ONE $R^{2\times 2}$ TEST OF CONVEXITY

$$p(x) := x^4 + (x^4)^T$$

$$\frac{1}{2}p(\mathcal{X}) + \frac{1}{2}p(\mathcal{Y}) - p(\frac{1}{2}\mathcal{X} + \frac{1}{2}\mathcal{Y}) \succeq 0?$$ **Try matrices**

$$\mathcal{X} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathcal{Y} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \tfrac{1}{2}\mathcal{X} + \tfrac{1}{2}\mathcal{Y} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\frac{1}{2}p(\mathcal{X}) = \frac{1}{2}\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^4 + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^4\right] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$\tfrac{1}{2}p(\mathcal{Y}) = \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix}$ and $p(\tfrac{1}{2}\mathcal{X} + \tfrac{1}{2}\mathcal{Y}) = \begin{pmatrix} 22 & 15 \\ 15 & 12 \end{pmatrix}$. **Then**

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} - \begin{pmatrix} 22 & 15 \\ 15 & 12 \end{pmatrix} \succeq 0$$ **i.e., is PSD**

Convex Matrix Inequalities vs Linear MIs

Scott McCullough **University of Florida**

Victor Vinnikov **Ben Gurion University**

Mihai Putinar **Ben Gurion University**

Nick Slinglend **UC San Diego**

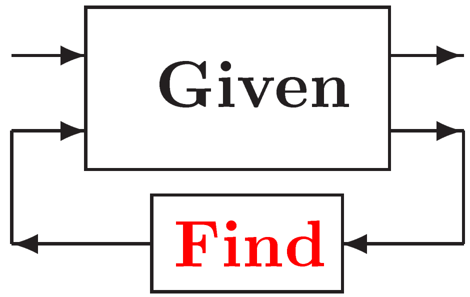
Camino, de Oliveira, Shopple, Skelton **UCSD**

I am Bill Helton **UCSD**

NCAlgebra[a]

[a] Helton, Stankus (CalPoly SanLObispo), Miller (Archemedies Inc)

Systems problems → Matrix Ineq



Many such problems Eg. H^∞ control

Example: Get Riccati expressions like

$$AX + XA^T - XBB^TX + CC^T \succ 0$$

OR linear matrix inequalities Linear Matrix Inequalities like

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succ 0$$

which is equivalent to the Riccati inequality.

CONVEX MIs vs LMIs ?

Convex is good!

Given a function r and the associated Matrix Inequality

$$\gamma I - r(\vec{X}) \succ 0.$$

is there an Linear Matrix Inequality

$$\gamma \Lambda_0 \ - \ \mathcal{L}(\vec{X}) \succ 0.$$

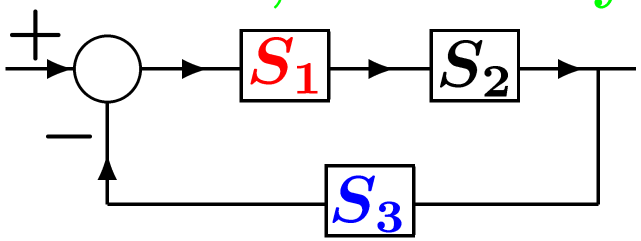
whose solutions $\vec{X}$ are the same as (a component of) the solutions to $\gamma I - r(\vec{X}) \succ 0$?

Some type of convexity for r is necessary.

DIMENSIONLESS FORMULAS

1. **DIMENSIONLESS FORMULAS – THIS TALK**

   Topology is fixed; but many systems . E.g.

   WANT FORMULAS: which hold regardless of the dimension of system S_1, S_2, S_3. E.g. almost all control text problems.

   – Get noncommutative formulas

DIMENSIONLESS INEQUALITIES

Does Scale Gracefully with Dimension

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succeq 0 \qquad (1)$$

Looks the same regardless of system size.

Dimension Dependent

If dimensions of the matrices A, B, C, X are specified write (1) with matrices $L_0, \ldots, L_m$ as,

$$\sum_{j=0}^{m} L_j x_j \succeq 0 \qquad (2)$$

unknown numbers x_j, (from $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$)

THE RANGE OF QUESTIONS

Characterize and analyze these situations:

	LMI	$CONVEX$
$DIMENSIONLESS(NC)$	**?**	**?**
$DIMENSION\ DEP.(COM)$	**?**	$usual\ \ convexity$

Q? Can we treat many more problems with convex techniques than LMI techniques?

