Math 193b, HANDOUT FOR HOMEWORK

Some Answers and Solutions

Here you will find some solutions and answers to problems answers to which are not given in the book.

1. ***Chapter 7***:

5. Certainly, the problem has a solution if the original probability multiplied by 1.1 does not exceed one. Let k be a certain age, and $\mu(u)$ and $\mu^*(u)$ be the old and the new force of mortality, respectively. Then, for the corresponding survival functions, the ratio

$$\frac{s^*(k)}{s(k)} = \frac{\exp\left\{-\int_0^k \mu^*(u)du\right\}}{\exp\left\{-\int_0^k \mu(u)du\right\}} = \exp\left\{\int_0^k [\mu(u)-\mu^*(u)]du\right\}.$$

Thus, for the l.-h.s. to be equal to 1.1, we should have

$$\int_0^k [\mu(u)-\mu^*(u)]du = \ln 1.1.$$

For example, this is true if $\mu(u)-\mu^*(u) = \frac{\ln 1.1}{k}$ for $u \in [0,k]$. Certainly, it is possible if $\mu(u) \geq \frac{\ln 1.1}{k}$.
If $k=1$, we should have $\mu(u)-\mu^*(u) = \ln 1.1$; that is, the force of mortality during the first year should be $\ln 1.1$ less.

8. Since $\mu(x) \to \infty$ as $x \to \omega$, The life time X cannot exceed ω; that is, $P(X > \omega) = 0$. For this to be true, the integral $\displaystyle\int_0^{\omega} \frac{du}{(\omega-u)^{\alpha}}$ should diverge, so we set $\alpha \geq 1$. Now, for $\alpha > 1$ and $x < \omega$, we have

$$s(x) = \exp\left\{-\int_0^x \frac{du}{(\omega-u)^{\alpha}}\right\} = \exp\left\{-\frac{1}{(\alpha-1)\omega^{\alpha-1}}\left(\frac{1}{(1-x/\omega)^{\alpha-1}}-1\right)\right\},$$

while for $\alpha = 1$, we have the uniform distribution on $[0,\omega]$. Indeed, in this case,

$$s(x) = \exp\left\{-\int_0^x \frac{du}{\omega-u}\right\} = \exp\left\{-\ln\left(\frac{\omega}{\omega-x)}\right)\right\} = 1-\frac{x}{\omega}.$$

10. Both functions are artificial. Nevertheless, the integral $\int_0^{\infty}\mu(x)dx$ converges in the first case, and diverges in the second. So, the first $\mu(x)$ cannot serve as a model for units with bounded lifetimes.

11.

(a) When multiplying a force of mortality by 3, we cube the corresponding probability, So, the answer is $0.5^3 = 0.125$.

(b) In general,

$$ {}_tp_x = \exp\left\{ -\int_x^{x+t} \mu(z)dz \right\}. $$

If we subtract from the force of mortality a number c, then the new probability equals

$$ {}_tp_x^* = \exp\left\{ -\int_x^{x+t} (\mu(z)-c)dz \right\} = \exp\left\{ ct - \int_x^{x+t} \mu(z)dz \right\} = e^{ct}\exp\left\{ -\int_x^{x+t} \mu(z)dz \right\} = e^{ct}\,{}_tp_x. $$

So, in our case, the new probability equals $e^{0.01\cdot 30}\cdot 0.05$.

16. **(a)** From l_0 newborn girls, on the average, $l_0 s_f(50) = l_0\frac{1}{\sqrt{2}}$ survive 50 years. For boys, this number is $l_0 s_m(50) = l_0\frac{2}{3}$. Hence, the ratio of these mean values is $\frac{3}{2\sqrt{2}} \approx 1.06$.

(As a matter of fact, the precise solution should be different. Let η_1 and η_2 be the number of survivors among men and women (both r.v.'s have binomial distributions). Then we should consider, for example, the distribution of η_1/η_2 given $\eta_2 > 0$, and in particular, the corresponding conditional expectation.)

Now, let B_1 be the event that a person chosen at random is a male, and B_2 is that it is a female. The by the Bayes formula, the probability that a person of age 50 taken at random is a man is

$$ P(B_1\,|\,X>50) = \frac{P(X>50\,|\,B_1)P(B_1)}{P(X>50\,|\,B_1)P(B_1)+P(X>50\,|\,B_2)P(B_2)} = \frac{\frac{1}{2}s_m(50)}{\frac{1}{2}s_m(50)+\frac{1}{2}s_f(50)} = \frac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{\sqrt{2}}} \approx 0.49. $$

Hence,

$$ \begin{aligned} {}_{20|10}q_{50} &\approx 0.49\,\frac{s_m(70)-s_m(80)}{s_m(50)} + 0.51\,\frac{s_f(70)-s_f(80)}{s_f(50)} \\ &= 0.49\,\frac{\sqrt{2/9}-\sqrt{1/9}}{2/3} + 0.51\,\frac{\sqrt{0.3}-\sqrt{0.2}}{\sqrt{1/2}} \approx 0.17. \end{aligned} $$

(b) In general, let B_1 be the event that a person chosen at random belongs to the first group, and B_2 – that it belongs to the second group. We know that $P(B_1) = w_1$ and $P(B_2) = w_2$. Certainly, $w_1 + w_2 = 1$.

