Here you will find some solutions and answers to problems answers to which are not given in the book.

1. \textit{Chapter 7}:

5. Certainly, the problem has a solution if the original probability multiplied by \( 1.1 \) does not exceed one. Let \( k \) be a certain age, and \( \mu(u) \) and \( \mu^*(u) \) be the old and the new force of mortality, respectively. Then, for the corresponding survival functions, the ratio

\[
\frac{s^*(k)}{s(k)} = \frac{\exp \left\{ -\int_0^k \mu^*(u) \, du \right\}}{\exp \left\{ -\int_0^k \mu(u) \, du \right\}} = \exp \left\{ \int_0^k (\mu(u) - \mu^*(u)) \, du \right\}.
\]

Thus, for the l.h.s. to be equal to 1.1, we should have

\[
\int_0^k (\mu(u) - \mu^*(u)) \, du = \ln 1.1.
\]

For example, this is true if \( \mu(u) - \mu^*(u) = \frac{\ln 1.1}{k} \) for \( u \in [0, k] \); that is, for the new mortality force, we will have

\[
\mu^*(u) = \mu(u) - \frac{\ln 1.1}{k}.
\]

Certainly, it is possible if \( \mu(u) \geq \frac{\ln 1.1}{k} \) for all \( u \in [0, k] \) (since \( \mu^*(u) \) should be non-negative).

If \( k = 1 \), we should have \( \mu(u) - \mu^*(u) = \ln 1.1 \); that is, the force of mortality during the first year should be \( \ln 1.1 \) less.

8. Since \( \mu(x) \to \infty \) as \( x \to \omega \), the life time \( X \) cannot exceed \( \omega \); that is, \( P(X > \omega) = 0 \). For this to be true, the integral \( \int_0^\omega \frac{du}{(\omega - u)\alpha} \) should diverge (be equal to infinity), so we set \( \alpha \geq 1 \). Now, for \( \alpha > 1 \) and \( x < \omega \), we have

\[
s(x) = \exp \left\{ -\int_0^x \frac{du}{(\omega - u)\alpha} \right\} = \exp \left\{ \frac{1}{(\alpha - 1)\omega^{\alpha - 1}} \left( \frac{1}{\left(1 - x/\omega\right)^{\alpha - 1}} - 1 \right) \right\},
\]

while for \( \alpha = 1 \), we have the uniform distribution on \([0, \omega]\). Indeed, in this case,

\[
s(x) = \exp \left\{ -\int_0^x \frac{du}{\omega - u} \right\} = \exp \left\{ -\ln \left( \frac{\omega}{\omega - x} \right) \right\} = 1 - \frac{x}{\omega}.
\]

10. Both functions are artificial. Nevertheless, the integral \( \int_0^\omega \mu(x) \, dx \) converges in the first case, and diverges in the second. So, the first \( \mu(x) \) cannot serve as a model for units with bounded lifetimes.

16. (a) From \( l_0 \) newborn girls, on the average, \( l_0 \sigma_f(50) = l_0 \frac{1}{\sqrt{2}} \) survive 50 years. For boys, this number is \( l_0 \sigma_m(50) = l_0 \frac{2}{\sqrt{2}} \). Hence, the ratio of these mean values is \( \frac{3}{2\sqrt{2}} \approx 1.06 \).
(As a matter of fact, the precise solution should be different. Let \( \eta_1 \) and \( \eta_2 \) be the number of survivors among men and women (both r.v.'s have binomial distributions). Then we should consider, for example, the distribution of \( \eta_1/\eta_2 \) given \( \eta_2 > 0 \), and in particular, the corresponding conditional expectation.)

