UNITARY SK_1 OF GRADED AND VALUED DIVISION ALGEBRAS, I

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Abstract. The reduced unitary Whitehead group SK_1 of a graded division algebra equipped with a unitary involution (i.e., an involution of the second kind) and graded by a torsion-free abelian group is studied. It is shown that calculations in the graded setting are much simpler than their nongraded counterparts. The bridge to the non-graded case is established by proving that the unitary SK_1 of a tame valued division algebra wih a unitary involution over a henselian field coincides with the unitary SK_1 of its associated graded division algebra. As a consequence, the graded approach allows us not only to recover results available in the literature with substantially easier proofs, but also to calculate the unitary SK_1 for much wider classes of division algebras over henselian fields.

1. Introduction

Motivated by Platonov's striking work on the reduced Whitehead group $\mathrm{SK}_1(D)$ of valued division algebras D, see [P$_2$, P$_4$], V. Yanchevskiĭ, considered the unitary analogue, $\mathrm{SK}_1(D,\tau)$, for a division algebra D with unitary (i.e., second kind) involution τ, see [Y$_1$, Y$_2$, Y$_3$, Y$_4$]. Working with division algebras over a field with henselian discrete (rank 1) valuation whose residue field also contains a henselian discrete valuation, and carrying out formidable technical calculations, he produced remarkable analogues to Platonov's results. By relating $\mathrm{SK}_1(D,\tau)$ to data over the residue algebra, he showed not only that $\mathrm{SK}_1(D,\tau)$ could be nontrivial but that it could be any finite abelian group, and he gave a formula in the bicyclic case expressing $\mathrm{SK}_1(D,\tau)$ as a quotient of relative Brauer groups. Over the years since then several approaches have been given to understanding and calculating the (nonunitary) group SK_1 using different methods, notably by Ershov [E], Suslin [S$_1$, S$_2$], Merkurjev and Rost [Mer] (For surveys on the group SK_1, see [P$_4$], [G], [Mer] or [W$_2$, §6].) However, even after the passage of some 30 years, there does not seem to have been any improvement in calculating SK_1 in the unitary setting. This may be due in part to the complexity of the formulas in Yanchevskiĭ's work, and the difficulty in following some of his arguments.

This paper is a sequel to [HaW] where the reduced Whitehead group SK_1 for a graded division algebra was studied. Here we consider the reduced unitary Whitehead group of a graded division algebra with unitary graded involution. As in our previous work, we will see that the graded calculus is much easier and more transparent than the non-graded one. We calculate the unitary SK_1 in several important cases. We also show how this enables one to calculate the unitary SK_1 of a tame division algebra over a henselian field, by passage to the associated graded division algebra. The graded approach allows us not only to recover most of Yanchevskiĭ's results in [Y$_2$, Y$_3$, Y$_4$], with very substantially simplified proofs, but also extend them to arbitrary value groups and to calculate the unitary SK_1 for wider classes of division algebras. There is a significant simplification gained by considering arbitrary value groups from the outset, rather than towers of discrete valuations. But the greatest gain comes from passage to the graded setting, where the reduction to arithmetic considerations in the degree 0 division subring is quicker and more transparent.

We briefly describe our principal results. Let E be a graded division algebra, with torsion free abelian grade group Γ_{E}, and let τ be a unitary graded involution on E. "Unitary" means that the action of τ on

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the center $\mathsf{T} = Z(\mathsf{E})$ is nontrivial (see §2.3). The *reduced unitary Whitehead group* for τ on E is defined as

$$\mathrm{SK}_1(\mathsf{E},\tau) \;=\; \big\{a \in \mathsf{E}^* \mid \mathrm{Nrd}_{\mathsf{E}}(a^{1-\tau}) = 1\big\}/\big\langle a \in \mathsf{E}^* \mid a^{1-\tau} = 1\big\rangle,$$

where $\mathrm{Nrd}_{\mathsf{E}}$ is the reduced norm map $\mathrm{Nrd}_{\mathsf{E}}\colon \mathsf{E}^* \to \mathsf{T}^*$ (see [HaW, §3]). Here, $a^{1-\tau}$ means $a\tau(a)^{-1}$. Let $\mathsf{R} = \mathsf{T}^\tau = \{t \in \mathsf{T} \mid \tau(t) = t\} \subsetneqq \mathsf{T}$ (see §2.3). Let E_0 be the subring of homogeneous elements of degree 0 in E; likewise for T_0 and R_0. For an involution ρ on E_0, $S_\rho(\mathsf{E}_0)$ denotes $\{a \in \mathsf{E}_0 \mid \rho(a) = a\}$ and $\Sigma_\rho(\mathsf{E}_0) = \langle S_\rho(\mathsf{E}_0) \cap \mathsf{E}_0^*\rangle$. Let n be the index of E, and e the exponent of the group $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$. Since $[\mathsf{T}:\mathsf{R}] = 2$, there are just two possible cases: either (i) T is unramified over R, i.e., $\Gamma_{\mathsf{T}} = \Gamma_{\mathsf{R}}$; or (ii) T is totally ramified over R, i.e., $|\Gamma_{\mathsf{T}} : \Gamma_{\mathsf{R}}| = 2$. We will prove the following formulas for the unitary SK_1:

