Motivic degree zero DT-invariants

(Behrend, Bryan, Szendrői)

Introduction: Recall from Bochao’s talk that if $X$ is a scheme (finite type over $\mathbb{C}$), the virtual Euler characteristic of $X$ is

$$
\chi_{\text{vir}}(X) := \sum_{k \in \mathbb{Z}} k \cdot \chi(\nu_X^{-1}(k))
$$

Behrend function of $X$.

This is sensitive to singularities.
scheme structure (important for enumerative geometry in \( \dim \geq 3 \)). For \( X \) a Calabi-Yau threefold, we saw that \( X_{\text{vir}}(\text{Hilb}^n X) \) is the degree zero DT invariant of \( X \).

Can this come from something "greater" (i.e., "motivic")?

If \( X \) is a \( \mathbb{C} \)-variety, a "virtual motive" of \( X \) is a class \( [X]_{\text{vir}} \) in "ring of motivic weights"
such that

\[ \chi ([X])_{\text{vir}} = \chi_{\text{vir}} (X). \]

(need to define for motivic weight)

We will construct a virtual motive for \( \text{Hilb}^n X \) for \( X \) a threefold over \( \mathbb{C} \) (not necessarily Calabi-Yau),

and because of motivation above call \( [\text{Hilb}^n X]_{\text{vir}} \) a motivic degree zero Donaldson-Thomas invariant.

This will use description of
\text{Hilb}^n \mathbb{C}^3$ as a degeneracy locus, and will give motivic Gött sche-like formulas for partition functions of threefolds.
What is the ring of motivic weights?

Let us first define the Grothendieck ring of varieties:

$$K_0(\text{Var}_\mathbb{C})$$ is the free abelian group on isomorphism classes of varieties over $\mathbb{C}$ with product given by taking Cartesian products modulo scissor relations:

- if $Y \subseteq X$ is a closed subvariety,

$$[X] = [Y] + [X \setminus Y]$$

in $K_0(\text{Var}_\mathbb{C})$. 
Some standard facts:

- if \( F \to E \to B \) is a Zariski-trivial fibration then
  \[
  [E] = [F] \cdot [B]
  \]

(since \( B \) is a variety, can use quasi-compactness to construct finite cover by trivializing open sets, then induct on \( \# \) of trivializing opens and use scissor relations cleverly)

- if \( X \) is stratified by disjoint locally closed subsets as \( X = \bigsqcup_{i=1}^n X_i \) then
  \[
  [X] = \sum_{i=1}^n [X_i]
  \]
(follows from scissor relation on equivariant argument)

- if \( f: X \rightarrow Y \) is a bijective morphism on \( \mathbb{C} \)-points then \([X] = [Y]\) (proof requires stratification property).

This last point can be used to show that \( K_0(\text{Var}_\mathbb{C}) \cong K_0(\text{Sch}_\mathbb{C}) = K_0(\text{Sp}_\mathbb{C}) \) where \( \text{Sch}_\mathbb{C}, \text{Sp}_\mathbb{C} \) are the categories of schemes and algebraic spaces respectively.

To get ring of motivic weights, just
define
\[ M_C = K_0(\text{Var}_C)[L^{-\frac{1}{2}}] \]
formally, where \( L = [A^1] \)
("Lefschetz motive").

Computations in \( M_C \) are very concrete using fibration property.

**Examples:**

For notation, set \([n]_L = (L^n - 1) \cdot (L^{n-1} - 1) \cdot \ldots \cdot (L - 1)\).

\[
\begin{bmatrix} n \\ k \end{bmatrix}_L = \frac{[n]_L}{[k]_L}.
\]
\[
\binom{\mathcal{G}_{\text{Ln}}(\mathbb{C})}{1} = \frac{\binom{n}{2}}{1 - \frac{1}{n}} \cdot \frac{\binom{n}{3}}{1 - \frac{2}{n}} \cdots \frac{\binom{n}{n-1}}{1 - \frac{n-1}{n}},
\]

since one can build an element of \( \mathcal{G}_{\text{Ln}} \) by first choosing a nonzero column \((\frac{1}{n}^n - 1)\), then choosing an independent second column \((\frac{1}{n}^n - \frac{2}{n})\), etc.

