

# Virtual $\chi_y$ -genera of Quot schemes on surfaces.

## 0. Introduction

$X$  smooth projective surface / C,  $\beta \in H_2(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ .

1)  $\overline{M}_{g,n}(X, \beta)$   $\rightsquigarrow$  GW invariant  $\Rightarrow$  k-theoretic GW

2)  $\overline{M}_{X,H}(\nu)$   $\rightsquigarrow$  VW invariant =  $e^{\text{vir}}(M)$   $\Rightarrow$  refined VW =  $\chi_y^{\text{vir}}(M)$   
 [GK, 2017]

3)  $\text{Quot}_X(\mathcal{E}, \nu)$   $\rightsquigarrow$  Quot scheme invariant =  $e^{\text{vir}}(\text{Quot})$   $\Rightarrow$  refined Quot scheme invariant  
 [OP, 2019]  $= \chi_y^{\text{vir}}(\text{Quot})$ .

Guiding Principle: Building blocks of sheaf theoretic moduli space on surface  $X$  are

Hilbert scheme of  $\begin{cases} \text{points} \\ \text{divisors} \end{cases} \times^{[n]} \text{Hilb}^n \rightarrow$  universal formula [EGL, 2001]  
 $\rightarrow$  SW invariant [DKO, 2007]

# 1. Virtual invariant of Quot scheme

$$\text{Quot}_X(C^N, \beta, n) := \left\{ [0 \rightarrow S \xrightarrow{\oplus N} Q \rightarrow Q \rightarrow 0] \mid \begin{array}{l} \text{rk}(Q) = 0, c_1(Q) = \beta, \\ X(X, Q) = n \end{array} \right\}$$

$$\left. \begin{array}{l} \text{Tor}_n = \text{Hom}(S, Q) \\ \text{Ob}_S = \text{Ext}^1(S, Q) \end{array} \right\} \text{difference} = v.\dim(\text{Quot}) = Nn + \beta^2.$$

$$\text{Ob}_S^{>1} = \text{Ext}^2(S, Q) \stackrel{\text{SD}}{=} \text{Hom}(Q, S \otimes k_X)^\vee = 0$$

$\Rightarrow \exists$  2-term PGT  $\Rightarrow [\text{Quot}]^{\text{vir}}, \Theta_{\text{Quot}}^{\text{vir}}, T_{\text{Quot}}^{\text{vir}}, \Sigma_{\text{Quot}}^{\text{vir}} \dots$

[FG, 2010]

$$\begin{aligned} e^{\text{vir}}(\text{Quot}) &:= \sum_{[\text{Quot}]^{\text{vir}}} \text{C}_d(T_{\text{Quot}}^{\text{vir}}) \in \mathbb{Z} \\ &\text{refinement} \downarrow \\ x_{-y}^{\text{vir}}(\text{Quot}) &:= \sum_{p=0}^{\infty} (-y)^p X(\text{Quot}, \Theta_{\text{Quot}}^{\text{vir}} \otimes \Lambda^p \Sigma_{\text{Quot}}^{\text{vir}}) \in \mathbb{Z}[y] \\ &y=1 \\ &q=0 \\ \text{Ell}_{-y, q}^{\text{vir}}(\text{Quot}) & \end{aligned}$$

2. Work of [OP, 2019] on  $e^{\text{vir}}(\text{Qut})$  & Conjecture.

Def.  $\sum_{X,N,\beta} e^{\text{vir}}(q) := \sum_{n \in \mathbb{Z}} q^n e^{\text{vir}}(\text{Qut}_X(\mathbb{C}^N, \beta, n)) \in \mathbb{Q}((q)).$

Main results of [OP] include:

$\exists$  explicit formula.

1)  $X$  any surface,  $\beta = 0 \Rightarrow \sum_{X,N,\beta=0} e^{\text{vir}}(q) = \bigcup_N k_x^2$

2)  $X$  minimal surface of general type with  $p_g > 0$ . (+ simply conn,  $\exists C \in |k_X|$ )  
 $\beta = \ell k_X$  ( $0 \leq \ell \leq N$ )

$$\Rightarrow \sum_{X,N,\beta=\ell k_X} e^{\text{vir}}(q) = q^{\ell(\ell-2)} \left( \frac{(-1)^{X(k_X)}}{\text{sw}(k_X)} \right)^{\ell} \cdot \sum_{\substack{J \subseteq [N] \\ |J| = N-\ell}} A_J^{k_X^2}$$

3)  $\tilde{X} \rightarrow X$  blow up of rational surface,  $\beta = E$ ,  $N = 1$

$$\Rightarrow \sum_{\tilde{X}, 1, E} e^{\text{vir}}(q) = q \cdot \left( \frac{(1-q)^2}{1-2q} \right)^{k_X^2} = \sum_{X, 1, E} e^{\text{vir}}(q).$$

Conj(OP)  $\times$  simply connected  $\Rightarrow \sum_{X,N,E}^{e^{\text{vir}}} (q) \in \mathbb{Q}(q) \subseteq \mathbb{Q}((q))$

subring  
i.e., coefficients are related.