(i) Suppose T/R is unramified:

- If E/T is unramified, then $\mathrm{SK}_1(\mathsf{E},\tau) \cong \mathrm{SK}_1(\mathsf{E}_0, \tau|_{\mathsf{E}_0})$ (Prop. 4.10).
- If E/T is totally ramified, then (Th. 5.1):
$$\begin{aligned}\mathrm{SK}_1(\mathsf{E},\tau) \;&\cong\; \big\{a \in \mathsf{T}_0^* \mid a^n \in \mathsf{R}_0^*\big\}/\{a \in \mathsf{T}_0^* \mid a^e \in \mathsf{R}_0^*\}\\ &\cong\; \big\{\omega \in \mu_n(\mathsf{T}_0) \mid \tau(\omega) = \omega^{-1}\big\}/\mu_e.\end{aligned}$$
- If $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ is cyclic, and σ is a generator of $\mathrm{Gal}(Z(E_0)/T_0)$, then (Prop. 4.13):
  - $\mathrm{SK}_1(\mathsf{E},\tau) \;\cong\; \{a \in E_0^* \mid N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathsf{E}_0}(a)) \in \mathsf{R}_0\}/\big(\Sigma_\tau(\mathsf{E}_0)\cdot\Sigma_{\sigma\tau}(\mathsf{E}_0)\big)$.
  - If E_0 is a field, then $\mathrm{SK}_1(\mathsf{E},\tau) = 1$.
- If E has a maximal graded subfield M unramified over T and another maximal graded subfield L totally ramified over T, with $\tau(\mathsf{L}) = \mathsf{L}$, then E is semiramified and (Cor. 4.11)
$$\mathrm{SK}_1(\mathsf{E},\tau) \;\cong\; \big\{a \in \mathsf{E}_0 \mid N_{\mathsf{E}_0/\mathsf{T}_0}(a) \in \mathsf{R}_0\big\} \Big/ \prod_{h \in \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)} \mathsf{E}_0^{*h\tau}.$$

(ii) If T/R is totally ramified, then $\mathrm{SK}_1(\mathsf{E},\tau) = 1$ (Prop. 4.5).

The bridge between the graded and the non-graded henselian setting is established by Th. 3.5, which shows that for a tame division algebra D over a henselian valued field with a unitary involution τ, $\mathrm{SK}_1(D,\tau) \cong \mathrm{SK}_1(\mathsf{gr}(D),\widetilde{\tau})$ where $\mathsf{gr}(D)$ is the graded division algebra associated to D by the valuation, and $\widetilde{\tau}$ is the graded involution on $\mathsf{gr}(D)$ induced by τ (see §3). Thus, each of the results listed above for graded division algebras yields analogous formulas for valued division algebras over a henselian field, as illustrated in Example 5.3 and Th. 5.4. This recovers existing formulas, which were primarily for the case with value group $\mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}$, but with easier and more transparent proofs than those in the existing literature. Additionally, our results apply for any value groups whatever. The especially simple case where E/T is totally ramified and T/R is unramified is entirely new.

In the sequel to this paper [W_3], the very interesting special case will be treated where E/T is semiramified (and T/R is unramified) and $\mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ is bicyclic. This case was the setting of essentially all of Platonov's specifically computed examples with nontrivial $\mathrm{SK}_1(D)$ [P_2, P_3], and likewise Yanchevskiĭ's unitary examples in [Y_3] where the nontrivial $\mathrm{SK}_1(D,\tau)$ was fully computed. This case is not pursued here because it requires some more specialized arguments. For such an E, it is known that $[\mathsf{E}]$ decomposes (nonuniquely) as $[\mathsf{I}\otimes_{\mathsf{T}}\mathsf{N}]$ in the graded Brauer group of T, where I is inertial over T and N is nicely semiramified, i.e., semiramified and containing a maximal graded subfield totally ramified over T. Then a formula will be given for $\mathrm{SK}_1(E)$ as a factor group of the relative Brauer group $\mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0)$ modulo other relative Brauer groups and the class of I_0. An exactly analogous formula will be proved for $\mathrm{SK}_1(\mathsf{E},\tau)$ in the unitary setting.

2. Preliminaries

Throughout this paper we will be concerned with involutory division algebras and involutory graded division algebras. In the non-graded setting, we will denote a division algebra by D and its center by K; this D is equipped with an involution τ, and we set $F = K^\tau = \{a \in K \mid \tau(a) = a\}$. In the graded setting, we will write E for a graded division algebra with center T, and $\mathsf{R} = \mathsf{T}^\tau$ where τ is a graded involution on E. (This is consistent with the notation used in [HaW].) Depending on the context, we will write $\tau(a)$ or a^τ for the action of the involution on an element, and K^τ for the set of elements of K invariant under τ. Our convention is that $a^{\sigma\tau}$ means $\sigma(\tau(a))$.