2) \( [\text{Gr}(k,n)] = \binom{n}{k} \).
Using identity \( \binom{n}{2} - \binom{k}{2} - \binom{n-k}{2} \)

\[ = k(n-k), \text{ equivalent to showing} \]

\[ [Gr(k,n)] = \frac{[GL_n]}{\prod_{k=1}^{k(n-k)} [GL_k][GL_{n-k}]} . \]

\( Gr(k,n) \) has \( GL_n \)-action, and

if \( \Lambda \in Gr(k,n) \) is \( \xi x_{k+1} = \ldots = x_n = 0 \),

then stabilizer of \( \Lambda \) explicitly is

matrices of form
so follows from fibration property.

The following is known:

(Morrison - Bryan, Feit - Fine) Let \( C_n \) denote the reduced variety of pairs of commuting matrices in \( \text{End}(C^n) \times 2 \). Then

\[
\sum_{n > 0} \frac{[C_n]}{[\text{GL}_n]} t^n =
\]
\[
\prod_{m=1}^{\infty} \prod_{j=0}^{\infty} \frac{1}{(1 - tl_{1}^{-j} + t^{m})^{-1}} \quad \text{in} \quad \mathcal{M}_{\mathbb{C}} \left[ (1 - tl_{1}^{-n})^{-1} : n \geq 1 \right].
\]

**Specializations of motivic classes.**

Can recover lots of invariants of varieties via homomorphisms from \( \mathcal{M}_{\mathbb{C}} \) to other rings.

Deligne's mixed Hodge structure gives

**E-polynomial homomorphism**

\[
E : \mathcal{M}_{\mathbb{C}} \rightarrow \mathbb{Z}[x, y, (xy)^{-\frac{1}{2}}]
\]

\[p, q, r, \ldots \in \mathbb{N}, \ldots, 109, 1, \ldots \]
\[ X \text{ variety} \mapsto \sum_{i} x_i^p q^9 \sum (-1)^i \dim H^i \left( H(X, \mathbb{Q}) \right) \]

\[ \left( xy \right)^{-\frac{1}{2}} \mapsto \left( xy \right)^{-\frac{1}{2}} \]

Specialize \( x = y = -q^{1/2} \), \( (xy)^{1/2} = q^{1/2} \), get weight polynomial

\[ W_{\text{mc}} \rightarrow \mathbb{Z} \left[q^{\pm \frac{1}{2}}\right], \]

which just gives Poincaré polynomial in variable \( q^{\frac{1}{2}} \) for \( X \) smooth projective.

Finally, setting \( q^{\frac{1}{2}} = -1 \) gives
Euler characteristic

\[ \chi : \mathcal{M}^0 \to \mathbb{Z}. \]

Power structure for motivic weights:

A power structure on a ring \( R \) is a map

\[ R \times (1 + t R[[t]]) \to 1 + t R[[t]] \]

\[ (m, A(t)) \mapsto A(t)^m \]

satisfying some usual exponential properties

\[ (A(t))^0 = 1, \quad A(t)^1 = A(t), \quad A(t)^{m+n} = A(t)^m \cdot A(t)^n, \quad \text{etc...} \]
For \( R = K_0(\text{Var}_c) \), Gusein-Zade et al constructed a power structure uniquely characterized by

\[
(1 - t)^{-[X]} = \sum_{n=0}^{\infty} [\text{Sym}^n X \cdot t^n]
\]

for \( X \) a variety. Briefly state construction:

If

\[
\mathcal{A}(t) = 1 + \sum_{i \geq 1} A_i t^i, \quad X \text{ variety},
\]

then

\[
\mathcal{A}(t)^{[X]} = 1 + \sum_{\lambda} \prod_{i} \left[ (\prod_{i} X_{\lambda_i}^{a_i} \cdot \Delta) \cdot (\prod_{i} A_{\lambda_i}^{a_i}) \right] t^{\lambda}
\]

d partition

where \( \lambda \) runs over all partitions of all lengths,
\[ \alpha = 1^{a_1} 2^{a_2} 3^{a_3} \ldots \], \Delta is the "big diagonal" in \( \mathcal{X}^{\alpha} \) (all pts in product where some pair of factors is equal, so \( \mathcal{X}^{\alpha} \setminus \Delta \) is tuples of distinct points), and \( \pi_1 G_\alpha \) maps the input to the orbit space under the permutation action by \( G_\alpha := \prod_{i} S_{\alpha_i} \).

