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Towards Making LMIs Automatically

1 Introduction

In control theory applications of optimization problems, one comes across constraints on noncommuting variables in the form of matrix inequalities involving rational functions of matrix variables. The inequality

$$PA + A^TP + (PB + C^TD)R^{-1}(PB + C^TD)^T + C^TC < 0$$

is an example of an inequality involving a rational function of several matrix variables. Since matrix multiplication does not commute, we must deal with these rational functions of matrix varaibles as rational functions in noncommuting indeterminants when manipulating these expressions algebraically. When these inequalities involving rational functions of matrix variables can be written in the form of linear matrix inequalities, there are reliable numerical algorithms for finding optimal solutions in a feasiblity domain. To this end, it is in our interest to study the relationships between positivity sets of rational functions in noncommuting variables and LMIs.

Consider $x = (x_1, \ldots, x_g)$ a vector of noncommuting indeterminants. Define a **NC linear pencil** (NC stands for noncommutative) as a function, $L(x, x^T)$, that can be written as

$$L(x, x^T) = A_0 + \sum_{i=1}^{g} A_i x_i + F_i x_i^T$$

for some matrices $A_0, A_i, F_i \in \mathbb{R}^{m\times m}$ for some m. For example if $x = (x_1, x_2)$, then

$$L(x, x^T) := \begin{pmatrix} x_1 + x_1^T & x_2 \\ x_2^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_1 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_1^T + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x_2^T + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is an NC linear pencil. Also notice that a Schur complement of $L(x, x^T)$ is

$$r(x, x^T) := x_1 + x_1^T - x_2 x_2^T.$$

Using some basic facts about Schur complements we can conclude that for any tuple of $m\times m$ matrices (X, X^T),

$$L(X, X^T) > 0 \text{ iff } r(X, X^T) > 0.$$

The types of realizations that we will be discussing involve writing NC rational functions as Schur complements of NC linear pencils.

Theorem 3.1 *Suppose that $r(x, x^T)$ is a symmetric NC rational function. Then there exists a symmetric NC linear semi-pencil $L(x, x^T)$ such that $r(x, x^T)$ is the Schur complement of $L(x, x^T)$. In the case that all of the x_i are symmetric, there exists a symmetric NC linear pencil $L(x)$ whose Schur complement is $r(x)$.*

The fact that all symmetric NC rational functions can be realized as the Schur complement of a symmetric NC linear pencil is not a new result (e.g. Berstel & Reutenauer, 1988.) The question of how one can construct such a linear pencil is answered in this paper. We will give a more constructive proof that all symmetric NC rational functions are Schur complements of NC linear pencils. From this easy proof an algorithm for construcing the pencil in question readily follows. The resulting algorithm has been implemented under the NC Algebra package for Mathematica.

To prove that all symmetric NC rational functions are realizable as Schur complements of linear pencils, we will first need a different type of realization. We say that a rational function, $r(x, x^T)$, has a **CGB representation** if there exist some $d \in \mathbb{N}$, $B, C \in \mathbb{R}^d$ and a $d \times d$ NC linear pencil $G(x, x^T)$ so that

$$r(x, x^T) = C^T G(x, x^T)^{-1} B.$$

For example consider again

$$r(x, x^T) := x_1 + x_1^T - x_2 x_2^T.$$

Using Lemma 2.1 one can see that

$$r(x, x^T) = C^T G(x, x^T)^{-1} B$$

when

$$C = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad G(x, x^T) = \begin{pmatrix} x_1 + x_1^T & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & y^T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Theorem 2.2 *All NC rational functions have a CGB representation.*

Given a symmetric rational function we use this CGB representation to find a symmetric NC linear pencil whose Schur complement is the given rational function.

