Math 193a, HANDOUT FOR HOMEWORK

Some Answers and Solutions

Here you will find some solutions and answers to problems answers to which are not given in the book.

1. ***Chapter 1***:

12. The only graphs which are not given in Section 3.1.3, are the graph of $\ln x$ and the parabola $2ax - x^2$.

18. (a) The utility function $u(x)$ is concave, so the customer is a risk averter.

(b) The function is convex, and the customer is a risk lover.

(c) Let $y = w - G$. We have $E\{u(w-\xi)\} = 0.9 \cdot u(100) + 0.05 \cdot u(50) + 0.05 \cdot u(0) = 0.9 \cdot 50 + 0.05 \cdot 50 \cdot \frac{3}{4} = 46.875$. Then one should consider the inequality $y - 0.005y^2 \geq 46.875$, which gives $y \geq 75$, and $G_{\max} = 25$. It has been proved in Section 3.4.2 that $G_{\max} \geq E\{\xi\}$, and this is certainly the case here since $E\{\xi\} = 7.5$.

(d) By (3.2.4) and Jensen's inequality, $u_1(w_1) = E\{u_1(w_1 + H_{\min} - \xi)\} \leq u_1(w_1 + H_{\min} - E\{\xi\})$. So, $w_1 \leq w_1 + H_{\min} - E\{\xi\}$, and $H_{\min} \geq E\{\xi\}$. Again by (3.2.4), writing H instead of $H_{\min}$, we have $\sqrt{300} = E\{\sqrt{300 + H - \xi}\} = 0.9 \cdot \sqrt{300+H} + 0.05 \cdot \sqrt{250+H} + 0.05 \cdot \sqrt{200+H}$. An approximate solution (which may be obtained even by Excel) is ≈ 8.05.

(e) In this case, $\sqrt{300} = E\{\sqrt{300+H-\xi}\} = \frac{1}{100}\int_0^{100} \sqrt{300+H-x}dx = \frac{2}{300}[(300+H)^{3/2} - (200+H)^{3/2}]$. The numerical solution is ≈ 50.70.

(f) The new function $u(w) = 200w - w^2 + 349 = 200(w - 0.005w^2) + 349$, that is, a linear transformation of the function from (a), so the answer is the same.

19. In this case, we proceed from (3.2.8). In Example 3.2-1, we have computed already that $E\{u(w-\xi)\} = 2/3$. On the other hand, $E\{u(w - \frac{1}{2}\xi - G)\} = \int_0^1 (2(1-G-\frac{1}{2}x) - (1-G-\frac{1}{2}x)^2)dx = -\frac{175}{300} + \frac{5}{2}(1-G) - (1-G)^2$. Solving the quadratic equation $-\frac{175}{300} + \frac{5}{2}(1-G) - (1-G)^2 = \frac{2}{3}$, we come to $G \approx 0.31$.

22. The preference order.

26. All functions under consideration are concave except Case 1: the function $u(x) = x^{\alpha}$ is concave only for $\alpha \leq 1$.

27. The function is convex, so the person is a risk lover. When comparing two r.v.'s, X and Y, one should verify the inequality $E\{e^{a(w+X)}\} \geq E\{e^{a(w+Y)}\}$. The term e^{aw} cancels, and the inequality is equivalent to $E\{e^{aX}\} \geq E\{e^{aY}\}$.

30. (a-i) They do not differ since $u_1(x)$ is a linear transformation of $u_2(x)$.

(a-ii) Since $E\{u_1(X)\} = -2E\{u_2(X)\} + 3$, the preference orders are opposite: if $X \succsim Y$ for John, then $X \precsim Y$ for Mary.

(b) Mary is more risk averse since $u_2(x)$ is "more concave" (for example, you may graph the functions $u_1(x)$ and $u_2(x)$). Certainly, such a reasoning is heuristic, and since we did not consider in class what "more concave" means rigorously, students are suggested to take it at a heuristic level.

A rigorous approach is connected with the notion of relative risk aversion (see Section 3.4.3). In our case, for $u(x) = x^{\alpha}$, the relative risk aversion coefficient $R_r = 1-\alpha$. Hence, the less α is, the larger is the relative risk aversion coefficient. It is optional in this course.

38. Comparing the two distributions under consideration, we see that it looks like we "took" a probability mass of 0.2 from mass 0.5, and moved 0.1 to the right and 0.1 to the left. In this case, the dispersion of the distribution gets larger, and our intuition tells us that in the risk aversion case, the distribution should get "worse".