Can extend in straightforward way to power structure on \( \mathcal{M}_0 \).

Also have \( \text{Exp} \) map

\[
\text{Exp}: \ t \mathcal{M}_0[[[t]]] \to 1 + t \mathcal{M}_0[[[t]]]
\]

\[
\sum_{n=1}^{\infty} [A_n] t^n \mapsto \prod_{n=1}^{\infty} (1 - t^n)^{-[A_n]}
\]
Definitions seem weird, but make sense for partition functions:

For $X$ a variety, define

$$H_X(t) = 1 + \sum_{n=1}^{\infty} [\text{Hilb}^n X] t^n,$$

and

$$H_{A_d}^0(t) = 1 + \sum_{n=1}^{\infty} [\text{Hilb}_0^n A^d] t^n.$$

Then if $d = \dim X$, can show

$$H_X(t) = H_{A_d}^0[\mathcal{X}]$$

i.e., stratum $\text{Hilb}^2 X$ of subschemes of $X$. 
w/ $\alpha_i$ pts of multiplicity $i$ is motivically what it should be: the product over $i$ of $\alpha_i$ degree $i$ fuzzy points $([ \text{Hilb}_0 A^d])$ supported at $\alpha_i$ distinct reduced points in $X (X^{\alpha_i} \Delta)$.

**Picture:** $\alpha = k^d \Delta$, $\text{Hilb}_{k^d} X = \ldots$
degree $k$ point

$X^\alpha_k \setminus \Delta$

$(\text{mod } \Sigma_k)$
$\left(\mathbb{C}^3\right)^{[n]}$ as a degeneracy locus

**Theorem:** $\text{Hilb}^n \mathbb{C}^3$ can be described as the degeneracy locus $\mathbb{E}df_n = 0^3$ for some regular map $f_n: M_n \to \mathbb{C}$ where $M_n$ is a smooth ambient variety.

**Proof:** We claim that $\text{Hilb}^n \mathbb{C}^3$ has the following explicit description:

$$\left\{ (A, B, C, v) \mid A, B, C \in \text{End}(\mathbb{C}^n) \right\}$$

subject to $v \in \mathbb{C}^n$, $[A, B] = [A, C] = [B, C] = 0$,
v should span \( C^n \) under \( \rho \)

polynomial action of \( A, B, C \)

\[
/ \text{GL}_n(C)
\]

\[
g \cdot (A, B, C, \nu) = (gAg^{-1}, gBg^{-1}, gCg^{-1}, g\nu)
\]

What's the identification?

Points of \( \text{Hilb}^n C^3 \) are just ideals \( I \subset C[x, y, z] \) s.t.

\[
\dim_c (C[x, y, z] / I) = n.
\]
Given such an $I$, first identify $C[x, y, z]/I \cong C^n$ (then mod out by $G_{\text{lin}}(C)$ action to make canonical) and construct $(A, B, C, \nu)$ as follows:

\[ A, B, C := \text{respectively } x^*, y^*, z^*, \]

\[ \nu := 1 \in C[x, y, z]/I \]

Conversely, given $(A, B, C, \nu)$, recover $I \triangleq C[x, y, z]$ as:
\[ \ker \left( C[x,y,z] \ni f \mapsto f(A,B,C)v \right) \]

Can convince self that these two maps are well-defined and mutually inverse, so have set-theoretic equality (on \( C \)-points). How to show scheme structure?