Denote by $w_1(x)$ the probability that a person taken at random from people of age x (we know her/his age!) belongs to the first group; $w_2(x)$ is defined respectively. By the the Bayes formula,

$$ w_1(x) = P(B_1\,|\,X>x) = \frac{P(X>x\,|\,B_1)P(B_1)}{P(X>x\,|\,B_1)P(B_1)+P(X>x\,|\,B_2)P(B_2)} = \frac{w_1\cdot {}_xp_0^{(1)}}{w_1\cdot {}_xp_0^{(1)} + w_2\cdot {}_xp_0^{(2)}}. \tag{1} $$

Similarly,

$$ w_2(x) = \frac{w_2\cdot {}_xp_0^{(2)}}{w_1\cdot {}_xp_0^{(1)} + w_2\cdot {}_xp_0^{(2)}}. \tag{2} $$

Then the conditional survival function

$$ {}_tp_x = w_1(x)\,{}_tp_x^{(1)} + w_2(x)\,{}_tp_x^{(2)}. \tag{3} $$

22. The distribution of $T(x)$ is a mixture of exponential distributions but the weights depend on x. We may use general formulas (1)-(3) or proceed as follows. Let w_i, $\mu_i, i = 1,2$, be the original weight and force of mortality for the ith group. Then

$$\begin{aligned} P(T(x) > t) &= P(X > x+t \,|\, X > x) \\ &= \frac{w_1 \exp\{-\mu_1(x+t)\} + w_2 \exp\{-\mu_2(x+t)\}}{w_1 \exp\{-\mu_1 x\} + w_2 \exp\{-\mu_2 x\}} \\ &= w_1(x) \exp\{-\mu_1 t\} + w_2(x) \exp\{-\mu_2 t\}, \end{aligned}$$

where

$$w_i(x) = \frac{w_i \exp\{-\mu_i x\}}{w_1 \exp\{-\mu_1 x\} + w_2 \exp\{-\mu_2 x\}},\ i = 1,2.$$

For the data in the exercise,

$$\begin{aligned} w_1(20) &= \frac{0.3 \exp\{-20/50\}}{0.3 \exp\{-20/50\} + 0.7 \exp\{-20/80\}} \approx 0.27, \\ w_2(20) &= 1 - w_1(20) \approx 0.73. \end{aligned}$$

The answer is natural. The forces of mortality in the two groups are different, so the share of people from, for instance, the first group who attain an age x depends on x.

25. We write

$$\begin{aligned} E\{K(x)\} &= E\{K(x) \,|\, T(x) < 1\} P(T(x) < 1) + E\{K(x) \,|\, T(x) \geq 1\} P(T(x) \geq 1) \\ &= 0 \cdot P(T(x) < 1) + E\{K(x) \,|\, K(x) \geq 1\} P(T(x) \geq 1) \\ &= E\{1 + K(x+1)\} p_x = (1 + e_{x+1}) p_x. \end{aligned}$$

Thus,

$$e_x = (1 + e_{x+1}) p_x.$$

2. ***Chapter 8***:

1. **(a)** Regarding A_x, the nearest payment may occur only at $t = 1$. So, $A_x \le v$. Formally, $A_x = E\{v^{K+1}\} \le E\{v^{0+1}\} = v$.

We cannot write the same for $\bar{A}_x$ since the payment may occur at any $t > 0$. Formally, $\bar{A}_x = E\{e^{-\delta T}\}$, and the bound $e^{-\delta}$ does not apply.

(1b) If $\delta = 0$, the APV of the whole life insurance with unit benefit is equal to one, and does not depend on the age of the insured. If $\delta > 0$, then, in a typical situation, $\bar{A}_x$ or A_x are increasing in x because the older a person is, the shorter her/his future lifetime, and hence the future unit payment is discounted "to less of an extent". Formally, it is easy to see if – for example, in the continuous time case – we integrate (2.1.1) by parts:

$$\bar{A}_x = \int_0^\infty e^{-\delta t} d(-{}_tp_x) = 1 - \delta \int_0^\infty e^{-\delta t}\, {}_tp_x dt. \tag{4}$$

So, *if* ${}_tp_x$ is decreasing in x for all t, then $\bar{A}_x$ is increasing.

In the discrete time case, the counterpart of the integration by parts may look as follows:

$$\begin{aligned} A_x &= \sum_{k=0}^{\infty} v^{k+1} P(K(x) = k) = \sum_{k=0}^{\infty} v^{k+1} \left({}_kp_x - {}_{k+1}p_x\right) \\ &= v\sum_{k=0}^{\infty} v^k\, {}_kp_x - \sum_{k=0}^{\infty} v^{k+1}\, {}_{k+1}p_x \\ &= v\sum_{k=0}^{\infty} v^k\, {}_kp_x - \sum_{m=1}^{\infty} v^m\, {}_mp_x \\ &= 1 + v\sum_{k=0}^{\infty} v^k\, {}_kp_x - \sum_{m=0}^{\infty} v^m\, {}_mp_x = 1 - (1-v)\sum_{k=0}^{\infty} v^k\, {}_kp_x. \end{aligned}$$

However, there may be "dangerous" years such that if a person survives them, the expected remaining lifetime gets larger, and hence the APV of a whole life insurance gets smaller.

Rigorously, it may be shown with use of (2.3.3) and (2.3.6). For example, from (2.3.6) with $n = 1$, we have $A_x = A^1_{x:\overline{1}|} + v\,p_x A_{x+1} = v(1-p_x) + vp_x A_{x+1} \ge v(1-p_x)$. Assume that p_x is very small (the x is a dangerous year). Then, since $A_x \le v$ for all x's, A_x is close to v. On the other hand, by the same formula, $A_{x+1} = v(1-p_{x+1}) + vp_{x+1}A_{x+2} \le v(1-p_{x+1}) + v^2 p_{x+1}$, and if p_{x+1} is not small, A_{x+1} may be essentially less than v.

4. In Exercise **7**-28, we have shown that $P(K = k) = pq^k$, where $p = 1 - e^{-\mu}$. By (**0**.4.3.1), the m.g.f. $M_{K+1}(z) = pe^z/(1-qe^z)$. Consequently,

$$A_x = M_{K+1}(-\delta) = e^{-\delta}\frac{1-e^{-\mu}}{1-e^{-\mu}e^{-\delta}} = e^{-\delta}\frac{1-e^{-\mu}}{1-e^{-(\mu+\delta)}}.$$

Using the fact that $e^x = 1 + x + o(x)$ for $x \to 0$, we can write

$$A_x = (1+\delta+o(\delta))(1+\mu+o(\mu))(1+\mu+\delta+o(\mu+\delta))^{-1} \sim \frac{\mu}{\mu+\delta}$$

for $\mu + \delta \to 0$. (Since μ and δ are positive, from $\mu+\delta \to 0$ it follows that $\mu \to 0$ and $\delta \to 0$.)