Let us justify this in the EUM case. Let u be a utility function, and let X_1 and X_2 be r.v.'s with the respective distributions. We have $E\{u(X_1)\} = 0.1u(1)+0.2u(2)+0.5u(3)+0.2u(4)$ and $E\{u(X_2)\} = 0.1u(1)+0.3u(2)+0.3u(3)+0.3u(4)$. Then $E\{u(X_1)\}-E\{u(X_2)\} = -0.1u(2)+0.2u(3)-0.1u(4) = 0.2\left[u(3)-\frac{1}{2}(u(2)+u(4))\right] \geq 0$ if $u(x)$ is concave.

39. This exercise is a generalization of Exercise 38. Let u be a utility function, a r.v. X_1 has the original distribution, and X_2 has the transformed distribution. Then $E\{u(X_1)\} = \sum_k u(x_k)p_k$. The corresponding representation $E\{u(X_2)\}$ is the same except the terms with $k = i-1$, i, and $i+1$. Consequently, the difference $E\{u(X_1)\}-E\{u(X_2)\} = -\Delta u(x_{i-1})+2\Delta u(x_i)-\Delta u(x_{i+1}) = 2\Delta\left[u(x_i)-\frac{1}{2}(u(x_{i-1})+u(x_{i+1}))\right]$.

If x's are equally spaced, then $x_i = \frac{1}{2}(x_{i-1}+x_{i+1})$, and $E\{u(X_1)\}-E\{u(X_2)\}$ is positive due to the concavity of $u(x)$.

Note that, for a particular i, all x's do not need to be equally spaced but rather we should have $x_{i+1}-x_i = x_i - x_{i-1}$ only for this particular i .

61. (a) Since $F(x) = 1-e^{-x/m}$, from (5.1.4) it follows that $\lambda = \int_d^{\infty} e^{-x/m}dx = me^{-d/m}$. On the other hand, $\lambda = m/2$, so $d = m\ln 2 \approx 0.69m$.

(b) In this case, $F(x) = \frac{x}{2m}$, and $\lambda = \int_d^{2m}\left(1-\frac{x}{2m}\right)dx = m\left(1-\frac{d}{2m}\right)^2$. Since $\lambda = \frac{m}{2}$, we have $d = 2(1-\frac{1}{\sqrt{2}})m \approx 0.59m$.

In the first case, the deductible is somewhat larger. It is not surprising, since the exponential r.v. has a greater dispersion and may assume large values.

2. ***Chapter 2***:

1. (a) The student should look over Table (**0**.2.6.1).

(b) Since (1.1.3) is the particular case of (1.1.10), it suffices to consider the Γ-density. Let a r.v. $X_{1\nu}$ have the density $f_{1\nu}(x) = \frac{1}{\Gamma(\nu)}x^{\nu-1}e^{-x}$ for $x \geq 0$, and $= 0$ otherwise. Consider the r.v. $X_{a\nu} = \frac{1}{a}X_{1\nu}$. In accordance with (**0**.2.6.1), $X_{a\nu}$ has the density $\frac{a^{\nu}}{\Gamma(\nu)}x^{\nu-1}e^{-ax}$ for $x \geq 0$, and $= 0$ otherwise. This is the density $f_{a\nu}(x)$ from (1.1.10). We have arrived at it just changing the scale.

5. (a) If X has the d.f. $F_1(x) = x^{\gamma}/(1+x^{\gamma})$, then by virtue of (**0**.2.6.1), the r.v. $Y = \theta X$ has the d.f. $F(x) = (x/\theta)^{\gamma}/(1+(x/\theta)^{\gamma})$.

7. $F_Y(x) = P(X \leq x/k) = 1-q+qF_{\xi}(x/k)$; $E\{Y\} = kq\mu$, and $Var\{Y\} = k^2(qv^2+q(1-q)\mu^2)$.

19. (b) ***Solution:*** Note that $f_{S_2}(x-y) = x-y$ if $0 \leq x-y \leq 1$, which is equivalent to $x-1 \leq y \leq x$, and $f_{S_2}(x-y) = 2-x+y$ if $1 \leq x-y \leq 2$, which is equivalent to $x-2 \leq y \leq x-1$. Otherwise, $f_{S_2}(x-y) = 0$.

Keeping this in mind, we have for $x \leq 1$,

$$f_{S_3}(x) = \int_0^x (x-y)dy = x^2 - \frac{x^2}{2} = \frac{x^2}{2}.$$

For $1 \leq x \leq 2$,

$$f_{S_3}(x) = \int_0^{x-1} (2-x+y)dy + \int_{x-1}^1 (x-y)dy = [-2x^2+6x-3]/2.$$

For $2 \leq x \leq 3$,

$$f_{S_3}(x) = \int_{x-2}^1 (2-x+y)dy = [x^2-6x+9]/2.$$

Otherwise, $f_{S_3}(x) = 0$.

