition is satisfied w. r. to any point in C, and so is 4-Intercepts from b condition for any b in C.

BAD NEWS:

I am pretty sure that I have an example of a curve of degree 5 consisting of 6 unnested ovals and a pseudoline such that for one of the regions C lying in the exterior of all ovals the RZminus2 condition is satisfied wrt some points in C and violated wrt other points.

The moral of the matter seems to be that unlike the RZ condition which necessarily forces the right geometric configuration of the ovals, the RZminus2 does not. It can hold for some points in C and not for others, and most likely also wrt all points in C, just because of the way the ovals twist around and not because of their nesting.

So the RZminus2 condition is too weak. What is needed is some way to count the zeroes ‘‘with signs’’, as when one counts degrees or intersection numbers in topology. I am not sure how to carry this out, though. I should say that I think that the geometric condition of being enclosed by [d/2]-1 ovals is quite reasonable by itself. My main problem with it is that I do not quite see how to generalize it to handle singular curves (not to mention the higher dimensional case).

From vinnikov@cs.bgu.ac.il Mon Jun 13 03:50:22 2005 Date: Mon, 13 Jun 2005 13:43:29 +0300 (IDT) From: Victor Vinnikov <vinnikov@cs.bgu.ac.il> To: Bill Helton <helton@math.ucsd.edu> Cc: Victor Vinnikov <vinnikov@cs.bgu.ac.il> Subject: Re: your mail

Dear Bill,

A correction to ‘‘GOOD NEWS’’ --- it is J with at most 1 negative eigenvalue.

Regarding ‘‘BAD NEWS’’, the example is essentially from

N. A’Campo, Sur la premi‘ere partie du seizi‘eme probl‘eme de Hilbert, S‘eminaire Bourbaki 1978/79, n. 537, p. 537-02

Here is the description. I follow A’Campo’s notation and I will fax you the page from his paper later. It would be indeed nice to plot it in MMa to make sure that everything works.

Let C_3 be the cubic curve $y^2-p(x)=0$, where p is a cubic polynomial with 3 distinct real roots (e.g., $y^2 - x (x - 1) (x-2) = 0$). Let L_1 and L_2 be straight lines which are close to tangents to C_3 at the two symmetric real inflection points and each of which intersects C_3 in 3 real points. The curve V in question is given by $C_3 \cdot L_1 \cdot L_2 = \epsilon$ with ϵ small of a suitable sign.

It consists of a pseudo line denoted I and 6 unnested ovals denoted $II,\ldots,VII$. C is the region bounded by the pseudoline I and the oval II.

Notice

that C lies in the exterior of all ovals. So a J in a symmetric pencil representation of C has at least 2 negative eigenvalues.

[Notice also that the 4-Intercepts from b condition seems to hold for any b in C.]

Now, move each of the lines L_1 and L_2 in parallel towards the oval II till it touches the oval. Denote the resulting lines $L_1’$ and $L_2’$. It is quite clear that for each point in the (bounded) subregion of C bounded by $L_1’$, $L_2’$ and II, the RZminus2 condition is satisfied. It is also quite clear that there are points in C so that the RZminus2 condition fails.

One additional remark on ‘‘GOOD NEWS’’: it is conceivable (I think) that the statement fails if one requires real symmetric pencil representations rather than self adjoint.

Best regards, Victor

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> what is the example?

> i would like to plot it.

> is it in an MMa file?

> bill