Thm(L)  $\times$  any surface with  $(P_2 > 0)$   $\Rightarrow \left\{ \begin{array}{l} \sum_{X,N,E}^{e^{\text{vir}}} (q) \in \mathbb{Q}(q) \\ \sum_{X,N,E}^{x^{\text{vir}}} (q) \in \mathbb{Q}(y)(q) \end{array} \right. \quad y=1$ .

key ideas:

1) Refined Quot scheme invariant  $\longleftrightarrow$  SW invariant.

e.g.  $X$  min. general type,  $P_S > 0 \Rightarrow \text{SW}(k_X) = (-1)^{X(\Theta_X)}$

2) Multiplicative universal formula (under certain condition  $\star$ )

e.g.  $U_N, A_J$

3) Blow up formula (under  $\star$ ) Is blow up term rational?

$$\text{e.g. } \sum_{X,1,E}^{e^{\text{vir}}}(q) = \boxed{q} \cdot \sum_{X,1,E=0}^{(q)}(q)$$

### 3. Seiberg-Witten invariants of [DK0]

$X$  smooth proj surface,  $\beta \in H^1(X, \mathbb{Z})$ .

$$\text{Hilb}_X^\ell := \left\{ D \subseteq X \text{ effective divisor s.t. } c_1(\mathcal{O}_X(D)) = \beta \right\}.$$

$$T_\alpha = H^0(\mathcal{O}_D(D)), \quad \text{Ob}_s = H^1(\mathcal{O}_D(D))$$

$$\Rightarrow \exists \text{ 2-term POT of } v_{\mathcal{D}_\beta} = \frac{\beta(\beta-k)}{2}.$$

Consider  $\begin{cases} L_\beta := (\mathcal{O}_{\text{Hilb}_X^\ell}(D)) \Big|_{\text{Hilb}_X^\ell \neq \emptyset} & \nexists h = c_1(L_\beta) \\ AJ : \text{Hilb}_X^\ell \rightarrow \text{Pic}^\ell & \text{sending } D \mapsto \mathcal{O}_X(D) \end{cases}$

$$\text{Define } SW(\beta) := \sum_{k=0}^{\infty} AJ_* \left( h^k [\text{Hilb}_X^\ell]^{\text{vir}} \right) \in H_*(\text{Pic}^\ell) \simeq \bigwedge^X H^1(X, \mathbb{Z}).$$

$$\begin{aligned} \text{If } v_{\mathcal{D}_\beta} = 0, \text{ then } \bigwedge^0 H^1(X, \mathbb{Z}) &\simeq \mathbb{Z} \\ SW(\beta) &= \deg [\text{Hilb}_X^\ell]^{\text{vir}} \end{aligned}$$

Principle

VW theory  
(Mochizuki)

Quot scheme theory  
(Virtual localization  
+ some calculation)

4. Correct assumption  $\nRightarrow$  Main result.

Def (L) We say  $\beta$  is SW length N if

$$[\forall \beta = \beta_1 + \dots + \beta_N \text{ with } \text{SW}(\beta_i) \neq 0 \quad \forall i] \Rightarrow [\nu d_{\beta_i=0} = 0 \quad \forall i]$$

$\exists$  many examples (all cases in [OP] + new)

e.g.  $\boxed{A} \times \text{any surface}, \beta = 0 \Rightarrow \beta : \text{SW length N}$ .

$\because$  Only effective decomposition:  $\beta = 0 + \dots + 0 \nRightarrow \nu d_{\beta_i=0} = 0$ .

New  $\boxed{B}$   $\pi: X \rightarrow C$  relatively minimal elliptic surface  $\Rightarrow \beta : \text{SW length N}$ .  
 $\beta$ : supported on the fibers (i.e.,  $\pi_* F = 0$ )

- o  $k_X \sim \text{rat'l mult of } F$
- o Zariski Lemma

$\boxed{C} \times \text{any surface with } p_g > 0 \Rightarrow \forall \beta : \text{SW length N}.$   
[DKo]

$\boxed{D} \quad \beta : \text{SW length N} \xrightarrow{\hookleftarrow} \tilde{\beta} + lE : \text{SW length N}.$

Main Thm (L) Assume  $\beta$ : SW length  $N$ .