21. Since a is a common scale parameter, without loss of generality we may set $a = 1$. Since $f_{1\nu}(x-y) = 0$ for $y > x$, by (2.1.3) and (1.1.10), for $x > 0$,

$$\begin{aligned} f_S(x) &= \int_0^\infty f_{1\nu_1}(x-y) f_{1\nu_2}(y)dy = \int_0^x \frac{1}{\Gamma(\nu_1)}(x-y)^{\nu_1-1}e^{-(x-y)}\frac{1}{\Gamma(\nu_2)}y^{\nu_2-1}e^{-y}dy \\ &= \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}x^{\nu_1+\nu_2-1}e^{-x}\int_0^x \left(1-\frac{y}{x}\right)^{\nu_1-1}\left(\frac{y}{x}\right)^{\nu_2-1}\frac{1}{x}dy. \end{aligned}$$

By the variable change $z = y/x$,

$$\begin{aligned} f_S(x) &= \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}x^{\nu_1+\nu_2-1}e^{-x}\int_0^1 (1-z)^{\nu_1-1}z^{\nu_2-1}dz \\ &= \frac{\Gamma(\nu_1+\nu_1)}{\Gamma(\nu_1)\Gamma(\nu_2)}B(\nu_1,\nu_2)f_{1(\nu_1+\nu_2)}(x), \end{aligned}$$

where the constant $B(\nu_1,\nu_2) = \int_0^1 (1-z)^{\nu_1-1}z^{\nu_2-1}dz$. We know that $f_S(x)$ is a density, and hence $\int_0^\infty f_S(x)dx = 1$. On the other hand, $f_{1(\nu_1+\nu_2)}(x)$ is also a density and $\int_0^\infty f_{1(\nu_1+\nu_2)}(x) = 1$ either. Then the expression $\dfrac{\Gamma(\nu_1+\nu_1)}{\Gamma(\nu_1)\Gamma(\nu_2)}B(\nu_1,\nu_2)$ must be equal to one. So, we have proved that $f_S(x) = f_{1(\nu_1+\nu_2)}(x)$ and, on the way, that $B(\nu_1,\nu_2) = \dfrac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1+\nu_1)}$. In Mathematics, the function $B(\cdot,\cdot)$ is called the Beta-function or the Euler integral.

30. The parameter $a = 1/m$. Hence, in view of (**0**.4.3.5), $M_\xi(z)$ exists for $z < 1/m$. It is noteworthy that the inequality is strict.

33. The function in Fig.10a cannot be a m.g.f. The function $g(z)$ in Fig.10c is not a m.g.f. either. The function in Fig.10b looks as a m.g.f.

34. For Fig.11a, the corresponding r.v.'s have a negative mean , while for Fig.11b, it is positive.

35. $M'(0) = E\{X\} = m$, $M''(0) = E\{X^2\} = m^2 + \sigma^2$.

36. The means are the same, $Var\{X_1\} < Var\{X_2\}$.

47. (a)

$$\theta \approx \frac{2.326\sqrt{22295}}{3300} \approx 0.105$$

48. (d)

$$\theta \approx \frac{1.645\sqrt{195}}{100} \approx 0.229$$

(f) $\theta_{\text{new}} \approx \frac{0.229}{\sqrt{2}} \approx 0.162.$

Additional Problems:

(a) $0.3\,e^{7z}+0.7e^{11z}$, $0.7\,e^{7z}+0.3e^{11z}$, $(0.3\,e^{7z}+0.7e^{11z})(0.7\,e^{7z}+0.3e^{11z})$

(b) The Γ-distribution with $a = 1/3$ and $\nu = 30$.

(c) You will wait for when the current customer in service is served, and three before you. So, $n = 4$, and the waiting time is *Gamma*-distributed with $\nu = 4\times 1 = 4, a = 1/2$. In the second case $\nu = 4\times 2 = 8$.

3. ***Chapter 3***:

2. If $N \equiv n$, then $M_N(z) = e^{zn}$. Therefore, (1.5) implies $M_S(z) = M_N(\ln M_X(z)) = \exp\{(\ln M_X(z))n\} = M_X^n(z)$.