Just check functor of points explicitly (i.e. functor \( \text{Hilb}^n C^3 \) is represented by the set, we will show later that set has scheme structure).
Call set $H$, then have flat family 
$H \rightarrow H$ of $0$-dimensional subschemes of $C^3$ given by associating 
to $(A, B, C, v) \in H$

$$\ker \{ k[x, y, z] \rightarrow f(A, B, C, v). \}$$

To check representability, suppose

$\tilde{\pi} : \tilde{Z} \rightarrow U$ is flat family of $0$-dimensional length $n$ subschemes of $C^3$. If $U = U \cup U_x$ is open

cover trivializing $\tilde{\pi}_* \mathcal{O}_{\tilde{Z}}$, can use
multiplication by $x, y, z$ to define maps $U_x \rightarrow \widetilde{H}$ (H w/o $/GL_n(\mathbb{C})$) and these glue when modding out by $GL_n(\mathbb{C})$ since the trivializations of $\pi_* O_{\mathbb{Z}}$ on individual $U_x$ are related by $GL_n(\mathbb{C})$, so passes to global unique (need to check) map $\phi: U \rightarrow \widetilde{H}$ s.t. $\phi^* \mathcal{H} = \mathcal{Z}$, so $\mathcal{H}$ represents the functor and hence is...
Hilb^n C^3 if H is a scheme (which we will show). Nakajima has details of representability proof for C^2.

Embedding of Hilb^n C^3 as critical locus:

Want to construct smooth variety M_n and regular map f_n : M_n \rightarrow C s.t. H = \{ \text{det}_{f_n} = 0 \}. This will give scheme structure of H and hence also prove that
$H = \text{Hilb}^n \mathbb{C}^3$ as schemes.

Set $U_n \subseteq \text{End}(\mathbb{C}^n)^{\otimes 3} \otimes \mathbb{C}$ to be the subset of $(A, B, C, v)$ such that $v$ spans $\mathbb{C}^n$ under iterated action by $A, B, C$.

This is an open stability condition. can be rewritten as $U'(A, B, C, v)$ s.t. there does not exist a proper subspace $S \subseteq \mathbb{C}^n$ s.t. $TS \subset S$ and
\( \mathcal{V} \in S \) for any of \( T = A, B, C' \).

Thus \( U_n \) is an open subset of an affine space. If we have \( \text{GL}_n(C) \oplus \text{End}(C^n) \oplus C^n \) in usual way then \( U_n \) is set of GIT stable points for this action linearized by character \( X: \text{GL}_n(C) \to C^* \)

\[ g \mapsto \det(g) \]

(can check explicitly via \( 1 - PS \)).
Stability condition implies action of $GL_n(C)$ on $U_n$ is free, so GIT quotient

$$M_n := U_n / GL_n(C)$$

is smooth quasi-projective variety.

$H$ is cut out by the additional equations $[A, B] = [A, C] = [B, C] = 0$ (equations are conjugation - equivariant).

Define $\tilde{f}_n : U_n \to C$

$$(A, B, C, \nu) \mapsto \text{Tr} ([A, B] C)$$
This map is conjugation-invariant, so descends to regular map

\[ f_n : M_n \to C. \]

Working explicitly in coordinates shows that \( df_n = 0 \) if and only if

\[ [A, B] = [A, C] = [B, C] = 0. \]

Explicitly,

\[ \text{Tr}( [A, B] C) = \sum \sum (A_{ij} B_{jk} - B_{ij} A_{jk}) C_{ki}. \]

So

\[ \partial_n \text{Tr}( [A, B] C) = \sum A_{ij} B_{jk} - B_{ij} A_{jk}, \]
\[ \partial_{k_i} \partial_{j_i} \partial_{j_k} = \partial_{j_i} \partial_{j_k} \partial_{k_i} \]

the \((i, j, k)\)th entry in \([A, B]\), so that \[df_n = 0 \implies [A, B] = 0.\]

Index-juggling shows the same for \(\partial_{A_{ij}}, \partial_{B_{ij}}\), so that \(df_n = 0\) implies commutation and conversely.

This concludes the proof.

(What fails for \(C^1\) and higher?)
Virtual motive of $\text{Hilb}^n \mathbb{C}^3$

Given a regular map $f : M \to \mathbb{C}$ with $M$ quasi-projective satisfying certain torus-equivariance properties, one can use motivic integration (whatever that means...) to define an absolute motivic vanishing cycle of $f$ and show it can be computed as

$$[\nu_f] = [f^{-1}(1)] - [f^{-1}(0)]_{\text{central fiber}}.$$  

Suppose $Z = \sum df = \mathcal{O}_X^3 \subset X$, and set

$$[Z]_{\text{vir}} = \prod_{-\dim X}^{\dim X} [\nu_f].$$

By
computing fibrewise Euler characteristics in terms of Milnor fibres, can show that

$$X([Z]_{\text{vir}}) = X_{\text{vir}}(Z).$$

(All of this is not easy and relies on lots of other work on motivic integration, arc spaces).