1.1 Notation

Again, let $x = \{x_1, \ldots, x_g\}$ denote noncommuting indeterminants. Let $\mathcal{N}_*(x)$ denote the free $\mathbb{R}$-algebra on the $2g$ generators $\{x, x^T\} = \{x_1, \ldots, x_g, x_1^T, \ldots, x_g^T\}$, i.e. the noncommutative polynomials on those $2g$ generators. The algebra has a natural involution determined by $x_j \mapsto x_j^T$, $x_j^T \mapsto x_j$, and supposing that w is a word in $\{x, x^T\}$, say $w = z_1 \cdots z_n$, then $w^T = z_n^T \cdots z_1^T$. We say that $p(x, x^T) \in \mathcal{N}_*(x, x^T)$ is **symmetric** if $p(x, x^T) = (p(x, x^T))^T$. Let $p(x, x^T)^{-1}$ denote the inverse of $p(x, x^T)$ satisfying $p(x, x^T)^{-1} p(x, x^T) = 1 = p(x, x^T) p(x, x^T)^{-1}$ and thus $(p(x, x^T)^{-1})^T = (p(x, x^T)^T)^{-1}$. Let the **NC rational functions of $\{x, x^T\}$ with real coefficents**, [1] denoted by $\mathcal{R}_*(x)$, be the closure of $\mathcal{N}_*(x)$ under finite

[1]We do not address the complicated issue of when two rational expressions are the same. Fortunately it is not needed here since there may be many different realizations for a single rational function.

numbers of inversions, products, transposes, and sums. Define $r(x, x^T)$ to be **symmetric** if $(r(x, x^T))^T = r(x, x^T)$. When the x_i are assumed to be symmetric denote the polynomials as $\mathcal{N}(x)$ and the rationals as $\mathcal{R}(x)$.

This article describes elementary constructions of "system realizations" of multivariable noncommutative symmetric rational functions. The type of system realization that we produce is most easily defined as a Schur complement of a symmetric NC linear pencil where these new terms are defined as below.

For a $d \times d$ matrix $L(x, x^T)$ call L a **NC linear semi-pencil** if the entries of L are polynomials in $\mathcal{N}_*(x)$ of degree one or less. We say that a $d \times d$ matrix $L(x)$ is a **NC linear pencil** if the entries of L are polynomials in $\mathcal{N}(x)$ of degree one or less.

Recall that if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a block 2×2 matrix, then

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS_1^{-1}CA^{-1} & -A^{-1}BS_1^{-1} \\ -S_1^{-1}CA^{-1} & S_1^{-1} \end{pmatrix} \tag{1.1}$$ it:e11

when A and $S_1 = D - CA^{-1}B$ are invertible. Or equivalently,

$$M^{-1} = \begin{pmatrix} S_2^{-1} & -S_2^{-1}BD^{-1} \\ -D^{-1}CS_2^{-1} & D^{-1} + D^{-1}CS_2^{-1}BD^{-1} \end{pmatrix} \tag{1.2}$$ it:e12

when D and $S_2 = A - BD^{-1}C$ are invertible. The matrices S_1 and S_2 are called **Schur complements** of M.

1.2 Symmetric Rational Functions in Symmetric Indeterminants

Suppose that $x = \{x_1, \ldots, x_g\}$ is a vector of symmetric noncommuting indeterminants. In addition suppose that

$$L(x) = A_0 + A_1x_1 + \cdots + A_gx_g$$

is a symmetric NC linear pencil where $A_i \in \mathbb{R}^{d \times d}$ are symmetric for all i. We say that $L(x)$ is **pinned** provided that there exists some $v \in \mathbb{R}^d$ such that $A_iv = 0$ for all $1 \leq i \leq g$. We call v the **vector pinning the pencil**. Otherwise the pencil is **unpinned**.

Consider a **symmetric descriptor realization**

$$r(x) = D + C(J - L(X))^{-1}C^T$$

for some NC linear pencil $L(x)$ with $A_0 = 0$ of a given symmtric rational function $r(x)$. We say that the realization is **observable** if the following condition is satisfied

$$CJA^wv = 0 \quad \text{ for all words w}$$

implies $v = 0$. **Controllable** means that the span of $(JA^w)^TC^T$ is all of $\mathbb{R}^d$. The realization is called **minimal** if it is both controllable and observable. By symmetry controllability and observability are equivalent.