4. The number of claims with priority for separate insurances are the r.v.'s

$$X_i = \begin{cases} 0 & \text{with probability } e^{-\lambda}, \\ 1 & \text{with probability } 1-e^{-\lambda}, \end{cases} \quad i = 1,2.$$

The sum

$$S = X_1 + X_2 = \begin{cases} 0 & \text{with probability } e^{-\lambda}e^{-\lambda} = e^{-2\lambda}, \\ 1 & \text{with probability } 2e^{-\lambda}(1-e^{-\lambda}), \\ 2 & \text{with probability } (1-e^{-\lambda})^2. \end{cases}$$

On the other hand, for the joint insurance, the number of claims is a Poisson r.v. with parameter 2λ. So, the number of priority claims is the r.v.

$$S' = \begin{cases} 0 & \text{with probability } e^{-2\lambda}, \\ 1 & \text{with probability } e^{-2\lambda}2\lambda, \\ 2 & \text{with probability } 1-e^{-2\lambda}-e^{-2\lambda}2\lambda. \end{cases}$$

We see that $P(S=0) = P(S'=0)$, while $P(S=1) > P(S'=1)$, since the inequality $2e^{-\lambda}(1-e^{-\lambda}) > e^{-2\lambda}2\lambda$ is equivalent to the inequality $e^{\lambda}-1 > \lambda$.

As a matter of fact, the last conclusion is true for any particular distribution of X's. Indeed, for the corresponding events we have $\{S=0\} = \{S'=0\}$, $\{S=1\} \supset \{S'=1\}$, and hence $\{S=2\} \subset \{S'=2\}$.

Thus, $P(S \le S') = 1$, $P(S < S') > 0$. So, for the same price, the joint insurance covers more claims. (It makes sense also to mention the first stochastic dominance.)

6. **(a)** $\lambda = 231\cdot 0.01 + 124\cdot 0.05 + 347\cdot 0.03 = 18.92$, $\bar{p}_n = \frac{\lambda}{231+124+347} \approx 0.02695$.

12. **(a)** If ξ is the size of a damage and d is the deductible, then the number of claims is a Poisson r.v. $\widetilde{\Lambda}$ with parameter $\widetilde{\lambda} = \lambda P(\xi > d) = \lambda e^{-ad} = 300e^{-2/10} \approx 245.62$. So, $E\{\widetilde{\Lambda}\} = Var\{\widetilde{\Lambda}\} \approx 245.62$**.**

(b) $P(\widetilde{\Lambda} \le 230) = \sum_{k=0}^{230} e^{-\widetilde{\lambda}}\widetilde{\lambda}^k/k!$. Since $\sqrt{Var\{\widetilde{\Lambda}\}} \approx 15.67$, the number 230 differs from $E\{\widetilde{\Lambda}\}$ approximately by the standard deviation, so the probability is not small.

(c) Since $E\{\widetilde{\Lambda}\}$ is large, the distribution of $\widetilde{\Lambda}$ is close to the corresponding normal distribution. Since the difference $E\{\widetilde{\Lambda}\}-230$ is close to the standard deviation, $P(\widetilde{\Lambda}\leq 230)\approx\Phi(-1)\approx 0.159$. (More precisely, one may write $P(\widetilde{\Lambda}\leq 230)=P(\frac{1}{15.67}(\widetilde{\Lambda}-245.62)\leq\frac{1}{15.67}(230-245.62))\approx P(\frac{1}{15.67}(\widetilde{\Lambda}-245.62)\leq -0.997)\approx\Phi(-1)\approx 0.159$, but it is not necessary.)

21. By (1.1) and (1.3), for the total payment S, we have $E\{S\}=4\cdot 50=200$, $Var\{S\}=2\cdot 50+16\cdot 10^2=1700$. If the distribution of S is closely approximated by the Γ-distribution with parameters a, ν, then $\frac{\nu}{a}=200$, and $\frac{\nu}{a^2}=1700$. Then $a=\frac{200}{1700}\approx 0.118$, and $\nu\approx 23.53$. Using Excel, we get that $P(S>250)\approx 1-\Gamma(250,0.118,23.53)\approx 0.117$.

28. In the book and below, the symbol $\stackrel{d}{=}$ means "equal by distribution", that is, the distributions of the corresponding r.v.'s are the same.

Let $\lambda=E\{N_1\}=E\{N_2\}$, and let N be a Poisson r.v. with parameter 2λ. Consider r.v.'s Y_i mutually independent, independent of N, and assuming values ± 1 with probabilities $\frac{1}{2}$. Let $W=Y_1+...+Y_N$, let $\widetilde{N}_1$ be the number of Y's assuming the value 1, and $\widetilde{N}_2$ be the number of Y's assuming the value -1.

(We may view this as the situation of two portfolios with the respective number of claims $\widetilde{N}_1$ and $\widetilde{N}_2$. The size of all claims in the first portfolio equals 1, while in the second, it is -1.)

By Proposition 10, $\widetilde{N}_1$ and $\widetilde{N}_2$ are independent Poisson r.v.'s with parameter λ, that is, the vector $(\widetilde{N}_1,\widetilde{N}_2)\stackrel{d}{=}(N_1,N_2)$. On the other hand, $W\stackrel{d}{=}\widetilde{N}_1-\widetilde{N}_2$, and hence $W\stackrel{d}{=}N_1-N_2$.