Hence, for small μ and δ, the quantities A_x and $\bar{A}$ are close.

6. In this case, $\overline{A}_x = \frac{\mu}{\mu+\delta} = \frac{1}{1+\delta/\mu}$. If $\overline{A}_x = \frac{1}{2}$, then $\frac{\delta}{\mu} = 1$, and ${}^2\overline{A}_x = \frac{1}{1+2\delta/\mu} = \frac{1}{3}$. Thus, $Var\{Z\} = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{12}$.

13. If T is uniform, then—in view of the assumption we made in Section 2.1.3—formula (2.1.6) is precise. It is interesting to see it directly. In Example 1.1-1, we got $\overline{A}_x = \frac{1-e^{-s}}{s}$, where $s = (\omega - x)\delta$. From Example 1.1-2, setting $v = e^{-\delta}$, we obtain that $A_x = \dfrac{v(1-v^{\omega-x})}{(\omega-x)(1-v)} = \dfrac{\delta e^{-\delta}}{1-e^{-\delta}} \cdot \dfrac{(1-e^{-s})}{s} = \dfrac{\delta}{e^{\delta}-1} \cdot \dfrac{(1-e^{-s})}{s} = \dfrac{\delta}{i} \cdot \overline{A}_x$. This is not the only possible example. For any distribution of T with a density that is constant in intervals $(k, k+1)$, formula (2.1.6) is precise.

21. We proceed from Exercise **7**-22. Since for the constant force of mortality, $\overline{A}_x = \frac{\mu}{\mu+\delta}$, in our case,

$$\overline{A}_x = w_1(x)\frac{\mu_1}{\mu_1+\delta} + w_2(x)\frac{\mu_2}{\mu_2+\delta}.$$

Thus, $\overline{A}_{20} \approx 0.27\frac{0.02}{0.02+\delta} + 0.73\frac{0.0125}{0.0125+\delta} \approx 0.223$, and ${}^2\overline{A}_x = 0.27\frac{0.02}{0.02+2\delta} + 0.73\frac{0.0125}{0.0125+2\delta} \approx 0.126$ for $\delta = 0.05$. By the double rate rule, $Var\{Z\} \approx 0.126 - (0.223)^2 \approx 0.076$.

22. We have

$A_{30} = 0.25$, $A_{50} = 0.4$, and $A_{30:\overline{20}|} = 0.55$. Then, by (2.3.6) and (2.4.7), $0.25 = (0.55 - {}_{20}E_{30}) + {}_{20}E_{30} \cdot 0.4$, which gives ${}_{20}E_{30} = 0.5$. Then $e^{-\delta 20}0.95 = 0.5$, and $\delta \approx 0.032$.

29. The first group is healthier since for *any* t, the probability to attain age $x+t$ is larger for the first group. In a certain sense, people from the first group live longer, and then for any insurance, the APV for the first group should be smaller; more precisely, not larger.

Rigorously, let T be a lifetime, and let Ψ be a random moment of payment. Assume that $\Psi = \psi(T)$ where $\psi(x)$ is a non-decreasing function. This is the case for all insurances we considered. Then, integrating by parts, for the APV we have

$$\begin{aligned} A &= E\{e^{-\delta\Psi}\} = \int_0^\infty e^{-\delta\psi(t)} dP(T \le t) = -\int_0^\infty e^{-\delta\psi(t)} dP(T > t) \\ &= e^{-\delta\psi(0)}P(T > 0) + \int_0^\infty P(T > t) de^{-\delta\psi(t)}. \end{aligned}$$

Formally, $\psi(t)$ may be non-differentiable at one point (as $\psi(t) = \min(t, n)$) but for integration it does not matter. In any case, the function $e^{-\delta\psi(t)}$ is non-increasing, and hence the differential $de^{-\psi(t)} \le 0$. Therefore the choice of a larger function $P(T > t)$ leads to a non-larger quantity A.

33. **(a)** *All* characteristics we consider may be represented as $E\{\exp\{-\delta\Psi\}\}$, where Ψ is the corresponding random payment time. If $P(0 < \Psi < \infty) > 0$ (that is, Ψ is not equal to zero or infinity with probability one), then $E\{\exp\{-\delta\Psi\}\}$ is decreasing in δ.

(b) All functions are continuous in δ; so, it suffices to consider the case $\delta = 0$. The present value of \$1 to be paid in future is \$1. However, we must not forget that for some insurances (for example, for a term insurance), with a positive probability, there will be no payments. Hence, the APV equals one times the probability that there will be a payment. In particular, as $\delta \to 0$,

$$\begin{aligned} \lim A_x &= \lim \overline{A}_x = \lim A_{x:\overline{n}|} = \lim \overline{A}_{x:\overline{n}|} = 1, \\ \lim A^{1}_{x:\overline{n}|} &= \lim \overline{A}^{1}_{x:\overline{n}|} = P(K(x) < n) = {}_n q_x, \\ \lim {}_{m|}A_x &= \lim {}_{m|}\overline{A}_x = \lim_n E_x = P(T(x) > m) = {}_m p_x. \end{aligned}$$

(c) Rigorously, we should compute $\lim_{\delta\to 0} E\{\exp\{-\delta\Psi\}\}$. We should not forget that Ψ may assume an infinite value with a positive probability, and consequently, the r.v. $Z = \exp\{-\delta\Psi\}\}$ may be equal zero with the same probability. So, $\lim_{\delta\to 0} E\{\exp\{-\delta\Psi\}\} = E\{\mathbf{1}_{\{\Psi<\infty\}}\} = P(\Psi < \infty)$. (One may pass the limit inside the expectation by the Lebesgue dominated convergence theorem, which is certainly out of the scope of this book.)

34. For any number δ, the function $e^{-\delta t}$ is convex (in another terminology, concave upward), so by Jensen's inequality (see p.100), $E\{\exp\{-\delta\Psi\}\} \geq \exp\{-\delta E\{\Psi\}\}$. Thus, when replacing Ψ by $E\{\Psi\}$, we underestimate the APV. In particular, $A_x \geq \exp\{-\delta e_x\}$ and $\bar{A}_x \geq \exp\{-\delta \overset{\circ}{e}_x\}$.