$\exists$  universal series  $U_N, V_{N,i}, W_{N,ij} \in \mathbb{Q}(y, e^{\mu_1}, \dots, e^{\mu_N})((z))$  s.t.

The generating series of equivariant refined Quot scheme invariants is

$$\sum_{x,N,\beta}^{vir} (q | w_1, \dots, w_N) = q^{\frac{-\beta \cdot k_x}{2}} \sum_{\substack{\beta = \beta_1 + \dots + \beta_N \\ s.t. \quad v \delta \beta_i = 0}} SW(\beta_1) \dots SW(\beta_N) U_N \cdot \prod_{1 \leq i \leq N} V_{N,i} \cdot \prod_{1 \leq i < j \leq N} W_{N,ij}$$

↓  
 Hilb<sub>x</sub>'s  
 ↓  
 X<sub>[m]</sub>'s

(Rmk.  $\exists$  similar formula for monopole contribution to refined VW invariants: )

[Laaarakker, 2018]  $X$  surface with  $b_2 > 0, b_1 = 0$ .

$$\sum_{x,N,\beta}^{vw} (q) = A^{\chi(X)} \cdot \beta_x^{k_x^2} \sum_{\substack{\beta = \beta_1 + 2\beta_2 + \dots + (N-1)\beta_{N-1} \\ (\text{mod } N \cdot H^2(X; \mathbb{Z}))}} SW(\beta_1) \dots SW(\beta_{N-1}) \prod_{i < j} C_{i,j}^{(\beta_i, \beta_j)}$$

↓  
 known!

(Sketch of proof of main thm)  $\beta$ : Sh length  $N$

In general, geometry of higher  $N$  Quot scheme is poorly understood.

Let  $C^x \cong C^N$  with distinct weights  $w_1, \dots, w_N \Rightarrow C^x \cong \text{Quot}_x(C^N, \beta, n)$   
 (Similar as  $C^x \cong \mathbb{P}^{N-1}$ )

$$\left( \text{Quot}_x(C^N, \beta, n) \right)^{C^x} = \bigsqcup_{\substack{\beta = \beta_1 + \dots + \beta_N \\ n = n_1 + \dots + n_N}} \text{Quot}_x(C^1, \beta_1, n_1) \times \dots \times \text{Quot}_x(C^1, \beta_N, n_N)$$

Induced obstruction theory of each fixed locus splits into factors.

Question: What is  $[\text{Quot}_x(C^1, \beta, n)]^{\text{vir}}$ ?

[GSY, 2017] virtual class of nested Hilbert scheme.

$\exists$  identification  $\text{Quot}_x(C^1, \beta, n) = X^{[m]} \times \text{Hilb}^\beta = X_\beta^{[m, 0]}$  where  $m = n + \frac{\beta(\beta+k)}{2}$ .

[F, 1968]

Prop(L)  $[\text{Quot}_x(C^1, \beta, n)]^{\text{vir}} \stackrel{\textcircled{1}}{=} [X_\beta^{[m, 0]}]^{\text{vir}} \stackrel{\textcircled{2}}{=} e\left(\underbrace{\mathcal{O}_{\mathbb{P}^m}^{[m, 0]}}_{\text{rank } m}\right) \cap \left([X^{[m]}] \times [\text{Hilb}_x^\beta]^{\text{vir}}\right)$

pf) ①: compare  $T^{\text{vir}}$ . ② Special case of [GT, 2019]

By virtual localization of [GP, 1999],

$$\begin{aligned} x_{x_2}^{\text{vir}}(Q_{\text{unt}_X}(\beta, \beta, n)) &= \sum_{\beta = \beta_1 + \dots + \beta_N} \sum_{n = n_1 + \dots + n_N} \left\{ \begin{array}{c} (\text{sth}) \\ X^{[m_1]} \times \dots \times X^{[m_N]} \times [\text{Hilb}^{\beta_1}]^{\text{vir}} \times \dots \times [\text{Hilb}^{\beta_N}]^{\text{vir}} \end{array} \right. \\ &= \sum_{\substack{\beta = \beta_1 + \dots + \beta_N \\ \text{s.t. } \nu d\beta_i = 0}} SW(\beta_1) \dots SW(\beta_N) \sum_{n = n_1 + \dots + n_N} \left\{ \begin{array}{c} (\text{sth}) \\ X^{[m_1]} \times \dots \times X^{[m_N]} \end{array} \right. \\ [\text{Hilb}^{\beta_i}]^{\text{vir}} &= SW(\beta_i)[pt] \end{aligned}$$

$$\Rightarrow \sum_{X, N, \beta} x_{x_2}^{\text{vir}}(q|w, \dots, w_N) = q^{-\beta \cdot k_X} \sum_{\substack{\beta = \beta_1 + \dots + \beta_N \\ \nu d\beta_i = 0}} SW(\beta_1) \dots SW(\beta_N) \sum_{m_1, \dots, m_N \geq 0} \left\{ \begin{array}{c} (\text{sth}) \\ X^{[m_1]} \times \dots \times X^{[m_N]} \end{array} \right.$$

[ EGL 2001 ]

$\exists$  multiplicative universal formula.