Due to recent work of Helton and collaborators we know the following result:

Theorem 1.1 (HMV in prep) *Consider a symmetric descriptor realization*

$$r(x) = D + C(J - L(x))^{-1}C^T.$$

of r which is controllable (so observable as well). Assume is matrix convex to the extent that there exists an $\epsilon > 0$ so that

$$\frac{1}{2}r''(X)[H] = C(J - L(X))^{-1}L(H)(J - L(X))^{-1}L(H)(J - L(X))^{-1}C^T$$

is finite and positive semidefinite for all $|X| < \epsilon$ and all H.

(1) The symmetric pencil $J - L(X)$ is unpinned implies J is positive definite. Wlog we can take $J = I$.

(2) More generally even if the realization is pinned, then define

$$\alpha_0 := \begin{pmatrix} A_1 & A_2 & \dots A_g \end{pmatrix}$$

and let P_{α_0} denote the orthogonal projection onto the range of α_0, then $P_{\alpha_0} J P_{\alpha_0}$ is positive semidefinite. Moreover when r is a single NC rational function rather than a matrix of NC rational functions, the pinning space has dimension at most one, so the codimension of Range α_0 is at most one. This implies that J has at most one negative eigenvalue.

Given a symmetric convex rational function $r(x)$ it is not difficult to show that the algorithm presented in this paper will produce a descriptor realization. If that realization were minimal and unpinned, then as a consequence of the above result the sublevel sets for $r(x)$ will be equivalent to LMIs. So we face two clear problems with our realization.

1. Our realization is not necessarily minimal. The issue of taking our realization and writing an equivalent minimal realization is not difficult however.

2. After we find a minimal realization it may will be a pinned realization. The issue of "unpinning" a realization is still open.

If we can indeed unpin minimal realizations, then the algorithm presented in this paper would be a first step towards making LMIs automatically from sublevel sets of symmetric rational functions.

1.3 LMIs

sec:lmi

In many practical optimization problems when one can write inequality constraints in the form of linear matrix inequalities, there are very reliable numerical algorithms to solve the optimization problem. Our overall goal would be to write a set of inequality constraints given as rational functions in the form of an LMI.

Given a rational function $r(x)$ in symmetric inderminants $x = (x_1, \dots, x_g)$ we can define the **positivity domain of** r to be

$$\mathcal{D}_r := \{X \in \mathbb{R}^{ng \times n} \text{ for some } n : r(X) > 0 \text{ and } X_i^T = X_i\}.$$

The component of $\mathcal{D}_r$ containing 0 we will denote as $\mathcal{D}_r^0$. Given a symmetric NC linear pencil $L(x)$ define similarly

$$\mathcal{D}_L := \{X \in \mathbb{R}^{ng\times n} \text{ for some } n : L(X) > 0 \text{ and } X_i^T = X_i\}.$$

Finding a symmetric NC linear pencil $L(x)$ so that $\mathcal{D}_L = \mathcal{D}_r^0$ is what is meant by writing the positivity set of a rational function as an LMI.

1.4 Outline

In Section 2 we will prove the existence of a representation of the functions in $\mathcal{R}_*(x)$ that will be used in Section 3. We will prove the main realization theorem in Section 3. In Section 4 we will generalize the results of Sections 2 and 3 to include matrix valued rational functions. Finally, in Section 5, we will discuss briefly the algorithms that NC Algebra uses to find the representations of interest.

2 A Representation of Functions in $\mathcal{R}_*(x)$

In this section we will show that for every $r(x, x^T) \in \mathcal{R}_*(x)$ there exist vectors C and B in $\mathbb{R}^d$ and a $d\times d$ NC linear semi-pencil $G(x, x^T)$ such that $r(x, x^T) = C^T G(x, x^T)^{-1}B$. We will call such a function $r(x, x^T)$ **CGB representable** and let $Z(x, x^T)$ denote the set of CGB representable functions in $\mathcal{R}_*(x)$.

2.1 Some Properties of $Z(x, x^T)$

Lemma 2.1

1. *Suppose that $r(x, x^T) \in \mathcal{N}_*(x)$ is a degree one or less polynomial. Then $r(x, x^T) \in Z(x, x^T)$.*

2. *Suppose that*

$$r_1(x, x^T) = C_1^T G_1(x, x^T)^{-1}B_1 \in Z(x, x^T) \ \textit{and} \ r_2(x, x^T) = C_2^T G_2(x, x^T)^{-1}B_2 \in Z(x, x^T)$$

for some NC linear semi-pencils $G_1(x, x^T)$ and $G_2(x, x^T)$. Then

   (a) $r_1(x, x^T) + r_2(x, x^T) \in Z(x, x^T)$,

   (b) $r_1(x, x^T)r_2(x, x^T) \in Z(x, x^T)$, *and*

   (c) $(r_1(x, x^T))^T \in Z(x, x^T)$.