Now, let \( B_1 \) be the event that a person chosen at random is a male, and \( B_2 \) is that it is a female. The by the Bayes formula, the probability that a person of age 50 taken at random is a man is

\[
P(B_1 | X > 50) = \frac{P(X > 50 | B_1) P(B_1)}{P(X > 50 | B_1) P(B_1) + P(X > 50 | B_2) P(B_2)} = \frac{\frac{1}{2} s_m(50)}{\frac{1}{2} s_m(50) + \frac{1}{2} s_f(50)} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{\sqrt{2}}} \approx 0.49.
\]

Hence,

\[
20|10 q_{50} \approx 0.49 \frac{s_m(70) - s_m(80)}{s_m(50)} + 0.51 \frac{s_f(70) - s_f(80)}{s_f(50)} = 0.49 \frac{\sqrt{2/9} - \sqrt{1/9}}{2/3} + 0.51 \frac{\sqrt{0.3} - \sqrt{0.2}}{\sqrt{1/2}} \approx 0.17.
\]

(b) In general, let \( B_1 \) be the event that a person chosen at random belongs to the first group, and \( B_2 \) - that it belongs to the second group. We know that \( P(B_1) = w_1 \) and \( P(B_2) = w_2 \). Certainly, \( w_1 + w_2 = 1 \).

Denote by \( w_1(x) \) the probability that a person taken at random from people of age \( x \) (we know her/his age!) belongs to the first group; \( w_2(x) \) is defined respectively. By the the Bayes formula,

\[
w_1(x) = P(B_1 | X > x) = \frac{P(X > x | B_1) P(B_1)}{P(X > x | B_1) P(B_1) + P(X > x | B_2) P(B_2)} = \frac{w_1 \cdot s P_0^{(1)}}{w_1 \cdot s P_0^{(1)} + w_2 \cdot s P_0^{(2)}}.
\]  

Similarly,

\[
w_2(x) = \frac{w_2 \cdot s P_0^{(2)}}{w_1 \cdot s P_0^{(1)} + w_2 \cdot s P_0^{(2)}}.
\]

Then the conditional survival function

\[
t P(t) = w_1(x) \cdot P_1^{(1)} + w_2(x) \cdot P_1^{(2)}.
\]

22. The distribution of \( T(x) \) is a mixture of exponential distributions but the weights depend on \( x \). We may use general formulas (1)-(3) above or proceed as follows. Let \( w_i, \mu_i \), \( i = 1, 2 \), be the original weight and force of mortality for the \( i \)th group. Then

\[
P(T(x) > t) = P(X > x + t | X > x) = \frac{w_1 \exp(-\mu_1(x + t)) + w_2 \exp(-\mu_2(x + t))}{w_1 \exp(-\mu_1 x) + w_2 \exp(-\mu_2 x)} = w_1(x) \exp(-\mu_1 t) + w_2(x) \exp(-\mu_2 t),
\]

where

\[
w_i(x) = \frac{w_i \exp(-\mu_i x)}{w_1 \exp(-\mu_1 x) + w_2 \exp(-\mu_2 x)}, \quad i = 1, 2.
\]
For the data in the exercise,

\[
\begin{align*}
  w_1(20) &= \frac{0.3 \exp\{-20/50\}}{0.3 \exp\{-20/50\} + 0.7 \exp\{-20/80\}} \approx 0.27, \\
  w_2(20) &= 1 - w_1(20) \approx 0.73.
\end{align*}
\]

The answer is natural. The forces of mortality in the two groups are different, so the share of people from, for instance, the first group who attain an age \(x\) depends on \(x\).

25. We write

\[
E\{K(x)\} = E\{K(x) \mid T(x) < 1\}P(T(x) < 1) + E\{K(x) \mid T(x) \geq 1\}P(T(x) \geq 1)
\]

so, from (2.3.6) with \(n\) lifetime gets larger, and hence the APV of a whole life insurance gets smaller.