The m.g.f.

$$\begin{aligned}M_{N_1-N_2}(z) &= M_{N_1}(z)M_{-N_2}(z)=\exp\{\lambda(e^z-1)\}\exp\{\lambda(e^{-z}-1)\}\\ &= \exp\left\{\lambda\left(e^z+e^{-z}-2\right)\right\}.\end{aligned}$$

Note that we could obtain the same result using the representation for $M_{N_1-N_2}(z)$ above and (1.6), writing

$$\begin{aligned}M_W(z) &= \exp\{2\lambda(M_Y(z)-1)\}=\exp\left\{2\lambda\left(\frac{1}{2}e^z+\frac{1}{2}e^{-z}-1\right)\right\}\\ &= \exp\left\{\lambda\left(e^z+e^{-z}-2\right)=M_{N_1-N_2}(z).\right\}.\end{aligned}$$

29. Let N be a Poisson r.v. such that $E\{N\}=300$. Consider r.v.'s Y_i mutually independent, independent of N, and assuming values 3 and -5 with respective probabilities $\frac{1}{3}$ and $\frac{2}{3}$. Let $W=Y_1+...+Y_N$, let $\widetilde{N}_1$ be the number of Y's assuming the value 3, and $\widetilde{N}_2$ be the number of Y's assuming the value -5.

(We may view this as the situation of two portfolios with the respective numbers of claims $\widetilde{N}_1$ and $\widetilde{N}_2$. The size of all claims in the first portfolio equals 3, and in the second it equals -5.)

By Proposition 10, $\widetilde{N}_1$ and $\widetilde{N}_2$ are independent Poisson r.v.'s with parameters $\lambda_1=\frac{1}{3}300=100$ and $\lambda_2=\frac{2}{3}200=200$, respectively. Then, the vector $(\widetilde{N}_1,\widetilde{N}_2)\stackrel{d}{=}(N_1,N_2)$. On the other hand, it is clear that $W\stackrel{d}{=}3\widetilde{N}_1-5\widetilde{N}_2$, and hence $W\stackrel{d}{=}3N_1-5N_2$.

33. **(a)** N has the Poisson distribution with $\lambda=\lambda_1+\lambda_2=10\cdot 1^2+10\cdot 2^2=50$. Thus, $E\{N\}=Var\{N\}=50$. Using software we get $P(N\leq 50)\approx 0.5375$.

(b) Since $p_1=\frac{10}{50}$, in accordance with (3.2.7), $P(N_1<11\,|\,N=50)=B(10,\frac{1}{5},50)$, where $B(x;p,n)$ is the binomial d.f. with parameters p,n. Excel gives 0.58356.

(c) This is the compound Poisson distribution of a r.v. $W = Y_1 + ... + Y_N$ where N is Poisson with $\lambda = 50$, and the independent Y's take values 100 and 300 with probabilities $p_1 = 0.2$ and $p_2 = 0.8$, respectively.

(d) $E\{S\} = 50(100 \cdot 0.2 + 300 \cdot 0.8) = 13000$; $Var\{S\} = 50(100^2 \cdot 0.2 + 300^2 \cdot 0.8) = 3,700,000$. In accordance with (1.6),

$$M_S(z) = \exp\left\{50\left(e^{100z}\frac{1}{5} + e^{300z}\frac{4}{5} - 1\right)\right\} = \exp\left\{10e^{100z} + 40e^{300z} - 50\right\}.$$

34. **(a)** Since $p_1 = 0.2$ and $p_2 = 0.8$, we have $S \stackrel{d}{=} W$, where W is defined as in Exercise 33c above with

$$Y = \begin{cases} 100 & \text{with probability } \frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}, \\ 200 & \text{with probability } \frac{1}{5} \cdot \frac{2}{3} + \frac{4}{5} \cdot \frac{1}{2} = \frac{8}{15}, \\ 300 & \text{with probability } \frac{4}{5} \cdot \frac{1}{2} = \frac{2}{5}. \end{cases}$$

(b) $E\{S\} = 50E\{Y\} = 50 \cdot \frac{700}{3} = \frac{35000}{3}$; $Var\{S\} = 50E\{Y^2\} = 50 \cdot 58000 = 29 \cdot 10^5$.

(c) This follows from (3.1.13) where N_1, N_2, N_3 are the numbers of Y's which assumed the values 100, 200, 300, respectively. Consequently, $E\{N_i\} = E\{N\}p_i$. Namely, $E\{K_1\} = 50 \cdot \frac{1}{15} = \frac{10}{3}$, $E\{N_2\} = 50 \cdot \frac{8}{15} = \frac{80}{3}$, $E\{N_3\} = 50 \cdot \frac{2}{5} = 20$.