3. *Suppose that $r(x, x^T) = C^T G(x, x^T)^{-1}B$ for some NC linear semi-pencil $G(x, x^T)$ and that $r(x, x^T) \neq 0$. Then $r(x, x^T)^{-1} \in Z(x, x^T)$.*

The proof gives constructions for each of the above items.

1. Suppose that $r(x, x^T) \in \mathcal{N}_*(x)$ and r is degree one or less. Then

$$r(x, x^T) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -r(x, x^T) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.3} \quad \boxed{\texttt{e23}}$$

   So $r \in Z(x, x^T)$.

2. (a) Notice that $r_1(x, x^T) + r_2(x, x^T) =$

$$\begin{pmatrix} C_1^T & C_2^T \end{pmatrix} \begin{pmatrix} G_1(x, x^T) & 0 \\ 0 & G_2(x, x^T) \end{pmatrix}^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{2.4} \quad \boxed{\texttt{e24}}$$

   So $r_1(x, x^T) + r_2(x, x^T) \in Z(x, x^T)$.

   (b) Notice that

$$\begin{pmatrix} -G_1(x, x^T) & B_1 C_2^T \\ 0 & G_2(x, x^T) \end{pmatrix}^{-1} = \begin{pmatrix} -G_1(x, x^T)^{-1} & G_1(x, x^T)^{-1} B_1 C_2^T G_2(x, x^T)^{-1} \\ 0 & G_2(x, x^T)^{-1} \end{pmatrix}.$$

   So then

$$\begin{pmatrix} C_1^T & 0 \end{pmatrix} \begin{pmatrix} -G_1(x, x^T) & B_1 C_2^T \\ 0 & G_2(x, x^T) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \tag{2.5} \quad \boxed{\texttt{e25}}$$

$$= C_1^T (G_1(x, x^T)^{-1} B_1 C_2^T G_2(x, x^T)^{-1}) B_2 = r_1(x, x^T) r_2(x, x^T).$$

   Thus $r_1(x, x^T) r_2(x, x^T) \in Z(x, x^T)$.

   (c) Since

$$(r_1(x, x^T))^T = B_1^T G_1(x, x^T)^{-T} C_1, \tag{2.6} \quad \boxed{\texttt{ep24}}$$

   we have that $(r_1(x, x^T))^T \in Z(x, x^T)$.

3. Notice that by equation it:e11 1.1,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -G(x, x^T) & B \\ C^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (C^T G(x, x^T)^{-1} B)^{-1} = r(x, x^T)^{-1} \tag{2.7} \quad \boxed{\texttt{e26}}$$

   when $r(x, x^T) \neq 0$. Thus when $r(x, x^T) \neq 0$, we have $r(x, x^T)^{-1} \in Z(x, x^T)$. *q.e.d.*

2.2 The Existence Theorem

it:p22 **Theorem 2.2** *Every $r(x, x^T) \in \mathcal{R}_*(x)$ is CGB representable.*

Proof:
By Lemma 2.1.1 we see that all NC polynomials of degree one or less are CGB representable. Thus since each $p(x) \in \mathcal{N}_*(x)$ can be written as a finite sum of finite products of degree one or less NC polynomials and by Lemma 2.1.2, we have that $\mathcal{N}_*(x) \subseteq Z(x, x^T)$. Therefore by the closure properties of Lemma 2.1.2 and Lemma 2.1.3 and the construction of $\mathcal{R}_*(x)$ we have that $\mathcal{R}_*(x) \subseteq Z(x, x^T)$. *q.e.d.*

3 The Pencil Result

s3

In this section we will prove that any symmetric $r(x, x^T) \in \mathcal{R}_*(x)$ can be written as a Schur complement of some symmetric NC linear semi-pencil.