However, there may be “dangerous” years such that if a person survives them, the expected remaining lifetime gets smaller. Rigorously, it may be shown with use of (2.3.3) and (2.3.6). For example, from (2.3.6) with \(n=1\), we have \(A_x = A_x^1 + \sum_{k=0}^{\infty} p_k A_{x+1} = v(1 - p_x) + v p_k A_{x+1} \geq v(1 - p_x)\). Assume that \(p_x\) is very small (the \(x\) is a dangerous year). Then, since \(A_x \leq v\) for all \(x\)’s, \(A_x\) is close to \(v\). On the other hand, by the same formula,
\[ A_{x+1} = v(1 - p_{x+1}) + vp_{x+1}A_{x+2} \leq v(1 - p_{x+1}) + v^2p_{x+1}, \] and if \( p_{x+1} \) is not small, \( A_{x+1} \) may be essentially less than \( v \).

4. In Exercise 7-28, we have shown that \( P(K = k) = p d^k \), where \( p = 1 - e^{-\mu} \). By (0.4.3.1), the m.g.f. \( M_{K+1}(z) = pe^{z}/(1 - qe^z) \). Consequently,

\[
A_x = M_{K+1}(-\delta) = e^{-\delta} \frac{1 - e^{-\mu}}{1 - e^{-\mu}e^{-\delta}} = e^{-\delta} \frac{1 - e^{-\mu}}{1 - e^{-\mu+\delta}}.
\]

Using the fact that \( e^x = 1 + x + o(x) \) for \( x \to 0 \), we can write

\[
A_x = (1 + \delta + o(\delta))(1 + \mu + o(\mu))(1 + \mu + \delta + o(\mu + \delta))^{-1} \sim \frac{\mu}{\mu + \delta}
\]

for \( \mu + \delta \to 0 \). (Since \( \mu \) and \( \delta \) are positive, from \( \mu + \delta \to 0 \) it follows that \( \mu \to 0 \) and \( \delta \to 0 \).)

Hence, for small \( \mu \) and \( \delta \), the quantities \( A_x \) and \( \overline{A}_x \) are close.

6. In this case, \( \overline{A}_x = \frac{\mu}{\mu + \delta} = \frac{1}{1 + \delta/\mu} \). If \( \overline{A}_x = \frac{1}{2} \), then \( \frac{\delta}{\mu} = 1 \), and \( 2\overline{A}_x = \frac{1}{1+2\delta/\mu} = \frac{1}{3} \). Thus, \( Var\{Z\} = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12} \).

13. If \( T \) is uniform, then—in view of the assumption we made in Section 2.1.3—formula (2.1.6) is precise. It is interesting to see it directly. In Example 1.1-1, we got \( \overline{A}_x = e^{-\omega - \delta} \), where \( \omega = (\omega - x)\delta \). From Example 1.1-2, setting \( v = e^{-\delta} \), we obtain that \( A_x = \frac{v(1 - v^{\omega - x})}{(\omega - x)(1 - v)} = \frac{\delta e^{-\delta}}{1 - e^{-\delta}} \cdot \frac{(1 - e^{-s})}{s} = \frac{\delta}{e^\delta - 1} \cdot \frac{(1 - e^{-s})}{s} = \frac{\delta}{e^\delta - 1} \cdot \frac{1}{A_x} \).

This is not the only possible example. For any distribution of \( T \) with a density that is constant in intervals \( (k, k+1) \), formula (2.1.6) is precise.

21. We proceed from Exercise 7-22. Since for the constant force of mortality, \( \overline{A}_x = \frac{\mu}{\mu + \delta} \), in our case,

\[
\overline{A}_x = w_1(x) \frac{\mu_1}{\mu_1 + \delta} + w_2(x) \frac{\mu_2}{\mu_2 + \delta}.
\]

Thus, \( \overline{A}_{30} \approx 0.27 \frac{0.02}{0.02 + 0.025} + 0.73 \frac{0.0125}{0.025 + 0.025} \approx 0.223 \), and \( 2\overline{A}_x = 0.27 \frac{0.02}{0.02 + 0.025} + 0.73 \frac{0.0125}{0.025 + 0.025} \approx 0.126 \) for \( \delta = 0.05 \).