Ultimately we want Symbolic Algorithms

NOTATION

COLORS:

Black = knowns A, B

Colors like red green are variables $x \quad h$

Caps are matrices $X \quad H$

Substitute the matrix X **for variable** x **is** $x \to X$

g-tuples of variables $\vec{x} = \{x_1, \cdots, x_g\}$

g-tuples of symmetric matrices $\vec{X} = \{X_1, \cdots, X_g\}$;
$\vec{X} \in (\mathbb{S}\mathbf{R}^{n \times n})^g$.

Polynomial

$$p(x_1, x_2) := x_1^4 + x_2^4 + 7x_1$$

Geometrically matrix convex $\quad 0 \leq \alpha \leq 1$

$$r(\alpha \vec{X} + (1-\alpha)\vec{Y}) \quad \preceq \quad \alpha\, r(\vec{X}) + (1-\alpha)\, r(\vec{Y})$$

Matrix Convexity: Plugging any matrices into Hessian $r''(\vec{x})[h]$ of r wrt $\vec{x}$ gives a PosSD matrix.

> **Geometric and matrix convexity are the same, provided $r(\vec{x})$ is smooth and there are no constraints on $\vec{x}$. (cf. Merino+H, CDC1997)**

Notation: $\quad r''(x_1, \ldots, x_m)[h_1, \ldots, h_m]$

ONE $R^{2\times 2}$ TEST OF CONVEXITY

$$p(x) := x^4 + (x^4)^T$$

$$\frac{1}{2}p(\mathcal{X}) + \frac{1}{2}p(\mathcal{Y}) - p(\frac{1}{2}\mathcal{X} + \frac{1}{2}\mathcal{Y}) \succeq 0?$$ **Try matrices**

$$\mathcal{X} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathcal{Y} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \tfrac{1}{2}\mathcal{X} + \tfrac{1}{2}\mathcal{Y} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\frac{1}{2}p(\mathcal{X}) = \frac{1}{2}\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^4 + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^4\right] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$\tfrac{1}{2}p(\mathcal{Y}) = \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix}$ and $p(\tfrac{1}{2}\mathcal{X} + \tfrac{1}{2}\mathcal{Y}) = \begin{pmatrix} 22 & 15 \\ 15 & 12 \end{pmatrix}$. **Then**

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} - \begin{pmatrix} 22 & 15 \\ 15 & 12 \end{pmatrix} \succeq 0$$ **i.e., is PSD**

NC CONVEX POLYNOMIALS

THEOREM: **(Ando $g = 1$, McCullough-H)**

Convex NC polynomials have degree 2 or less.

What Are Convex NC Rational Functions?

This Talk

NC RATIONALS and DOMAINS

EXAMPLE: $\quad F(x,y) = (x - y^{-1})^{-1}, \quad x = x^T, y = y^T$

is a sym NC rational function in symmetric vars.
Domain is $\{X, Y : \ X - Y^{-1} \ \textit{and} \ Y \ \textit{are invertible}\}$

Given a rational expression $r(\vec{x})$ **DEFINE: its formal domain**

$$\mathcal{F}_{r,\mathrm{for}} := \{\vec{X} : r(\vec{X}) \ \textit{is not infinite}\}$$

$\mathfrak{r}$ **rational function equiv. classes of rational express** r

$$\mathcal{F}_{\mathfrak{r}} := \cup_{\{r \ \textit{represents} \ \mathfrak{r}\}} \mathcal{F}_{r,\mathrm{for}}$$

$\mathcal{F}^0_{\mathfrak{r}}$ **is the component of** $\mathcal{F}_{\mathfrak{r}}$ **the domain of** $\mathfrak{r}$ **containing 0.**