3.1 The Realization Theorem

it:p31 **Theorem 3.1** *Suppose that $r(x, x^T) \in \mathcal{R}_*(x)$ is symmetric. Then there exists a symmetric NC linear semi-pencil $L(x, x^T)$ such that $r(x, x^T)$ is the Schur complement of $L(x, x^T)$. In the case that all of the x_i are symmetric, there exists a symmetric NC linear pencil $L(x)$ whose Schur complement is $r(x)$.*

Proof:
From Proposition it:p22 2.2, there exist $C_1, B_1 \in \mathbb{R}^d$ and a $d \times d$ NC linear semi-pencil $G_1(x, x^T)$ such that $\frac{1}{2} r(x, x^T) = C_1^T G_1(x, x^T)^{-1} B_1$. Define now

$$C := \begin{pmatrix} C_1 \\ 0 \end{pmatrix}, B := \begin{pmatrix} 0 \\ B_1 \end{pmatrix}, \text{ and } G(x, x^T) := \begin{pmatrix} 0 & (G_1(x, x^T))^T \\ G_1(x, x^T) & 0 \end{pmatrix}. \tag{3.8}$$

e37

Notice we have $\frac{1}{2} r(x, x^T) = C^T G(x, x^T)^{-1} B$ and G is symmetric. Consider now the symmetric NC linear semi-pencil $L(x, x^T)$ defined as follows

$$L(x, x^T) := \left(\begin{array}{c|cccccc} 0 & 1 & C^T & 1 & 0 & 0 & C^T \\ \hline 1 & 1 & B^T & 0 & 0 & 0 & 0 \\ C & B & D & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -B^T & 0 & 0 \\ 0 & 0 & 0 & -B^T & -D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -B^T \\ C & 0 & 0 & 0 & 0 & -B & -D \end{array}\right) \tag{3.9}$$

e38

where $D := G(x, x^T) + BB^T$. Now taking the Schur complement as in S_2 of equation it:e12 1.2 we have Schur complement$(L(x, x^T))$

$$= -\begin{pmatrix} 1 & C^T \end{pmatrix} \begin{pmatrix} 1 & B^T \\ B & D \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ C \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & B^T \\ B & D \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & C^T \end{pmatrix} \begin{pmatrix} 1 & B^T \\ B & D \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C \end{pmatrix}$$

Now by equation it:e11 1.1 we have

$$\begin{pmatrix} 1 & B^T \\ B & D \end{pmatrix}^{-1} = \begin{pmatrix} 1 + B^T G(x, x^T)^{-1} B & -B^T G(x, x^T)^{-1} \\ -G(x, x^T)^{-1} B & G(x, x^T)^{-1} \end{pmatrix}.$$

So Schur complement$(L(x, x^T)) = C^T G(x, x^T)^{-1} B + B^T G(x, x^T)^{-1} C = r(x, x^T)$ since r and G are symmetric. When the x_i are symmetric, $L(x) := L(x, x^T)$ will clearly be a symmetric NC linear pencil. *q.e.d.*

3.2 A Remark on Applications

In some engineering problems one might encounter a rational function similar to those in this paper. One possible difference between those encountered in practice and those discussed so far may be that some of the "indeterminants" are actually known. For example, one might be interested in a Riccati expression

$$R(X) := AX + XA^T - XBB^TX + C^TC$$

when A, B, C are known and $X = X^T$ is unknown. Treating each variable as an indeterminant, one can easily see that $R(X)$ is a rational function in the sense defined earlier. Thus there is a symmetric NC linear pencil whose Schur complement is equal to $R(X)$. However in the case that some of the variables in $x = (x_1, \ldots, x_g)$ are known, we may consider a modified definition of NC linear pencil. Instead of saying that the entries of the pencil are degree one or less polynomials, we can loosen the requirement on the entries and say that each term can have one or fewer factors that is an unknown variable. For example, looking back at the Ricatti scenario,

$$p_1(X) := AX + XA^T + C^TC$$

would be an acceptable entry in a linear pencil while

$$p_2(X) := XBB^TX$$

would not. Notice that this new definition of NC linear pencil would not change any of the results so far – it only expands the class of NC linear pencils. Indeed

$$L(X) := \begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & 1 \end{pmatrix}$$

would be a NC linear pencil whose Schur complement is $R(X)$.