3. ***Chapter 9***:

3. The symbol $\bar{a}_{20}$ denotes the *expected* present value of the whole life insurance on a twenty years old person whose lifetime is *random*, while $\bar{a}_{\overline{20|}}$ denotes the present value of a *certain* annuity paid during 20 . years (see p.478).

19. This immediately follows from (3.2.5) since $v^k \cdot {}_kp_x = {}_kE_x$. The formula (3.2.5) was obtained by the current payment technique (see Section 1.1).

20. **(a)** It is convenient to proceed from (1.1.7), i.e., from the formula

$$\bar{a}_x = \int_0^\infty e^{-\delta t} c_t \cdot {}_tp_x dt \tag{5}$$

in the continuous time case, and from (1.2.5), i.e., from the formula

$$\ddot{a}_x = \sum_{k=0}^{\infty} e^{-\delta k} c_k \cdot {}_kp_x \tag{6}$$

in the case of annuity-due. The characteristics $\bar{a}_x$ and $\ddot{a}_x$ are non-increasing in δ. If $c_k \cdot {}_kp_x \neq 0$ at least for one $k \neq 0$, then $\ddot{a}_x$ is decreasing. However, if for example, in a m-deferred insurance contract, ${}_mp_x = 0$, then ${}_{m|}\ddot{a}_x$ is just zero.

In the continuous time case, we need $c_t \cdot {}_tp_x$ to be positive in an interval of positive t's. Except the trivial case where it is not so, all annuity characteristics under consideration are decreasing in δ.

(b) If $\delta = \infty$, all future payments have zero values. So, for any payments c_t, we have $\bar{a}_x \to 0$ as $\delta \to \infty$. In the discrete time case, the first payment is provided immediately and its value is not discounted. So, for all payments c_k, we have $\ddot{a}_x \to c_0$. Thus, in the case $c_t \equiv 1$ for the quantities $\ddot{a}_x$, $\ddot{a}_{x:\overline{m|}}$, the limit equals one, while in the case of the deferred annuity ${}_{m|}\ddot{a}_x$, the limit is zero.

Consider limits as $\delta \to 0$. All characteristics are continuous functions of δ, so it suffices to set $\delta = 0$. If $\delta = 0$, then the present value of an annuity paid at unit rate is equal to the length of the payment period in the continuous time case, and to the number of payments in the discrete time case. We must also remember that the period (or the number) mentioned may equal zero. In particular, as $\delta \to 0$,

$$\begin{aligned}
\lim \bar{a}_x &= E\{T\} = \overset{\circ}{e}_x, \\
\lim \ddot{a}_x &= E\{K+1\} = e_x + 1, \\
\lim \bar{a}_{x:\overline{n}|} &= \lim(\bar{a}_x - {}_np_x\bar{a}_{x+n}) = \overset{\circ}{e}_x - {}_np_x\, \overset{\circ}{e}_{x+n}, && (7)\\
\lim \ddot{a}_{x:\overline{n}|} &= \lim(\ddot{a}_x - {}_np_x\ddot{a}_{x+n}) = (e_x+1) - {}_np_x(e_{x+n}+1) \\
&= e_x + {}_nq_x - {}_np_xe_{x+n}, && (8)\\
\lim {}_{m|}\bar{a}_x &= {}_mp_xE\{T(x+m\} = {}_mp_x\cdot\, \overset{\circ}{e}_{x+m}, \\
\lim {}_{m|}\ddot{a}_x &= {}_mp_x(E\{K(x+m)\}+1) = {}_mp_x(e_{x+m}+1).
\end{aligned}$$

(c) Probably, the easiest way to prove the formulas above rigorously is to pass the limit as $\delta \to 0$ inside the integral and the sum in (5) and (6), respectively. Considering, for example, $\lim {}_{m|}\bar{a}_x$, we may write that, as $\delta \to 0$,

$${}_{m|}\bar{a}_x = \int_m^\infty e^{-\delta t}\cdot {}_tp_xdt \to \int_m^\infty {}_tp_xdt = {}_mp_x\int_m^\infty {}_{t-m}p_{x+m}dt.$$

Making the variable change $s = t-m$, we have

$${}_{m|}\bar{a}_x \to {}_mp_x\int_0^\infty {}_sp_{x+m}ds = {}_mp_xE\{T(x+m)\}.$$

21. Clearly, $\ddot{a}_{\overline{x:\overline{n}|}} \geq \ddot{a}_x$ since unlike the case of the whole life annuity, the certain and life annuity provides the first n payments for sure. One may also compare (3.4.7) with (3.1.8).

If the probabilities ${}_kp_x$ are close to one for $k = 0,...,n-1$, then the expression (3.4.7) is close to (3.1.8). Theoretically, $\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_x$ if ${}_kp_x = 1$ for $k = 0,...,n-1$.

Now, $\lim_{\delta\to 0}\ddot{a}_{\overline{x:\overline{n}|}} = E\{\max\{n, K+1\}\}$. This follows from (3.4.1) and the general fact that the present value $Y = \Psi$ if $\delta = 0$ and $c_k \equiv 1$. Formally, this follows, for example, from (1.2.3).

22. For $\delta = 0$, we rewrite (3.2.6) as

$$E\{K(x)+1\} = E\{\min\{K(x\}+1,\, n\}\} + {}_np_xE\{K(x+n\}+1\}. \qquad (9)$$

On the other hand,

$$\begin{aligned}
E\{\min\{K(x)+1, n\}\} &= E\{K(x)+1-\max\{K(x)+1-n,\, 0\}\} \\
&= E\{K(x)\}+1-E\{\max\{K(x)+1-n,\, 0\}\} \\
&= E\{K(x)\}+1-{}_np_xE\{K(x)+1-n\,|\,T(x)\geq n\} \\
&= E\{K(x)\}+1-{}_np_x(E\{K(x+n)\}+1).
\end{aligned}$$

Substituting this into the r.-h.s. of (M-9), we come to the l.-h.s. of (M-9).