**Upshot**: For $Z = \exists! \text{df} = 0$ a moduli space of sheaves on a Calabi-Yau threefold (call $f$ superpotential or global Chern-Simons functional), the associated DT invariant is $X_{\text{vir}}(Z)$, so $[Z]_{\text{vir}}$ is motivic DT invariant.
For \( Z = \text{Hilb}^n \mathbb{C}^3 \) we've already explicitly computed super-potential, so use this to explicitly compute \([\text{Hilb}^n \mathbb{C}^3]_{\text{vir}}\).

We already know \([\text{Hilb}^n \mathbb{C}^3]_{\text{vir}}\) exists, and some arguments show that classes \([\text{Hilb}^n \mathbb{C}^3]_{\text{vir}}\) for every partition \( \lambda \) of \( n \) such that

\[
[\text{Hilb}^n \mathbb{C}^3]_{\text{vir}} = \sum_{\lambda+n} [\text{Hilb}_{\lambda \times n} \mathbb{C}^3]_{\text{vir}}.
\]

**Theorem:** Let \( Z_{\mathbb{C}^3}(t) \) be the (motivic) partition function \( Z_{\mathbb{C}^3}(t) = \sum_t [\text{Hilb}^n \mathbb{C}^3]_{\text{vir}} t^n \).

Then \( Z_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} (1 - (2 + k - m \cdot 12) t^m)^{-1} \).
Proof sketch: Set

\[ Y_n = \{(A, B, C, v) \mid \text{Tr}(\text{[A,B]} C) = 0 \} \]

\[ Z_n = \{(A, B, C, v) \mid \text{Tr}(\text{[A,B]} C) = 1 \} \]

\[ C \times \text{End}(C^n)^{\times 3} \times C^n. \]

We have an isomorphism

\[ (\text{End}(C^n)^{\times 3} \times C^n) \setminus Y_n \cong C^* \times Z_n \]

via \[ (A, B, C, v) \rightarrow (\text{Tr}(\text{[A,B]} C), \frac{A}{\text{Tr}(\text{[A,B]} C)}, B, C). \]

By scissor relations, this implies

\[ (1 - 1^3)([Y_n] - [Z_n]) = 1 \frac{3n^2 + n}{n} - 1[Y_n]. \]
Set $Y_n = Y_n' \cup Y_n''$ where $Y_n'$ is locus where $[A, B] = 0$ and $Y_n''$ is complement. Set $C_n$ to be variety of pairs of commuting matrices in $\text{End}(C^n)$. Get projections $Y_n \to C_n$

\[
Y_n'' \to \exists C^{2n^2} \setminus C_n^3
\]

(affine)

with simple $Y$ fibers, analyze to get relation $\omega_n = \Pi^{n(n+1)} [C_n]$, where

$\omega_n = [Y_n] - [Z_n]$. To add stability condition, define the $(A, B, C)$-span of $v \in C^n$
for $A, B, C \in \text{End}(C^n)$ to be the orbit of $v$ under $\mathbb{C}[x,y,z]$ acting via $A, B, C$. This is a linear subspace, so set

$$X_n^k = \{ (A, B, C, \nu^3) \mid \dim_{\mathbb{C}} (A, B, C) \cdot \text{span of } \nu = k^3 \},$$

$$Y_n^k = Y_n \cap X_n^k,$$

$$Z_n^k = Z_n \cap X_n^k.$$  

Ultimate goal is $[Y_n^k] - [Z_n^k]$.  

Have fibration $Y_n^k \to \text{Gr}(k, n)$ sending $(A, B, C, \nu)$ to $(A, B, C) \cdot \text{span of } \nu$.  