5. Applications of the Main Thm. (to each case  $\boxed{A} \sim \boxed{B}$ )

Thm A.  $\times$  any surface,  $\kappa = 0$ .

$$\sum_{x, N, 0}^{\infty} (q) = \overline{U}_N^{k_x^2} \text{ where}$$

$$\overline{U}_N := U_N \Big|_{w_1 = \dots = w_N = 0} = \frac{(1-q)(1-y^N q)}{(1-(1+y)^N q)^N} \prod_{i \neq j} (1 - (1+y)t_i + y t_i t_j) \in \mathbb{Q}(y)(q).$$

Here  $t_1, \dots, t_N$  are roots of  $q = t^N$ .  $\therefore \uparrow P_N$ : polynomial in  $y, q$ .

$$\text{e.g. } P_1 = 1$$

$$\cdot P_2 = 1 - (1+4y+y^2)q + y^2 q^2$$

$$\cdot P_3 = 1 - (2+9y+9y^2+2y^3)q + (1+9y+36y^2+58y^3+36y^4+9y^5+y^6)q^2 \\ - (2+9y+9y^2+2y^3)y^3 q^3 + y^6 q^4.$$

$$\star P_N(q, y) = (y^N q^2)^{N-1} \cdot P_N(q^{-1}, y^{-1}).$$

By taking  $y=1$ , it recovers [op].

Thm B.  $\pi: X \rightarrow C$  relatively minimal elliptic surface,  $\beta$ : supported on the fibers.

$$\sum_{X, N, \beta}^{X^{\text{vir}}} (\beta) = \sum_{\beta = \beta_1 + \dots + \beta_N} SW(\beta_1) \cdots SW(\beta_N).$$

s.t.  $\beta_i$ : rat'l multiple of  $F$

By [DKO], SW invariants above are computed by

$$SW(\beta_i) = \sum_{\beta_i = dF + \sum_j a_j F_j} (-1)^d \binom{2g-2 + \chi(\mathcal{O}_X)}{d}$$

s.t.  $d \geq 0, 0 \leq a_j < m_j$

where  $F_j$ 's are multiple fibers of multiplicity  $m_j$ .

Thm C.  $X$  minimal surface of general type with  $p_g > 0$ .

For any  $\rho = lk_x$  ( $0 \leq l \leq N$ ), we have

$$:= G_{N,l,g} \in Q(y)(q).$$

$$\sum_{x,N,lk_x}^{vir}(q) = q^{l(l-g)} \cdot (sw(k_x))^l \cdot \underbrace{\sum_{\substack{J \subseteq [N] \\ |J| = N-l}} A_J^{-g}}$$

where

$$A_J(\{t_j\}_{j \in J}) = \frac{(-1)^{s(N+1)}}{q^s \cdot N^s} \times \prod_{j \in J} \frac{t_j (1-(1+y)t_j)^N}{(1-t_j)(1-yt_j)} \times \prod_{j_1 \neq j_2} \frac{t_{j_2} - t_{j_1}}{1-(1+y)t_{j_1} + yt_{j_1}t_{j_2}}.$$

(Set  $s = N-l$ )

Note that  $sw(k_x) = (-1)^{X(k_x)}$  by [CK, 2013].

By taking  $y=1$ , we recover [op] & lift technical assumptions  
 (simply conn,  $\exists C \overset{\text{smooth}}{\in} \mathbb{C}[k_x]$ )

Thm D.  $X$  any surface.  $\mathbb{P}$ : SW length  $N$ .

(Blow up formula)

$\pi: \tilde{X} \rightarrow X$  blow up at a point with exceptional divisor  $E$ .