We can also substitute into (3.2.6) the results of Exercise 20, which leads to

$$e_x + 1 = e_x + {}_nq_x - {}_np_xe_{x+n} + {}_np_x(e_{x+n}+1\}.$$

As is easy to see, this is an identity.

23. Reasoning heuristically, we can write the counterparts of (3.2.6) and (3.2.8) immediately:

$$\bar{a}_x = \bar{a}_{x:\overline{n}|} + v^n \, {}_n p_x \bar{a}_{x+n}, \tag{10}$$

and

$$\bar{a}_{x:\overline{m}|} = \bar{a}_{x:\overline{n}|} + v^n \, {}_n p_x \, \bar{a}_{x+n:\overline{m-n}|} \text{ for all } m = n, n+1, \dots, \tag{11}$$

respectively. Relation (M-10) follows from (M-11) by setting $m = \infty$. The proof of (M-11) is similar to what we did in Section 3.2 and runs as follows:

$$\begin{aligned} \bar{a}_{x:\overline{m}|} &= \int_0^m e^{-\delta t} \, {}_t p_x dt = \int_0^n e^{-\delta t} \, {}_t p_x dt + \int_n^m e^{-\delta t} \, {}_t p_x dt \\ &= \bar{a}_{x:\overline{n}|} + \int_n^m e^{-\delta(t-n)} e^{-\delta n} \, {}_n p_x \cdot {}_{t-n} \, p_{x+n} dt \\ &= \bar{a}_{x:\overline{n}|} + e^{-\delta n} \cdot {}_n \, p_x \int_n^m e^{-\delta(t-n)} \, {}_{t-n} p_{x+n} dt. \end{aligned}$$

Under the variable change $s = t - n$, this implies that

$$\bar{a}_{x:\overline{m}|} = \bar{a}_{x:\overline{n}|} + e^{-\delta n} \cdot {}_n \, p_x \int_0^{m-n} e^{-\delta s} \, {}_s p_{x+n} dt = \bar{a}_{x:\overline{n}|} + e^{-\delta n} \cdot \, {}_n p_x \cdot \bar{a}_{x+n:\overline{m-n}|}.$$

24. (i)

$$\bar{a}_x = \bar{a}_{x:\overline{n}|} + v^n \, {}_n p_x \bar{a}_{x+n};$$

(ii)

$$\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + v^n \, {}_n p_x \ddot{a}_{x+n};$$

(iii)

$$\ddot{a}_x = \ddot{a}_{x:\overline{m}|} + {}_{m|}\ddot{a}_x$$

[relations (ii) and (iii) follows from each other in view of (3.3.7)];

(iv)

$$\bar{A}_x = 1 - \delta \bar{a}_x;$$

(v)

$$A_x = 1 - d\ddot{a}_x;$$

(vi)

$$A_{x:n} = 1 - d\ddot{a}_{x:\overline{n}|};$$

(vii)

$$A^1_{x:\overline{n}|} = 1 - d\ddot{a}_{x:n} - v^n \, {}_n p_x.$$

27. Let \$10,000 be a unit of money. The APV of the uncle's gift is $2\ddot{a}_{20}$, while the annuity Ann prefers is the 10-year deferred annuity whose APV is $c \cdot {}_{10}p_{20} e^{-\delta \cdot 10} \ddot{a}_{30}$, where c is the annual payment. We use the Illustrative Table which corresponds to the data on the total population of USA, 2002. We have

${}_{10}p_{20} = (l_{30}/l_{20}) = \frac{97743}{98675} \approx 0.9906$, $\ddot{a}_{20} \approx 22.48818$, and $\ddot{a}_{30} \approx 21.25547$. Then, for the two annuities to be equivalent, we should have

$$c = \frac{2\ddot{a}_{20}}{e^{-\delta\cdot 10}\,{}_{10}p_{20}\cdot \ddot{a}_{30}} \approx \frac{2\cdot 22.48818\cdot e^{0.04\cdot 10}}{0.9906\cdot 21.2554} \approx 3.1866.$$

So, if Ann attains the age of 30, she will annually get $\$31,866$.

28. **(a)** Let Ψ be the number of visits. The r.v. Ψ has the geometric distribution in the form (**0**.3.1.7) with $p = 0.1$. The present value of the total money spent is the r.v. $Y = \sum_{k=1}^{\Psi} v^{k-1}X_k$, where X_k is the random amount spent at the kth visit. Assuming X's and Ψ are independent, we write $E\{Y\,|\,\Psi\} = \sum_{k=1}^{\Psi} v^{k-1}E\{X_k\,|\,\Psi\} = 8\sum_{k=1}^{\Psi} v^{k-1} = 8\frac{1-v^{\Psi}}{1-v}$. Then

$$E\{Y\} = E\{E\{Y\,|\,\Psi\}\} = E\left\{8\frac{1-v^{\Psi}}{1-v}\right\} = \frac{8}{1-v}(1-E\{v^{\Psi}\}) = \frac{8}{1-v}(1-M_{\Psi}(\ln v)),$$

where $M_{\Psi}(z) = \dfrac{e^z p}{1-qe^z}$, the m.g.f. of Ψ (see Section **0**.4.3.2), and $q = 1-p$. Now, $M_{\Psi}(\ln v) = \dfrac{vp}{1-qv}$ (which is the generating function of Ψ), and

$$E\{Y\} = \frac{8}{1-v}(1-\frac{vp}{1-qv}) = \frac{8}{1-qv} = \frac{8}{1-0.9\cdot 0.96} \approx 58.82.$$