Convexity Algorithm

The theory of Camino - H- Skelton gives an algorithm

❶ **What is the region of convexity for:**

$$F(x,y) = (x - y^{-1})^{-1}, \qquad x = x^T, y = y^T$$

☞ **APPLY:** NCConvexityRegion[F, $\{x, y\}$]

$$\Updownarrow$$

$$\{2(x - y^{-1})^{-1}, \quad 2y^{-1}\}$$

☞ **CONCLUDE:** $F(X,Y)$ **is matrix convex**

$$\mathcal{G} := \{(X,Y) : Y > 0 \quad \text{and} \quad X - Y^{-1} > 0\}$$

ODDITY: Region of convexity for r is same as a component of domain of r.

$$r(\vec{x}) = r_0 + r_1(\vec{x}) + \ell(\vec{x})\ell(\vec{x})^T \tag{3}$$

r_0 is a scalar, r_1 and $\ell(\vec{x})$ are affine linear in x.

$$r(\vec{x}) = \Lambda(\vec{x})(I_d - [A_1x_1 + \cdots + A_gx_g])^{-1}\Lambda(\vec{x})^T \tag{4}$$

A_j are symm matrices, $\Lambda(\vec{x})$ is $1 \times d$ affine linear in x.

That is, ℓ, r_1 and Λ are NC linear pencils with r_1 a symmetric linear pencil.

r is NC Convex where

$$(I_d - [A_1x_1 + \cdots + A_gx_g]) \ is \ Pos \ Def$$

C. K. Li - Mathias and Vanderbai

REPRESENTATION EXAMPLE

$$F(x,y) = (x - y^{-1})^{-1} \quad \textit{is convex near } x = 2, y = 2$$

Represent

$r(x,y) = F(x+2, y+2)$, *is convex near* $x = 0, y = 0$,

in the form

$$r(\vec{x}) = \Lambda(\vec{x})(I_d - [A_1 x_1 + \cdots + A_g x_g])^{-1} \Lambda(\vec{x})^T$$

ANS: up to normalization $r(x,y) =$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\mathrm{T}} \left[\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} y \right]^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

REALIZATIONS

We say r has a butterfly realization provided there is a rational expression for r of the form

$$r(\vec{x}) = r_0 + r_1(\vec{x}) + \ell(\vec{x})\ell(\vec{x})^T \qquad .$$

$$\textbf{(BR)} \qquad +\Lambda(\vec{x})(J - [A_1 x_1 + \cdots + A_g x_g])^{-1}\Lambda(\vec{x})^T$$

$J^2 = I$, $\quad A_j$ are symmetric matrices, $\quad r_0$ is a scalar.

$\ell(\vec{x})$ and $\Lambda(\vec{x})$ and r_1 are affine linear in x, i.e. NC linear pencils with r_1 a symmetric linear pencil.

If $J = I$, then r is matrix convex where

$I - [A_1 X_1 + \cdots + A_g X_g]$ is Pos Semi Def

Main Thm says this (BR) is all Convex funcs.

ALL CONVEX NC RATIONALS

THEOREM (McCullough Vinnikov +H) Suppose r is a symmetric noncommutative rational function in symmetric variables $\vec{x}$ which is matrix convex near 0

THEN

Realization: r has a minimal unpinned butterfly realization r with J equal to the identity. Specifically

$$r(\vec{x}) = r_0 + r_1(\vec{x}) + \ell(\vec{x})\ell(\vec{x})^T + \Lambda(\vec{x})(I_d - [A_1 x_1 + \cdots + A_g x_g])^{-1}\Lambda(\vec{x})^T,$$

where r_1 and ℓ are linear in x and $\Lambda(\vec{x})$ is linear in x.

Singularities: The realization r has domain whose component containing 0, is

$$\mathcal{P} := \{\vec{X} \in \mathbb{S}^g : (I_d - [A_1 X_1 + \cdots + A_g X_g]) \textit{ is Pos Def}\}.$$

COROLLARIES

Recall Butterfly Realization:

$$r(\vec{x}) = r_0 + r_1(\vec{x}) + \ell(\vec{x})\ell(\vec{x})^T + \Lambda(\vec{x})(I_d - [A_1 x_1 + \cdots + A_g x_g])^{-1}\Lambda(\vec{x})^T,$$

where r_1 and ℓ are linear in x and $\Lambda(\vec{x})$ is linear in x.

$r(x)$ is a Poly: (Ando $g = 1$, McCullough-H)
Convex NC polynomials have degree 2 or less.