4 Matrix Valued Rational Functions

s4

In this section we will show that both Theorem 2.2 and Theorem 3.1 generalize to the case of matrix valued rational functions. By matrix valued rational functions we mean $m \times n$ matrices $W(x, x^T)$ with coefficents in $\mathcal{R}_*(x)$.

4.1 Generalizing Theorem 2.2

To begin, we must extend what is meant by CGB representable. If $W(x, x^T)$ is a $m \times n$ matrix valued rational function, then $W(x, x^T)$ is called CGB representable when there exist $d \in \mathbb{N}, \mathfrak{C}^T \in \mathbb{R}^{m \times d}, \mathfrak{B} \in \mathbb{R}^{d \times n}$, and a $d \times d$ NC linear semi-pencil $\mathfrak{S}(x, x^T)$ such that $W(x, x^T) = \mathfrak{C}^T \mathfrak{S}(x, x^T)^{-1} \mathfrak{B}$.

it:p41 **Theorem 4.1** *Every matrix valued rational function is CGB representable.*

Proof:
it:p22
Suppose that $W(x,x^T) = (r_{i,j}(x,x^T))_{i,j=1}^{m,n}$ for some $r_{i,j}(x,x^T) \in \mathcal{R}_*(x)$. Then by Theorem 2.2 for each i,j there exist $C_{i,j}, B_{i,j} \in \mathbb{R}^{d_{i,j}}$ and a $d_{i,j} \times d_{i,j}$ NC linear semi-pencil $G_{i,j}(x,x^T)$ such that $r_{i,j}(x,x^T) = C_{i,j}^T G_{i,j}(x,x^T)^{-1} B_{i,j}$. Now define

$$\mathfrak{C}^T := \begin{pmatrix} C_{1,1}^T & \cdots & C_{1,n}^T & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & C_{2,1}^T & \cdots & C_{2,n}^T & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & C_{m,1}^T & \cdots & C_{m,n}^T \end{pmatrix},$$

$$\mathfrak{S}(x,x^T) := \begin{pmatrix} \begin{matrix} G_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_{1,n} \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \cdots & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} G_{2,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_{2,n} \end{matrix} & \cdots & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \cdots & \begin{matrix} G_{m,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G_{m,n} \end{matrix} \end{pmatrix},$$

$$\mathfrak{B} := \begin{pmatrix} B_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{1,n} \\ B_{2,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{2,n} \\ & \vdots & \\ B_{m,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{m,n} \end{pmatrix}.$$

Let $d = \sum_{i,j=1}^{m,n} d_{i,j}$. Notice that $\mathfrak{C}^T \in \mathbb{R}^{m \times d}, \mathfrak{B} \in \mathbb{R}^{d \times n}$, and $\mathfrak{S}(x,x^T)$ is a $d \times d$ NC linear semi-pencil. In addition it is an easy computation to show that $W(x,x^T) = \mathfrak{C}^T \mathfrak{S}(x,x^T)^{-1} \mathfrak{B}$. Thus we have that every matrix valued rational function is CGB representable. *q.e.d.*

4.2 Generalizing Theorem 3.1

it:p42 **Theorem 4.2** *Suppose that $W(x,x^T) \in M_m(\mathcal{R}_*(x))$ is symmetric. Then there exists a symmetric NC linear semi-pencil $L(x,x^T)$ such that $W(x,x^T)$ =Schur Complement($L(x,x^T)$). In the case that all of the x_i are symmetric, there exists a symmetric NC linear pencil $L(x)$ whose Schur complement is $W(x)$.*

Proof:
By Theorem 4.1 there exist $d \in \mathbb{N}, C_1^T \in \mathbb{R}^{m \times d}, B_1 \in \mathbb{R}^{d \times m}$, and a $d \times d$ NC linear semi-pencil $G_1(x, x^T)$ such that $\frac{1}{2}W(x, x^T) = C_1^T G_1(x, x^T)^{-1} B_1$. From this point the proof follows identically to that of Proposition 3.1. *q.e.d.*

5 Symbolic Computation of Realizations

The theorems in this paper convert readily to computer algorithms for finding realizations. In fact they were enhanced and implemented by John Shopple under the NC Algebra package for Mathematica.