(b) Making use of (**0**.7.3.2), we write

$$\begin{aligned} Var\{Y\} &= E\{Var\{Y\,|\Psi\}\} + Var\{E\{Y\,|\,\Psi\}\} = E\left\{\sum_{k=1}^{\Psi} v^{2(k-1)}Var\{X_k\}\right\} \\ +Var\left\{8\frac{1-v^{\Psi}}{1-v}\right\} &= E\left\{\frac{16}{12}\cdot\frac{1-v^{2\Psi}}{1-v^2}\right\} + \frac{64}{(1-v)^2}Var\{v^{\Psi}\} \\ &= \frac{4}{3}\cdot\frac{1}{1-v^2}(1-E\{v^{2\Psi}\}) + \frac{64}{(1-v)^2}(E\{v^{2\Psi}\}-(E\{v^{\Psi}\})^2). \end{aligned} \qquad (12)$$

As was already noted, $E\{v^{\Psi}\} = \frac{vp}{1-qv}$. Then $E\{v^{2\Psi}\} = \dfrac{v^2p}{1-qv^2}$. One can insert it into (12) and get a general formula, but we will restrict ourselves to a particular answer. For $v = 0.96$ and $p = 0.1$, we have $E\{v^{\Psi}\} \approx 0.706$ and $E\{v^{2\Psi}\} \approx 0.540$. Then, as is easy to calculate using (12), $Var\{Y\} \approx 1670.38$.

29. **(a)** Let $\$100,000$ be a unit of money, and $d = 1-e^{-\delta}$, where $\delta = 0.03$. Following (3.1.4) and (3.1.5), we have

$$\ddot{a}_{30} = \frac{1}{d}(1-A_{30}) = \frac{1}{1-e^{-0.03}}(1-0.2) \approx 27.07,$$

and

$$Var\{Y\} = \frac{1}{d^2}({}^2A_{30}-(A_{30})^2) = \left(\frac{1}{1-e^{-0.03}}\right)^2(0.09-0.2^2) \approx 57.24.$$

(b) Making use of (3.2.3) and the results of Exercise **8**-20, we have

$$\ddot{a}_{30:\overline{35}|} = \frac{1}{d}(1 - A_{30:\overline{35}|}) \approx \frac{1}{1-e^{-0.03}}(1-0.386) \approx 20.77.$$

For the variance of the 35-year term insurance, we got $Var\{Z\} \approx 0.030$. Then, for the corresponding annuity,

$$Var\{Y\} = \left(\frac{1}{1-e^{-0.03}}\right)^2 \cdot 0.03 \approx 34.35.$$

36. The combination of two temporal annuities, namely, the combination with the APV $20\ddot{a}_{20:\overline{45}|} + 10\ddot{a}_{20:\overline{5}|}$, pays 30 for the first 5 years and 20 for the next 40 years. We should subtract the 20-year deferred 25-year temporal annuity whose APV is $10 \cdot {}_{20|}\ddot{a}_{20:\overline{25}|}$, which amounts to $10v^{20}\,{}_{20}p_{20}\ddot{a}_{40:\overline{25}|}$. Thus, the total APV is $20\ddot{a}_{20:\overline{45}|} + 10\ddot{a}_{20:\overline{5}|} - 10v^{20}\,{}_{20}p_{20}\ddot{a}_{40:\overline{25}|}$.

On additional problem #2. Let $n = 100$, $Y_i;\ i = 1,\dots n$, be the present value of the annuity to be paid to the ith client, and $S = Y_1 + \cdots + Y_n$. We have computed above the quantities $m = E\{Y_i\}$ and $\sigma^2 = Var\{Y\}$ in both cases ((**a**)and (**b**)) above. Denote by H the initial size of the fund. We want to have a fund such that $P(S \le H) \ge 0.99$. Using the normal approximation, we have

$$P(S \le H) = P\left(\frac{S-mn}{\sigma\sqrt{n}} \le \frac{H-mn}{\sigma\sqrt{n}}\right) = P\left(S^* \le \frac{H-mn}{\sigma\sqrt{n}}\right) \approx \Phi\left(\frac{H-mn}{\sigma\sqrt{n}}\right) \ge 0.99.$$

So, if $q_{0.99}$ is the 0.99-quantile of the standard normal distribution (≈ 2.33), then

$$\frac{H-mn}{\sigma\sqrt{n}} \ge q_{0.99},$$

and hence

$$H \ge mn + q_{0.99}\sigma\sqrt{n}.$$

It remains to insert the particular values of $q_{0.99}, m, \sigma$, and n.

4. ***Chapter 10***:

4. **(i)** $\overline{P}_x = \overline{A}_x \big/ \bar{a}_x$ and $P_x = A_x \big/ \ddot{a}_x$. Since $\bar{a}_x \le \ddot{a}_x$, and $\overline{A}_x \ge A_x$ (see Exercises **8**-27 and **9**-15), $\overline{P}_x \ge P_x$.

(ii) The logic is the same. The premiums $\overline{P}_{x:\overline{n}|} = \overline{A}_{x:\overline{n}|} \big/ \bar{a}_{x:\overline{n}|}$ and $P_{x:\overline{n}|} = A_{x:\overline{n}|} \big/ \ddot{a}_{x:\overline{n}|}$. Then $\overline{P}_{x:\overline{n}|} \ge P_{x:\overline{n}|}$ because $\overline{A}_{x:\overline{n}|} \ge A_{x:\overline{n}|}$, and $\bar{a}_{x:\overline{n}|} \le \ddot{a}_{x:\overline{n}|}$.

(iii) $P_x = A_x \big/ \ddot{a}_x$ and $P_{x:\overline{n}|} = A_{x:\overline{n}|} \big/ \ddot{a}_{x:\overline{n}|}$. Clearly, $A_x \le A_{x:\overline{n}|}$, and $\ddot{a}_x \ge \ddot{a}_{x:\overline{n}|}$. Hence, $P_x \le P_{x:\overline{n}|}$.

(iv) $P^1_{x:n} = A^1_{x:\overline{n}|} \big/ \ddot{a}_{x:\overline{n}|}$ and $P_{x:n} = A_{x:\overline{n}|} \big/ \ddot{a}_{x:\overline{n}|}$. The denominators are the same, while $A^1_{x:\overline{n}|} \le A_{x:\overline{n}|}$, so $P^1_{x:n} \le P_{x:n}$.