For any  $0 \leq l \leq N$ , we have

$$\sum_{\tilde{X}, N, \mathbb{P}+lE}^{X^{\text{vir}}} (q) = \left( q^l \cdot B|_{N,l} \right) \cdot \sum_{X, N, \mathbb{P}}^{X^{\text{vir}}} (q) \quad \text{where}$$

$$B|_{N,l} = \sum_{\substack{J \subseteq [N] \\ |J|=N-l}} A_J \in Q(y)(q).$$

$$\text{e.g. } B|_{N,N} = 1, \quad B|_{N,N-1} = \frac{1}{1-y} \cdot \frac{(1-y^N) - (1-y^N)q}{(1-q)(1-y^Nq)}.$$

For any given  $N, l, q$ , it is easy to compute  $G_{N,l,q}$ .  $\nearrow$   
 $\nearrow q=0$ .

$$[\text{Thm } A \sim D] \Rightarrow \left[ \sum_{X,N,R}^{x \in \mathbb{Z}} (q) \in \mathbb{Q}(y)(q) \quad \text{if } X \text{ is surface with } p_2 > 0 \right]$$

(proof) Let  $X$  be surface with  $p_2 > 0$ . Then blow up formula apply b/c  $p_2 > 0$ .

Since  $B|_{N,R} \in \mathbb{Q}(y)(q)$ , we may assume  $X$ : minimal surface.

- $\text{kod} = 0$  :  $k3$  or abelian  $\Rightarrow \rho = 0$  is only SW class  $\Rightarrow$  Thm [A]
- $\text{kod} = 1$  : minimal elliptic surface  $\Rightarrow \rho = \text{rat'l mult. of } F$   
with  $p_2 > 0$  are only SW class  $\Rightarrow$  Thm [B]. -
- $\text{kod} = 2$  : minimal surface of general type with  $p_2 > 0$   $\Rightarrow \rho = k_X$  is only SW class  $\Rightarrow$  Thm [C]

Question : What happen if  $p_2 = 0$  ?

[JOP, 2020]  $X$  simply connected surface. Then  $\sum_{X,N=1}^{e \in \mathbb{Z}} (q) \in \mathbb{Q}(q)$ .

In fact, they study more general descendent series.

## 6. Reduced invariant of k3 surfaces.

X k3 surface. Usual Quot scheme invariants are trivial b/c

$\exists$  surjective restriction

$$O_{bs} = \text{Ext}^1(S, Q) \xrightarrow{\delta} \underbrace{\text{Ext}^2(Q, Q)}_{\cong 0} \xrightarrow{\text{tr}} H^2(\theta_x) \cong \mathbb{C}.$$

$$\therefore \text{Ext}^2(\theta_x^{\oplus N}, Q) = 0 \quad \therefore \text{Hom}(Q, Q \otimes k_x) \hookrightarrow H^0(k_x)$$

$\therefore$  We consider reduced obstruction theory with

$$\underline{\text{red. } O_{bs}} := \ker(O_{bs} \rightarrow \mathbb{C})$$

(Thm(L)) X k3 surface,  $\beta$ : primitive, big, nef.  $\cancel{N=1}$ .

$$\Rightarrow X_y^{\text{red}}(\text{Quot}_x(\mathbb{C}^l, \beta, n)) = X_y(\mathcal{E}^{[m]})$$

where  $\mathcal{E} \rightarrow \mathbb{P}$  is universal curve in class  $\beta$   $\nRightarrow m = n + (g-1)$ .

Rmk. 1) Above thm was proven in [op] on the level of Euler characteristic.

2)  $\mathcal{E}^{[m]}$  is proven to be smooth  $\nRightarrow h^{1,2}(\mathcal{E}^{[m]})$  are computed by [KY, 2000].

Together with Thm + KY formula

Cor. With previous assumption

$$\sum_{x, \beta}^{\text{red}} (t) = \sum t^n \bar{x}_x^{\text{red}} (\text{Quot}_x(C, \beta, n)) \text{ is given by}$$

coefficient of  $q^2$  in

$$\left( \frac{1}{t + \frac{1}{t} - \sqrt{y} - \frac{1}{\sqrt{y}}} \cdot \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^8 (1-q^n y) (1-q^n/y) (1-q^n \sqrt{y}/t) (1-q^n t/\sqrt{y}) (1-q^n t/\sqrt{y}) (1-q^n + \sqrt{y}) (1-q^n + \sqrt{y})} \right)$$

In particular,  $\sum (t)$  is rat'l in  $t$  variable. Kawai-Yoshioka

## 7. Further questions

- 1) Quot scheme invariants  
    ? ↗ VW invariant  
    ? ↗ Stable pair invariant
- 2)  $X$  with  $p_g = 0 \Rightarrow$  rationality question ?.
- 3) Reduced invariant of K3 surface with higher  $N / \beta$  non-primitive.  
    Define non-trivial reduced invariant of abelian surface.
- 4) Relative theory  $\text{Quot}_{X/D}^{(v)} \rightarrow$  degeneration formula.

⋮