(b) We provide induction. Assume that
\[ tP_x = P_x \cdot P_{x+1} \cdots P_{x+t-1} \cdots P_{x+t-1} \cdots P_{x+t-1} . \]
Then
\[ t+1P_x = P(T(x) > t)P(T(x) > t+1 | T(x) > t) = tP_x \cdot P(T(x+t) > 1) \]
\[ = tP_{x+1} \cdot P_{x+t} = tP_x \cdot P_{x+1} \cdots P_{x+t-1} \cdot P_{x+t} . \]

14. In view of (1.2.6) and the memoryless property, \( 20|10q_{50} = 20p_{50} - 30p_{50} = P(X > 20) - P(X > 30) = e^{-20/75} - e^{-30/75} \approx 0.0956. \)

15. The distribution is continuous and \( (1 - x/100)^{1/2} = 0.5 \) for \( x = 75. \) Next, taking into account that 75 is a median, we have
\[ 10\text{10q}_{75} = 10p_{75} - 20p_{75} = \frac{1}{s(75)} [s(85) - s(95)] \]
\[ = 2[(1 - 0.85)^{1/2} - (1 - 0.95)^{1/2}] \approx 0.327. \]
Using (1.2.10) and the fact that \( s(x) = 0 \) for \( x > 100, \) we write
\[ \bar{e}_x = \frac{1}{s(x)} \int_x^\infty s(y)dy = \frac{1}{s(x)} \int_x^{100} s(y)dy. \]
Then \( \bar{e}_{75} = 2 \int_{75}^{100} (1 - y/100)^{1/2} dy = \frac{30}{3}. \)

16. (a) From \( l_0 \) newborn girls, on the average \( l_0s_f(50) = l_0\frac{1}{\sqrt{2}} \) survive 50 years. For boys, this number is \( l_0\frac{2}{\sqrt{2}}. \) Hence, the ratio of these mean values is \( \frac{3}{2\sqrt{2}} \approx 1.06. \)

(As a matter of fact, the precise solution should be different. Let \( \eta_1 \) and \( \eta_2 \) be the number of survivors among men and women (both r.v.'s have binomial distributions). Then we should consider, for example, the distribution of \( \eta_1/\eta_2 \) given \( \eta_2 > 0, \) and in particular, the corresponding conditional expectation.)

Now, by the Bayes formula, the probability that a person of age 50 taken at random is a man is
\[ \frac{1}{2}s_m(50) + \frac{1}{2}s_f(50) = \frac{3}{3} + \frac{1}{\sqrt{2}} \approx 0.49. \]
Hence, \( 20|10q_{50} \approx 0.49 \frac{s_m(70) - s_m(80)}{s_m(50)} + 0.51 \frac{s_f(70) - s_f(80)}{s_f(50)} = 0.49 \frac{\sqrt{2/9} - \sqrt{1/9}}{2/3} + \frac{0.51 \sqrt{0.3} - \sqrt{0.2}}{\sqrt{1/2}} \approx 0.17. \)

(b) In general, the probability that a newborn chosen at random will survive \( x \) years is \( w_1 \cdot x_{p_0}^{(1)} + w_2 \cdot x_{p_0}^{(2)}, \) and the Bayes formula gives
\[ w_i(x) = \frac{w_i \cdot x_{p_0}^{(i)}}{w_1 \cdot x_{p_0}^{(1)} + w_2 \cdot x_{p_0}^{(2)}}. \quad (7.1) \]
The conditional survival function

\[ t_p_x = P(X > x + t | X > x) = \frac{w_1 \cdot x^{t} P_0^{(1)} + w_2 \cdot x^{t} P_0^{(2)}}{w_1 \cdot x P_0^{(1)} + w_2 \cdot x P_0^{(2)}} \]

\[ = \frac{w_1 \cdot x^{t} P_0^{(1)} + w_2 \cdot x^{2} P_0^{(2)}}{w_1 \cdot x P_0^{(1)} + w_2 \cdot x P_0^{(2)}} = w_1(x) t_p_x^{(1)} + w_2(x) t_p_x^{(2)}. \]  

(7.2)

17. \( s(60) = \sqrt{0.4}. \) For \( \lambda_0 = 100, \) we have \( E\{L(60)\} = 100\sqrt{0.4} \approx 63.25, \) \( Var\{L(60)\} = 100\sqrt{0.4}(1 - \sqrt{0.4}) \approx 23.25, \) \( P(L(60) = k) = \binom{100}{k}0.4^{k/2}(1 - \sqrt{0.4})^{100-k}. \)

18. Indeed, \( n p_x = P(T(x) > m)P(T(x) > n | T(x) > m) = m p_x \cdot P(T(x+m) > n - m) = m p_x \cdot n - m p_{x+m} \) for \( m \leq n. \)

19. \( 20p_{50} = s p_{50} \cdot 15p_{55}. \) Hence, \( s p_{50} = \frac{0.8}{0.85} \approx 0.94. \)

20. We have \( t_p_x = u p_x \cdot t - u p_{x+u} \) for \( u \leq t. \) Then \( t_p_x \leq t - u p_{x+u} \) since \( u p_x \leq 1. \) In our particular case, \( x = 60, t = 10, u = 5. \)

21. (a) We may assume the force of mortality to be constant if we suppose that the flow of accidents satisfy the memoryless property. (b) In view of (1.2.2), we need to know \( \mu(x) \) on \([30,40].\) If \( \mu(x) \) is a constant \( \mu \) on this interval, then for \( t \leq 10, \) we have \( t p_30 = \exp \left\{ \int_{30}^{30+t} \mu ds \right\} = \exp\{-\mu t\}, \) i.e., an exponential function. (c) In Example 1.2-4 we consider the period \([30,40],\) while in Example 1.2-6 we deal with the whole lifetime.

22. The distribution of \( T(x) \) is a mixture of exponential distributions but the weights depend on \( x. \) A student may use the general formulas (M-7.1) and (M-7.2) or proceed as follows. Let \( w_i, \mu_i, i = 1,2, \) be the original weight and the force of mortality for the \( i \)-th group. Then

\[ P(T(x) > t) = P(X > x + t | X > x) \]

\[ = \frac{w_1 \exp\{-\mu_1(x+t)\} + w_2 \exp\{-\mu_2(x+t)\}}{w_1 \exp\{-\mu_1x\} + w_2 \exp\{-\mu_2x\}} \]

\[ = w_1(x) \exp\{-\mu_1 t\} + w_2(x) \exp\{-\mu_2 t\}, \]

where

\[ w_i(x) = \frac{w_i \exp\{-\mu_i x\}}{w_1 \exp\{-\mu_1 x\} + w_2 \exp\{-\mu_2 x\}}, \ i = 1,2. \]

For the data in the exercise,

\[ w_1(20) = \frac{0.3 \exp\{-20/50\}}{0.3 \exp\{-20/50\} + 0.7 \exp\{-20/80\}} \approx 0.27, \]

\[ w_2(20) = 1 - w_1(20) \approx 0.73. \]

The answer is natural. The forces of mortality in the two groups are different, so the share of people from, for instance, the first group who attain an age \( x \) depends on \( x. \)