Idea: $r(\vec{x}) = r_0 + r_1(\vec{x}) + \ell(\vec{x})\ell(\vec{x})^T + \Lambda(\vec{x})\Lambda(\vec{x})^T$

Convexity Region: The function r is **matrix convex near 0** implies r is matrix convex on all of (the 0 component of) the domain of r:

$$\mathcal{P} = \{\vec{X} \in \mathbb{S}^g : (I_d - [A_1 X_1 + \cdots + A_g X_g]) \ is \ Pos \ Def\}.$$

Convexity Region r = A Component of Domain r

LMI REP THEOREM

THEOREM McCullough,Vinnikov, Bill) **If r is an NC sym rational function in symmetric variables x which is matrix convex near 0,**

THEN **there is a NC symmetric linear pencil $\mathcal{L}(\vec{x})$ and a PsD matrix Λ**

SUCH THAT **(the component of 0 of) the set**

$$\{\vec{X} \in \mathbb{S}^g : \gamma I - r(\vec{X}) \ is\ Pos\ Def\}$$

equals all solutions $\vec{X}$ to the LMI

$$\gamma \Lambda_0 - \mathcal{L}(\vec{X}) \ is\ Neg\ Def.$$

STATUS OF QUESTIONS

For Rational Functions

	LMI	$CONVEX$
$DIMLESS(NC)$	$CONVEX =$	$ButterFRep$
$DIM\ DEP.(COM)$	$n = 2, n > 2?$	$usual\ convexity$

Q? Can we treat more problems with convex techniques than LMI techniques?

A. FOR DIMENSIONLESS SYSTEMS PROBLEMS
CONVEX $\Leftrightarrow$ LMI

This is true under the restrictions of our set up.

We have some Symbolic Algorithms

PROOF

Note that the butterfly realization expresses

$$r - \gamma I = \quad \textit{Schur complement of } \mathcal{L}_\gamma(\vec{x}) :=$$

$$= \begin{pmatrix} -1 & 0 & \ell(\vec{x})^T \\ 0 & -(I - [\mathcal{A}_1 x_1 + \cdots + \mathcal{A}_g x_g]) & \Lambda(\vec{x})^T \\ \ell(\vec{x}) & \Lambda(\vec{x}) & r_0 - \gamma + r_1(\vec{x}) \end{pmatrix}$$

for any real number γ. GET

$$r(\vec{X}) - \gamma I \prec 0 \quad iff \quad \mathcal{L}_\gamma(\vec{X}) \prec 0$$

PROVIDED Singularities of $r(\vec{X})$ contain

. *the boundary of* $-(\, I - [\mathcal{A}_1 X_1 + \cdots + \mathcal{A}_g X_g]\,) \prec 0.$

This is what Singularities Theorem says is true.

PROOF OF REALIZATION

THM r **has a minimal Descriptor Realization** r **with** $J^2 = 1$ **and** $A_j \in \mathrm{R}^{n \times n}$ **symmetric. Specifically**

$$r(\vec{x}) = r_0 + C(J - \sum_j^g A_j x_j)^{-1} C^T, \qquad (5)$$

Schutzenberger, Fliess, Becker, Ball -Malikorn

FISHY "It's too good" Ex. It represents polynomials of degree 10 easily.

CONVEXITY vs $J = I$

LEMMA **Matrix convex + minimal implies $J = I$**

iff

there is no pinning vector η, ie. $\eta \in \cap_j \, Null(A_j)$.

PF: $g = 1$ Substitute all small symm $X \in \mathbb{R}^{n \times n}$ and all symm $H \in \mathbb{R}^{n \times n}$ $\quad x \to X \quad$ *and* $\quad h \to H$

$$0 \preceq r''(X)[H] =$$

$$[C \otimes I_n (J \otimes I_n - A \otimes X)^{-1} A \otimes H] \quad (J \otimes I_n - A \otimes X)^{-1} \quad [A \otimes H (J \otimes I_n - A \otimes X)^{-1} C^T \otimes I_n]$$

Can show $\left[A \otimes H (J - A \otimes X)^{-1} C^T\right]$ sweeps all but "a little" of $\mathbb{R}^{nd}$ $\implies (J - A \otimes X)^{-1}$ is Pos SemiDef, all X small, say $X = 0$. So $J \succ 0$.