11. The lifetime T is exponential with $\mu = 0.01$, so $P(K = k) = (1 - e^{-\mu})e^{-\mu k}$. The insurance may be represented as the sum of a 10-year term insurance with a benefit of 1000, and a 15-year term insurance with the same benefit.

The APV for the first insurance is

$$1000\cdot\sum_{k=0}^{9}v^{k+1}(1-e^{-\mu})e^{-\mu k}=1000\cdot(1-e^{-\mu})v\frac{1-(ve^{-\mu})^{10}}{1-ve^{-\mu}}\approx 72.857800.$$

For the second, it is

$$1000\cdot\sum_{k=0}^{14}v^{k+1}(1-e^{-\mu})e^{-\mu k}=1000\cdot(1-e^{-\mu})v\frac{1-(ve^{-\mu})^{15}}{1-ve^{-\mu}}\approx 95.594223.$$

The APV for the premium annuity-due may be computed by(**9**.3.2.5), which amounts to

$$\sum_{k=0}^{14}v^{k}e^{-\mu k}=\frac{1-(ve^{-\mu})^{15}}{1-ve^{-\mu}}\approx 10.112946.$$

Hence,

$$P\approx\frac{72.858+95.594}{10.113}\approx 16.657.$$

14. If $\delta=0$, then in view of what we got in Exercise 5 and (M-8),

$$P=\frac{{}_np_x\cdot e_{x+n}+1}{e_x+1-{}_np_x-{}_np_xe_{x+n}}.$$

17. Since we consider the event $\{L<0\}=\{Z-\pi_{\text{net}}(1+k)Y<0\}$, the size of the benefit does not matter: we can consider the problem for a unit benefit. We have $A_{50}\approx 0.32654$, $\ddot{a}_{50}\approx 17.1756$, so

$$\pi_{\text{net}}\approx\frac{0.32654}{17.1756}\approx 0.01901.$$

Next,

$$H^2={}^2A_{50}-(A_{50})^2\approx 0.13526-0.32654^2\approx 0.028632,$$

and $H\approx 0.169208$. Now, $d=1-e^{-0.04}\approx 0.039$, and

$$\pi_\gamma\approx\frac{0.32654+1.6448\cdot 0.1692/\sqrt{100}}{17.1756-1.6448\cdot 0.1692/(0.039\sqrt{100})}\approx 0.021526.$$

To find k, we should solve the inequality $(1+k)\pi_{\text{net}}\geq\pi_\gamma$. A solution is $k\geq 0.133$ or 13.3%.

18. We start with

$$\ddot{a}_x=\sum_{k=0}^{\infty}v^k\,{}_kp_x,$$

and

$$\ddot{a}_{x:\overline{n|}}=\sum_{k=0}^{n-1}v^k\,{}_kp_x.$$

It is given that, for all k, the probabilities ${}_kp_x$ are larger for the first group. Hence, $\ddot{a}_x$ and $\ddot{a}_{x:\overline{n|}}$ are also larger for the first group. Since $A_x=1-d\ddot{a}_x$, and $A_{x:\overline{n|}}=1-d\ddot{a}_{x:\overline{n|}}$, the characteristics A_x and $A_{x:\overline{n|}}$ are smaller for the first group. Because $P_x=A_x/a_x$ and $P_{x:\overline{n|}}=A_{x:\overline{n|}}/\ddot{a}_{x:\overline{n|}}$, both premiums are smaller for the first group.

19. Let $\mu(x) = \mu$, and $q = e^{-\mu}$. In Exercise **8**-4, we have shown that

$$A_x = v\frac{1-q}{1-qv}.$$

(In Exercise **8**-4. we wrote $e^{-\delta}$ for the discount factor v.) Then

$$1 - A_x = 1 - v\frac{1-q}{1-qv} = \frac{1-v}{1-qv} = \frac{d}{1-qv},$$

and

$$P_x = \frac{dA_x}{1-A_x} = v(1-q).$$

22. **(a)** First, we compute

$${}_{35}E_{30} = (l_{65}/l_{30})e^{-\delta \cdot 35} = (82609/97743)e^{-0.04\cdot 35} \approx 0.2084.$$

Then

$${}_{35|}a_{30} = {}_{35}E_{30} \cdot a_{65} \approx 0.2084 \cdot 12.6095 \approx 2.6280.$$

Next, we calculate

$$A^{1}_{30:\overline{35}|} = A_{30} - {}_{35}E_{30} \cdot A_{65} \approx 0.1666 - 0.2084 \cdot 0.5056 \approx 0.06123.$$

Then

$$A_{30:\overline{35}|} = A^{1}_{30:\overline{35}|} + {}_{35}E_{30} \approx 0.06123 + 0.2084 \approx 0.2696,$$

and

$$\ddot{a}_{30:\overline{35}|} = \frac{1 - A_{30:\overline{35}|}}{1 - \exp\{-0.04\}} \approx 18.6267.$$

Thus,

$$P_{\text{net}} = ({}_{35|}a_{30}/\ddot{a}_{30:\overline{35}|}) \approx \frac{2.6280}{18.6267} \approx 0.1411.$$

Adding 5%, we get ≈ 0.1482 per dollar, or \$7407 per year.