By the double rate rule, \( Var\{Z\} \approx 0.126 - (0.223)^2 = 0.076 \).

22. We have

\[
A_{30} = 0.25, A_{50} = 0.4, \text{ and } A_{30,35} = 0.55. \text{ Then, by (2.3.6) and (2.4.7), } 0.25 = (0.55 - 20E_{30}) + 20E_{30} \cdot 0.4, \text{ which gives } 20E_{30} = 0.5. \text{ Then } e^{-0.05} = 0.95, \text{ and } \delta \approx 0.032.
\]

29. The first group is healthier since for any \( t \), the probability to attain age \( x + t \) is larger for the first group. In a certain sense, people from the first group live longer, and then for any insurance, the APV for the first group should be smaller; more precisely, not larger.

Rigorously, let \( T \) be a lifetime, and let \( \Psi \) be a random moment of payment. Assume that \( \Psi = \psi(T) \) where \( \psi(x) \) is a non-decreasing function. This is the case for all insurances we considered. Then, integrating by parts, for the APV we have

\[
A = E\{e^{-\delta \Psi}\} = \int_0^\infty e^{-\delta \psi(t)} dP(T \leq t) = -\int_0^\infty e^{-\delta \psi(t)} dP(T > t)
\]

\[
= e^{-\delta \psi(0)} P(T > 0) + \int_0^\infty P(T > t) e^{-\delta \psi(t)} dt.
\]
Formally, $\psi(t)$ may be non-differentiable at one point (as $\psi(t) = \min(t, n)$) but for integration it does not matter. In any case, the function $e^{-\delta \psi(t)}$ is non-increasing, and hence the differential $de^{-\psi(t)} \leq 0$. Therefore the choice of a larger function $P(T > t)$ leads to a non-larger quantity $A$.

33. (a) All characteristics we consider may be represented as $E\{\exp\{-\delta \Psi\}\}$, where $\Psi$ is the corresponding random payment time. If $P(0 < \Psi < \infty) > 0$ (that is, $\Psi$ is not equal to zero or infinity with probability one), then $E\{\exp\{-\delta \Psi\}\}$ is decreasing in $\delta$.

(b) All functions are continuous in $\delta$; so, it suffices to consider the case $\delta = 0$. The present value of $1$ to be paid in future is $1$. However, we must not forget that for some insurances (for example, for a term insurance), with a positive probability, there will be no payments. Hence, the APV equals one times the probability that there will be a payment. In particular, as $\delta \to 0$,

$$
\lim A_x = \lim A_x = \lim A_x = \lim A_x = 1,
$$

$$
\lim A_x = \lim A_x = P(K(x) < n) = q_x,
$$

$$
\lim A_x = \lim A_x = \lim A_x = \lim A_x = P(T(x) > m) = P_x.
$$

(c) Rigorously, we should compute $\lim_{\delta \to 0} E\{\exp\{-\delta \Psi\}\}$. We should not forget that $\Psi$ may assume an infinite value with a positive probability, and consequently, the r.v. $Z = \exp\{-\delta \Psi\}$ may be equal zero with the same probability. So, $\lim_{\delta \to 0} E\{\exp\{-\delta \Psi\}\} = E\{1_{\{\Psi < \infty\}}\} = P(\Psi < \infty)$. (One may pass the limit inside the expectation by the Lebesgue dominated convergence theorem, which is certainly out of the scope of this book.)

34. For any number $\delta$, the function $e^{-\delta x}$ is convex (in another terminology, concave upward), so by Jensen’s inequality (see p.100), $E\{\exp\{-\delta \Psi\}\} \geq \exp\{-\delta E\{\Psi\}\}$. Thus, when replacing $\Psi$ by $E\{\Psi\}$, we underestimate the APV. In particular, $A_x \geq \exp\{-\delta e_x\}$ and $A_x \geq \exp\{-\delta \hat{e}_x\}$. 

5