The most basic algorithms coming from ideas presented in the paper come from Lemma 2.1. The corresponding functions manipulate given CGB representations in a way that corresponds to some algebraic manipulation of the given rational expression. The function `LPInverse` takes a CGB representation for a rational expression as input and outputs the CGB representation for the inverse of the given rational expression by implementing Equation 2.7. Similarly the function `LPTranspose` takes a CGB representation for some rational expression as input and outputs the CGB representation of the transpose of the given expression by implementing Equation 2.6. Given the CGB represenations for two rational expressions, the functions `LPCombinePlus` and `LPCombineTimes` output the CGB representation of the sum and product of the given rational expressions. The formulas for the output of these two functions can be read off of Equations 2.4 and 2.5.

The function `NCMultiRealization` takes as inputs a NC rational expression and a list of indeterminants and then outputs a CGB representation for the given rational expression. The algorithm is recursive. It first determines whether the rational expression is linear or not. If the expression is linear, then the CGB representation (given by equation 2.3) is returned. If not, then the function looks at the top operation (the `Head`) that is used in the definition of the expression, e.g. `Plus, inv, tp, NonCommutativeMultiply`, or `Times`. Then the function calls (and returns the output of) the appropriate function from `LPCombinePlus, LPInverse, LPTranspose,` and `LPCombineTimes`. The inputs for the CGB manipulating functions in the function calls from `NCMultiRealization` will be `NCMultiRealization` applied to each argument of the Head (thus the recursion.) As an example, suppose that

$$r(x, x^T) = x + x**x^T.$$

Here the $**$ represents `NonCommutativeMultiply` and the Mma `Head` of the representation for $r(x, x^T)$ is `Plus`. If `NCMultiRealization` is called with inputs $r(x, x^T)$ and $\{x, x^T\}$, then the function will return

$$\texttt{LPCombinePlus}\big[\texttt{NCMultiRealization}[x, \{x, x^T\}], \texttt{NCMultiRealization}[x**x^T, \{x, x^T\}]\big]$$

where `NCMultiRealization`$[x**x^T, \{x, x^T\}]$ returns

$$\texttt{LPCombineTimes}\big[\texttt{NCMultiRealization}[x, \{x, x^T\}], \texttt{NCMultiRealization}[x^T, \{x, x^T\}]\big].$$

Since the expressions x and x^T are linear, `NCMultiRealization` with inputs given by x or x^T and $\{x, x^T\}$ will return CGB representations given by Equation 2.3.

The function `CGBtoPencil` takes a CGB representation for some symmetric rational expression $r(x, x^T)$ and returns a symmetric NC linear semi-pencil whose Schur complement is $r(x, x^T)$. The algorithm can be read off from the proof of Theorem 3.1. In particular, the function redefines the given C, G, and B as in 3.8 also multiplying C by $\frac{1}{2}$. Then it returns a symmetric NC semi-pencil exactly as in equation 3.9.

The function `NCFindPencil` takes a symmetric rational expression and a list of indeterminants and outputs a symmetric NC linear semi-pencil whose Schur complement is the given rational expression. The function `NCFindPencil` calls `NCMultiRealization` with the given rational expression and list of indeterminants as input. The function then sends the output of `NCMultiRealization` to `CGBtoPencil` whose output is the desired realization.

There are also functions written to deal with the case of matrix valued rational functions. The functions `CGBtoPencilMatrix` and `NCFindPencilMatrix` are almost exactly as those above – the exceptions are in `NCFindPencilMatrix` the input is a matrix valued rational function and it calls `NCMatrixMultiRealization` rather than `NCMultiRealization`. The algothithm for `NCMatrixMultiRealization` follows from the proof of Theorem 4.1. Indeed if the matrix valued rational function $W(x, x^T) = (r_{i,j}(x, x^T))_{i,j=1}^{m,n}$ is input, then the function will find a CGB representation for each $r_{i,j}(x, x^T)$ using `NCMultiRealization`. It then returns a CGB representaion for $W(x, x^T)$ using the equations in the proof of Theorem 4.1.

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