27. First, as has been shown in Exercise 6, the benefit premium for an n-year term insurance is $\overline{P}^{1}_{x:\overline{n}|} = \overline{P}_{x:\overline{n}|} - \overline{P}_{x:\overline{n}|}^{\;\;1}$, where the last premium is that for the n-year pure endowment. Denoting, for simplicity and for a moment, this premium by $\overline{P}^{*}_{x:\overline{n}|}$, note that $\overline{P}^{*}_{x:\overline{n}|} = {}_{n}E_x/\bar{a}_{x:\overline{n}|}$. Now,

$$\begin{aligned}
{}_t\overline{V}^{1}_{x:\overline{n}|} &= \overline{A}^{1}_{x+t:\overline{n-t}|} - \overline{P}^{1}_{x:\overline{n}|}\bar{a}_{x+t:\overline{n-t}|} = \\
&= (\overline{A}_{x+t:\overline{n-t}|} - {}_{n-t}E_{x+t}) - (\overline{P}_{x:\overline{n}|} - \overline{P}^{*}_{x:\overline{n}|})\bar{a}_{x+t:\overline{n-t}|} \\
&= \overline{A}_{x+t:\overline{n-t}|} - \overline{P}_{x:\overline{n}|}\bar{a}_{x+t:\overline{n-t}|} - {}_{n-t}E_{x+t} + \overline{P}^{*}_{x:\overline{n}|}\bar{a}_{x+t:\overline{n-t}|} \\
&= {}_t\overline{V}_{x:\overline{n}|} - {}_{n-t}E_{x+t} + \overline{P}^{*}_{x:\overline{n}|}\bar{a}_{x+t:\overline{n-t}|},
\end{aligned}$$

where ${}_t\overline{V}_{x:\overline{n}|}$ is the reserve for the n-year endowment insurance. Using (2.3.5) and the above expression for $\overline{P}^{*}_{x:\overline{n}|}$, we have

$$\begin{aligned}
{}_t\overline{V}^{1}_{x:\overline{n}|} &= 1 - \frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} - {}_{n-t}E_{x+t} + {}_{n}E_x\frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} \\
&= (1 - {}_{n-t}E_{x+t}) - (1 - {}_{n}E_x)\frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}}.
\end{aligned}$$

For the exponential case, we follow what we did in Example 2.3.-2. Since ${}_nE_x = e^{-\delta n}e^{-n\mu} = e^{-(\delta+\mu)n}$,

$$ {}_t\overline{V}^{\,1}_{x:\overline{n}|} = (1-e^{-(\mu+\delta)(n-t)}) - (1-e^{-(\delta+\mu)n})\frac{1-e^{-(\delta+\mu)(n-t)}}{1-e^{-(\delta+\mu)n}} = 0, $$

which is not surprising in view of the memoryless property. Unlike the case of an endowment insurance, in the term insurance case, the "possibility that the payment will be soon made does not get larger" when the time is getting closer to the maturity of the contract.

28. If $\mu(x) = \mu$, then in accordance with (**9**.3.2.5),

$$ \ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} e^{-\delta k}e^{-\mu k} = \sum_{k=0}^{n-1} e^{-(\mu+\delta)k} = \frac{1-e^{-(\mu+\delta)n}}{1-e^{-(\mu+\delta)}}, $$

and, by (2.3.2),

$$ {}_kV_{x:\overline{n}|} = 1 - \frac{1-e^{-(\mu+\delta)(n-k)}}{1-e^{-(\mu+\delta)n}} = \frac{e^{(\mu+\delta)k}-1}{e^{(\mu+\delta)n}-1}. $$

29. If $\delta = 0$, then $\ddot{a}_x = E\{K(x)+1\}$, $\bar{a}_x = E\{T(x)\}$, $\ddot{a}_{x:\overline{n}|} = E\{\min\{K(x)+1,n\}\}$, and $\bar{a}_{x:\overline{n}|} = E\{\min\{T(x),n\}\}$. Then, in accordance with (2.3.2),

$$ {}_kV_x = 1 - \frac{E\{K(x+k)+1\}}{E\{K(x)+1\}} = 1 - \frac{e_{x+k}+1}{e_x+1} = \frac{e_x - e_{x+k}}{e_x+1}; $$

We have already obtained the similar formula for the full continuous case in (2.1.3). In the traditional notation, we write it as

$$ {}_t\overline{V}_x = 1 - \frac{E\{T(x+t)\}}{E\{T(x)\}} = \frac{\mathring{e}_x - \mathring{e}_{x+t}}{\mathring{e}_x}. $$

For an n-year endowment, using (2.3.2), (2.3.5), (M-7) and (M-8), we have

$$ {}_kV_{x:\overline{n}|} = 1 - \frac{E\{\min\{K(x+k)+1, n-k\}\}}{E\{\min\{K(x)+1,n\}\}} = 1 - \frac{e_{x+k} + {}_{n-k}q_{x+k} - {}_{n-k}p_{x+k}e_{x+n}}{e_x + {}_nq_x - {}_np_xe_{x+n}}, $$

and

$$ \begin{aligned} {}_t\overline{V}_{x:\overline{n}|} &= 1 - \frac{E\{\min\{T(x+t), n-t\}\}}{E\{\min\{T(x),n\}\}} = 1 - \frac{\mathring{e}_{x+t} - {}_{n-t}p_{x+t}\cdot \mathring{e}_{x+n}}{\mathring{e}_x - {}_np_x\cdot \mathring{e}_{x+n}} \\ &= \frac{\mathring{e}_x - \mathring{e}_{x+t} + ({}_{n-t}p_{x+t} - {}_np_x)\,\mathring{e}_{x+n}}{\mathring{e}_x - {}_np_x\cdot \mathring{e}_{x+n}}. \end{aligned} $$

30. For an m-year deferred life annuity, for $k \le m$,

$$ \begin{aligned} {}_kV_{\text{ben}} &= {}_{m-k|}\ddot{a}_{x+k} - P_{\text{ben}}\ddot{a}_{x+k:\overline{m-k}|} = {}_{m-k|}\ddot{a}_{x+k} - \frac{{}_{m|}\ddot{a}_x}{\ddot{a}_{x:\overline{m}|}}\cdot \ddot{a}_{x+k:\overline{m-k}|} \\ &= v^{m-k}\cdot {}_{m-k}p_{x+k}\cdot \ddot{a}_{x+m} - v^m\cdot {}_mp_x\cdot \ddot{a}_{x+m}\cdot \frac{\ddot{a}_{x+k:\overline{m-k}|}}{\ddot{a}_{x:\overline{m}|}} \\ &= v^{m-k}\cdot {}_{m-k}p_{x+k}\cdot \ddot{a}_{x+m}\left(1 - v^k\cdot {}_kp_x\cdot \frac{\ddot{a}_{x+k:\overline{m-k}|}}{\ddot{a}_{x:\overline{m}|}}\right). \end{aligned} $$