35. In this case, $S \stackrel{d}{=} W$, where W is defined as in Exercise 33c above with Y's having the density $f(x) = p_1 f_1(x) + p_2 f_2(x)$, where $f_1(x)$, $f_2(x)$ are the uniform densities on $[100,200]$ and $[200,300]$, respectively. Hence,

$$f(x) = \begin{cases} 0.2 \cdot \frac{1}{100} = 0.002 & \text{if } x \in [100,200], \\ 0.8 \cdot \frac{1}{100} = 0.008 & \text{if } x \in [200,300], \\ 0 & \text{otherwise.} \end{cases}$$

Then $E\{S\} = E\{N\}(p_1 m_1 + p_2 m_2) = 50(0.2 \cdot 150 + 0.8 \cdot 250) = 11500$, $Var\{S\} = E\{N\}(E\{Y^2\} = 50(0.2 \cdot (150^2 + \frac{1}{12}100^2) + 0.8 \cdot (250^2 + \frac{1}{12}100^2)) = 2766666.\bar{6}$, and

$$M_S(z) = \exp\left\{50\left(M(z) - 1\right)\right\},$$

where

$$\begin{aligned} M(z) &= p_1 M_1(z) + p_2 M_2(z) = 0.2 \cdot e^{100z}\frac{e^{100z} - 1}{100z} + 0.8 \cdot e^{200z}\frac{e^{100z} - 1}{100z} \\ &= \frac{e^{100z} - 1}{100z}(0.2 \cdot e^{100z} + 0.8 \cdot e^{200z}). \end{aligned}$$

36. The problem is similar to Exercise 35. In this case, $S \stackrel{d}{=} W$, where W is defined as in Exercise 33c above with Y's having the density $f(x) = w_1 f_1(x) + w_2 f_2(x)$, where $f_1(x)$, $f_2(x)$ are the uniform densities on $[100,300]$ and $[200,400]$, respectively. Hence,

$$f(x) = \begin{cases} 0.2 \cdot \frac{1}{200} = 0.001 & \text{if } x \in [100,200], \\ 0.2 \cdot \frac{1}{200} + 0.8 \cdot \frac{1}{200} = 0.005 & \text{if } x \in [200,300], \\ 0.8 \cdot \frac{1}{200} = 0.004 & \text{if } x \in [300,400], \\ 0 & \text{otherwise.} \end{cases}$$

Then $E\{S\} = E\{N\}(w_1m_1 + w_2m_2) = 50(0.2 \cdot 200 + 0.8 \cdot 300) = 14000$, $Var\{S\} = E\{N\}(E\{Y^2\} = 50(0.2 \cdot (200^2 + \frac{1}{12}200^2) + 0.8 \cdot (300^2 + \frac{1}{12}200^2)) = 4166666.\bar{6}$, and $M_S(z) = \exp\{50(M(z) - 1)\}$, where

$$\begin{aligned} M(z) &= w_1M_1(z) + w_2M_2(z) = 0.2 \cdot e^{100z}\frac{e^{200z}-1}{200z} + 0.8 \cdot e^{200z}\frac{e^{200z}-1}{200z} \\ &= \frac{e^{200z}-1}{200z}(0.2 \cdot e^{100z} + 0.8 \cdot e^{200z}). \end{aligned}$$

Additional problems:

(a) $\lambda_1\lambda_2, \lambda_1(\lambda_2 + \lambda_2^2)$.

(b) $1/4$.

4. ***Chapter 4***:

1. **(a)** Let us measure time in hours. Consider two intervals: $\Delta_1 = [0, 0.5]$ and $\Delta_2 = (0.5, 1]$. Since an interarrival time cannot be larger than one hour, if there were no arrivals in Δ_1, then there should be an arrival in Δ_2; that is, $P(N_{\Delta_2} = 0 \,|\, N_{\Delta_1} = 0) = 0$. On the other hand, $P(N_{\Delta_2} = 0 \,|\, N_{\Delta_1} = 1) > 0$ because if there was an arrival in Δ_1, the next arrival may occur after half an hour elapses.

(b) Similar to the above reasoning, $P(N_2 = 2 \,|\, N_{1.5} = 2,\, N_1 = 2) = 0$ since the condition means that there was no arrivals in $[1, 1.5]$. On the other hand, $P(N_2 = 2 \,|\, N_{1.5} = 2) > 0$ because there may be no arrivals during half an hour. Thus, the process is not Markov. Then it is not the process with independent increments because processes of the latter type have the Markov property.