UNPINNED and SINGULARITIES

If minimal and convex, then pinning vectors η are a one dim space. Can show J has at most 1 neg eigenvalue.

This allows conversion to an unpinned butterfly realization with $J = 1$ (several pages of algebra).

Minimality can be gotten by "cutting down".

SINGULARITIES:

$$r(\vec{X}) = r_0 + r_1(\vec{X}) + \ell(\vec{X})\ell(\vec{X})^T + \Lambda(\vec{X})(I - [A_1\vec{X}_1 + \cdots + A_g\vec{X}_g])^{-1}\Lambda(\vec{X})^T$$

is singular on the set

$$\textit{boundary of } \{\vec{X} : (I - [A_1 X_1 + \cdots + A_g X_g]) \succeq 0\}$$

Proof: Pain in the neck.

NONCOMMUTATIVE ANALYTIC
CO-ORDINATE CHANGES

NC CO-ORDINATE CHANGES

Variables are not symmetric.

An analytic polynomial map in g variables $\vec{x}$

$$\vec{x} \to p(\vec{x}) := \begin{pmatrix} p_1(\vec{x}) \\ \vdots \\ p_k(\vec{x}) \end{pmatrix}$$

Analytic means p contains no x_j^T. An NC symmetric hereditary polynomial q which is matrix positive, defines

$$\mathcal{D}_q := \{\vec{X} \in (\mathbb{R}^{n\times n})^g : \quad I \succeq q(\vec{X})\}.$$

Is there a bianalytic map p of

$\mathcal{D}_q$ onto the unit ball $\mathcal{D}_{x_1^T x_1 + \cdots + x_g^T x_g}$?

BiAnalytic Equivalence

Is there a bianalytic map p of

$\mathcal{D}_q := \{\vec{X} : \quad I \succeq q(\vec{X})\}$**onto the unit ball** $\mathcal{D}_{x_1^T x_1 + \cdots + x_g^T x_g}$**?**

Commutative $\mathbb{C}$ case $g = 1$: solved by Riemann Map Thm.

THEOREM **N. Slinglend + H**

SUPPOSE that $q(\vec{x})$ is a NC hereditary, matrix positive polynomial such that $q(0) = 0$.

THEN We can construct NC bianalytic map between $\mathcal{D}_q$ and $\mathcal{D}_{x_1^T x_1 + \cdots + x_g^T x_g}$ using an LMI or no such maps exists.

Surprising since construction is an LMI plus a rank constraint. Very nonconvex. We handle the rank constraint.

Ball Map Theorem

What are the NC analytic maps p of

$$\textit{the unit ball } \mathcal{D}_{x_1^T x_1 + \cdots + x_g^T x_g}$$

into

$$\textit{the unit ball } \mathcal{D}_{x_1^T x_1 + \cdots + x_k^T x_k}$$

which take the boundary into the boundary? $k \geq g$ We call these Joe Maps

Commutative case De Angelo 1990ish. Simple ans if $k < \frac{g}{2}$

THEOREM N. Slinglend + H

Every Ball Map is a linear isometry.

NC SEMI-ALGEBRAIC GEOMETRY

- **CLASSICAL COMMUTATIVE Suppose a polynomial p on $\mathbb{R}^n$ is positive on the domain, $\mathcal{G} \subset \mathbb{R}^n$, where another polynomial δ is positive. Must there be an algebraic relationship between p and δ?**

  Ans YES = Positivstellensatz 1920's - 1975's (active area today- has applications)

- **NONCOMMUTATIVE**

  Does this generalize to polynomials in NC variables

$$\vec{x} = \{x_1, \cdots, x_g\}$$

  Ans Often YES. McCullough, Putinar, H

ALGORITHMS and IMPLEMENTATIONS

1. **Convexity Checker - Camino, Skelton, H Public**
2. **Realization Builder: Convex Rational to LMI - Slinglend, Shopple in progress**
3. **Numerical matrix unknowns - Camino, Skelton, H in house**
4. **LMI Producer (uses existing methods on special problems) de Oliveira, H in house (out soon)**

Try NCAlgebra

END 2004