Linear algebra and fibration/scissor
\[\omega^k_n := \left[ Y^k_n \right] - \left[ Z^k_n \right] = \prod_{l=0}^{(n-k)(n+2k)} \left[ \begin{array}{c} \eta_l \\ \ell_l \end{array} \right] \prod_{l=0}^{(n-k)(n+2k)} \left[ C_{n-k} \right] \omega^k_k.\]

Since \( \omega^k_n = \omega_n - \sum_{k=0}^{n-1} \omega^k_n \) by stratification, use formulas to get

\[\omega^k_n = \prod_{l=0}^{n(n+1)} \left[ C_l \right] - \sum_{l=0}^{n-1} \left[ \begin{array}{c} \eta_l \\ \ell_l \end{array} \right] \prod_{l=0}^{(u-k)(n+2k)} \left[ C_{n-k} \right] \omega^k_k.\]

Since \( \left[Y_{f_n}\right] = \left[f_n^{-1}(1)\right] - \left[f_n^{-1}(0)\right] \)

\[= \left[ Z^n_n \right] - \left[ Y^n_n \right] = \left[ G_{L_n} \right] - \left[ G_{L_n} \right].\]
\[
\omega_n \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(n\right)}
\]\n
We've basically computed \([\text{Hilb}^n C^3]_{\text{vir}}\) in terms of \([C_n]\), which we know by Feit-Fine. Some algebra of above relations and putting into generating series gives

\[
C(t, 1^{1/2}) = Z C^3(t) C(t + 1^{-1/2})
\]

where \(C(t) = \sum_{n \neq 0} \frac{[C_n]}{[GL_n]} t^n\).
Solving for $Z_{\mathbb{C}^3}(t)$ and using Feit-Fine formula gives

$$Z_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} (1 - \frac{t^{2+k-m/2}}{m})^{-1},$$

as desired.

**Fun fact:** setting $\ell^{\frac{1}{2}} = -1$ in $Z_{\mathbb{C}^3}(t)$ recovers the MacMahon function for 3d-partitions, which we should expect.
Recall that we defined power structure on \( M_C \), defined in a way that

\[
\left( \sum_{n \geq 0} \left[ \text{Hilb}^n A^\text{dim} X \right] t^n \right)^{[X]} = \sum_{n \geq 0} \left[ \text{Hilb}^n X \right] t^n.
\]

Since \( X : M_C \to \mathbb{Z} \) is a homomorphism and the power structure is defined in terms of + and \( \cdot \) with \( \left[ \text{Hilb}^n A^\text{ad} \right] \),

we can replace classes with motivic virtual classes, so for \( X \) a threefold,
\[ Z_X(t) := \sum_{n \geq 0} \left[ \text{Hilb}^n X \right]_{\text{vir}} t^n \]

\[ = Z_{\mathbb{C}^3}(t) \]

Trick: know \( Z_{\mathbb{C}^3}(t) \), set \( X = \mathbb{C}^3 \)

in above to get

\[ Z_{\mathbb{C}^3}(t) = Z_{\mathbb{C}^3,0}(t) \]

Solve for \( Z_{\mathbb{C}^3,0}(t) \), and then compute formula for \( Z_X(t) = Z_{\mathbb{C}^3,0}(t) \).
Theorem: If \( \dim X = 3 \),

\[
Z_X(t) = \exp \left( \frac{t [X]_{\text{vir}}}{(1 - t^{1/2})^2 (1 - t^{-1/2})} \right).
\]

This can be rewritten in a way that depends on \( \dim X \) and gives the correct partition functions for \( \dim X = 0, 1, 2 \) (don't need virtual class since smooth).

Weird facts: This formula that depends on \( \dim X \) computes correct virtual class of \( [\text{Hilb}^n X] \) for \( n \leq 3 \) in all dimensions.
Up to a sign, $X$ of the partition function for $\dim X = d$ gives MacMahon's conjecture for $\#$ of $d$-dimensional partitions (expected by localization), but this conjecture is known to be only asymptotically correct.

Göttsche-like formula:

Let $M_g(t, q^2) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left( 1 - q^{k m} \right)^{g + \frac{1}{2} + k - \frac{m}{2}} t^{m-1}$

("refined MacMahon function"), then...
applying W-polynomial homomorphism to partition function $Z_X(t)$ (dim $X = 3$) gives

$$WZ_X(t) = \prod_{d=0}^{6} \frac{M_{d-3}(-t, -q^{\frac{1}{2}})}{d!} \frac{(-1)^d}{b^d}$$

for $X$ smooth projective threefold.