2. Assume that interarrival times are independent, and the expected value of the kth interarrival time equals k. Consider two intervals: $\Delta_1 = [0, 1]$ and $\Delta_2 = (1, 2]$. The probability $P(N_{\Delta_2} = 0 \,|\, N_{\Delta_1} = 1)$ should be much smaller than $P(N_{\Delta_2} = 0 \,|\, N_{\Delta_1} = 1000)$ because given that there were 1000 arrivals, the waiting time for the next arrival is 1001 times larger than the length of a unit interval, and the probability that there will be no arrival during a unit time interval is close to one.

Actually, we can state it rigorously if we observe that due to the memoryless property, $P(N_{\Delta_2} = 0 \,|\, N_{\Delta_1} = k - 1) = P(\tau_k > 1)$, where τ_k is the kth interarrival time. The last probability equals $\exp\{-1/k\} \to 1$ as $k \to \infty$.

12. **(a)** Since N_t is the process with independent increments, $Corr\{N_2, N_4 - N_2\} = 0$.

(b)

$$\begin{aligned} E\{N_tN_{t+s}\} &= E\{N_t(N_t + N_{t+s} - N_t\} = E\{N_t^2\} + E\{N_t(N_{t+s} - N_t)\} \\ &= E\{N_t^2\} + E\{N_t\}E\{(N_{t+s} - N_t)\} \\ &= (E\{N_t\})^2 + Var\{N_t\} + E\{N_t\}E\{(N_{t+s} - N_t)\} \\ &= (\lambda t)^2 + \lambda t + \lambda t \cdot \lambda s = \lambda t + \lambda^2 t(t+s). \end{aligned}$$

Next,

$$\begin{aligned} Cov\{N_t, N_{t+s}\} &= E\{N_tN_{t+s}\} - E\{N_t\}E\{N_{t+s}\} \\ &= \lambda t + \lambda^2 t(t+s) - \lambda t \cdot \lambda(t+s) = \lambda t, \end{aligned}$$

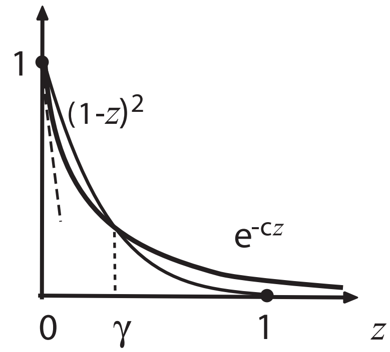

Figure 1:

and

$$Corr\{N_t, N_{t+s}\} = \frac{Cov\{N_t, N_{t+s}\}}{\sqrt{Var\{N_t\}Var\{N_{t+s}\}}} = \frac{\lambda t}{\sqrt{\lambda t \cdot \lambda(t+s)}} = \sqrt{\frac{t}{t+s}}.$$

(c) By the memoryless property, at the moment t, "everything starts over as from the beginning", and we can think about the mth arrival after time t. For a separate interarrival time τ, we have $E\{\tau\} = \frac{1}{\lambda}$, and $Var\{\tau\} = \frac{1}{\lambda^2}$. Consequently,

$$E\{T_{n+m}\,|\,N_t = n\} = t + E\{T_m\} = t + \frac{m}{\lambda},$$

and

$$Var\{T_{n+m}\,|\,N(t) = n\} = Var\{T_m\} = \frac{m}{\lambda^2}.$$

13. By virtue of (2.2.6), we should give an example of a function $\lambda(t)$ for which $\lim_{t\to\infty}\chi(t) = 1$. Let, for instance, $\lambda(t) = e^{-t}$. Then $\chi(\infty) = \int_0^\infty e^{-s}ds = 1$.

Another example may be $\lambda(t) = \dfrac{2}{\pi(1+t^2)}$ because $= \displaystyle\int_0^\infty \frac{1}{1+s^2}ds = \pi/2$.

17. Let time be measured in hours. If $\tau_1 = 1/2$, then after the first arrival, during the next half an hour, the intensity of arrivals is not high, and the probability that there will be no arrivals during half an hour is not small. However, if $\tau_1 = 1$, then after the first arrival, the intensity will become very high, and the probability that there will be no arrivals during half an hour will be very small.

More precisely, $P(\tau_2 > 0.5\,|\,\tau_1 = 0.5) = P(N_{(0.5,1]} = 0) = \exp\left\{-1\cdot\frac{1}{2}\right\} = \frac{1}{\sqrt{e}}$, while $P(\tau_2 > 0.5\,|\,\tau_1 = 1) = P(N_{(1,1.5]} = 0) = \exp\{-100\cdot\frac{1}{2}\} = e^{-50}$, which is a very small number.

5. ***Chapter 6***:

2. $\tilde{\phi}_n(u) \geq \phi_n(u)$.

5. (a) The derivative of $(1-z)^2$ at 0 is -2, and the derivative of e^{-cz} at 0 is $-c$. So, the exponential function intersects the parabola at a $\gamma > 0$ iff $c > 2$. See also the Fig.1 in this handout.

(c) It may be seen from Fig.1 in this handout. Rigorously, e^{-cz} is decreasing in c and both functions are continuous. Hence, the solution $\gamma = \gamma(c)$ is continuous and increasing in c. So, there exists $\gamma(\infty) = \lim_{c\to\infty}\gamma(c)$. The number $\gamma(c) < 1$ for all c. On the other hand, for instance, $\gamma(3) > 0$, and for $c > 3$, we can write

$$(1-\gamma(c))^2 = \exp\{-c\gamma(c)\} \le \exp\{-c\gamma(3)\} \to 0$$

as $c \to \infty$. Thus, $(1-\gamma(c))^2 \to 0$, and $\gamma(c) \to 1$.

Estimate (2.1.4) is not good in this case since for $c \to \infty$ the ruin probability converges to zero, while (2.1.4) gives only the bound e^{-u}.

Heuristically, the fact that the ruin probability vanishes while $c \to \infty$ should be clear. If c is very large, then the probability of ruin at the first step is very small, and at the beginning of the second step, the company starts with a very large surplus. Then the ruin probability is small due to Lundeberg's inequality.

We discuss a formal proof in Exercise 6c.

6. **(a)** Let us look at Fig.3b where for our case $\mu = m - c < 0$. The closer μ to zero, the smaller the solution p in Fig.3b. On the other hand, as we see in Fig.3a, for $\mu \ge 0$, the only solution is zero. So, we can guess that the solution p approaches zero as $\mu \to 0$.

If X is not degenerate, that it may take on values larger than m. In this case, if the premium is equal only to the mean value m, the company cannot function in the long run, since the deficit of payments is accumulating. The rigorous proof of this statements should consist in proving that the ruin probability in this case is one, which we do below.

7. **(a)** Let us look at Fig.3b where for our case $\mu = m - c < 0$. The larger c, the steeper the slope at the origin, and hence the larger the solution p in Fig.3b. So, we can guess that $p \to \infty$ as $\mu \to \infty$.

Heuristically, it is clear. A rigorous argumentation may repeat the corresponding arguments in Exercise 5c.

8. Setting $y = z/a$, we come to the equation $(1 + y(1+\theta)\nu)(1-y)^{\nu} = 1$. So, if y is a solution to the last equation, for the solution of (2.2.24), we should take $z = ay$.

13. (a) The information about λ is not needed. By (2.2.22),

$$\frac{1}{10\gamma}(e^{10\gamma} - 1) = 1 + (1+0.2)5\gamma.$$

A numerical solution is $\gamma \approx 0.053$, so $\psi(100) \le e^{-100\cdot 0.053} \approx 0.005$.

(b) We should solve the inequality

$$\exp(-0.053u) \le 0.05.$$

This implies $u \ge 57.62$.

14. **(a)** We have $m = 3$, and

$$M_X(z) = \frac{1}{4}e^{2z} + \frac{1}{2}e^{3z} + \frac{1}{4}e^{4z}.$$

Thus, (2.2.21) becomes $\frac{1}{4}e^{2z} + \frac{1}{2}e^{3z} + \frac{1}{4}e^{4z} = 1 + 1.1\cdot 3z$, or

$$e^{2z} + 2e^{3z} + e^{4z} = 4 + 13.2z. \qquad (1)$$

(b) No.

(c) Using software (for example Wolfram or even Excel), we get that a positive solution to (M-1) above is $\gamma \in [0.59, 0.6]$. Now, $E\{X^2\} = 9.5$ and $E\{X\} = 3$, so approximation (2.2.26) from the book gives

$$\gamma \approx \frac{2 \cdot 0.1 \cdot 3}{9.5} \approx 0.063.$$

(d) We write $e^{-\gamma u} \leq 0.03$, which gives $u \geq \frac{-1}{\gamma} \ln 0.3$. Since $-\ln(0.03) \leq 3.51$, we can write $u \geq 59.5 \geq \frac{3.51}{0.059}$.

(e) For the new distribution, the degree of "dispersion" is less, so we should expect at least a bit larger γ and a less ruin probability. Equation (2.2.22) becomes

$$\frac{e^{4z} - e^{2z}}{2z} = 1 + 1.1 \cdot 3z$$

Software leads to a solution $\gamma \in [0.06, 0.061]$. The rest is practically the same since the solution does not differ much from the previous.