In this section, we recall the notion of graded division algebras and collect the facts we need about them in §2.1. We will then introduce the unitary and graded reduced unitary Whitehead groups in §2.2 and §2.3.

2.1. **Graded division algebras.**

In this subsection we establish notation and recall some fundamental facts about graded division algebras indexed by a totally ordered abelian group. For an extensive study of such graded division algebras and their relations with valued division algebras, we refer the reader to [HW_2]. For generalities on graded rings see [NvO].

Let $\mathsf{R} = \bigoplus_{\gamma\in\Gamma} \mathsf{R}_\gamma$ be a graded ring, i.e., Γ is an abelian group, and R is a unital ring such that each R_γ is a subgroup of $(\mathsf{R}, +)$ and $\mathsf{R}_\gamma \cdot \mathsf{R}_\delta \subseteq \mathsf{R}_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. Set

$$\begin{aligned} \Gamma_{\mathsf{R}} &= \{\gamma \in \Gamma \mid \mathsf{R}_\gamma \neq 0\}, \text{ the grade set of } \mathsf{R};\\ \mathsf{R}^h &= \textstyle\bigcup_{\gamma\in\Gamma_{\mathsf{R}}} \mathsf{R}_\gamma, \text{ the set of homogeneous elements of } \mathsf{R}. \end{aligned}$$

For a homogeneous element of R of degree γ, i.e., an $r \in \mathsf{R}_\gamma \setminus \{0\}$, we write $\deg(r) = \gamma$. Recall that R_0 is a subring of R and that for each $\gamma \in \Gamma_{\mathsf{R}}$, the group R_γ is a left and right R_0-module. A subring S of R is a *graded subring* if $\mathsf{S} = \bigoplus_{\gamma\in\Gamma_{\mathsf{R}}}(\mathsf{S}\cap\mathsf{R}_\gamma)$. For example, the center of R, denoted $Z(\mathsf{R})$, is a graded subring of R. If $\mathsf{T} = \bigoplus_{\gamma\in\Gamma} \mathsf{T}_\gamma$ is another graded ring, a *graded ring homomorphism* is a ring homomorphism $f\colon \mathsf{R} \to \mathsf{T}$ with $f(\mathsf{R}_\gamma) \subseteq \mathsf{T}_\gamma$ for all $\gamma \in \Gamma$. If f is also bijective, it is called a graded ring isomorphism; we then write $\mathsf{R} \cong_{\mathrm{gr}} \mathsf{T}$.

For a graded ring R, a *graded left* R*-module* M is a left R-module with a grading $\mathsf{M} = \bigoplus_{\gamma\in\Gamma'} \mathsf{M}_\gamma$, where the M_γ are all abelian groups and Γ' is a abelian group containing Γ, such that $\mathsf{R}_\gamma \cdot \mathsf{M}_\delta \subseteq \mathsf{M}_{\gamma+\delta}$ for all $\gamma \in \Gamma_{\mathsf{R}}, \delta \in \Gamma'$. Then, Γ_{M} and M^h are defined analogously to Γ_{R} and R^h. We say that M is a *graded free* R-module if it has a base as a free R-module consisting of homogeneous elements.

A graded ring $\mathsf{E} = \bigoplus_{\gamma\in\Gamma} \mathsf{E}_\gamma$ is called a *graded division ring* if Γ is a torsion-free abelian group and every non-zero homogeneous element of E has a multiplicative inverse in E. Note that the grade set Γ_{E} is actually a group. Also, E_0 is a division ring, and E_γ is a 1-dimensional left and right E_0 vector space for every $\gamma \in \Gamma_{\mathsf{E}}$. Set $\mathsf{E}_\gamma^* = \mathsf{E}_\gamma \setminus \{0\}$. The requirement that Γ be torsion-free is made because we are interested in graded division rings arising from valuations on division rings, and all the grade groups appearing there are torsion-free. Recall that every torsion-free abelian group Γ admits total orderings compatible with the group structure. (For example, Γ embeds in $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ which can be given a lexicographic total ordering using any base of it as a $\mathbb{Q}$-vector space.) By using any total ordering on Γ_{E}, it is easy to see that E has no zero divisors and that E^*, the multiplicative group of units of E, coincides with $\mathsf{E}^h \setminus \{0\}$ (cf. [HW_2, p. 78]). Furthermore, the degree map

$$\deg\colon \mathsf{E}^* \to \Gamma_{\mathsf{E}} \tag{2.1}$$

is a group epimorphism with kernel E_0^*.

By an easy adaptation of the ungraded arguments, one can see that every graded module M over a graded division ring E is graded free, and every two homogenous bases have the same cardinality. We thus call M a *graded vector space* over E and write $\dim_{\mathsf{E}}(\mathsf{M})$ for the rank of M as a graded free E-module. Let

$\mathsf{S} \subseteq \mathsf{E}$ be a graded subring which is also a graded division ring. Then we can view E as a graded left S-vector space, and we write $[\mathsf{E}:\mathsf{S}]$ for $\dim_{\mathsf{S}}(\mathsf{E})$. It is easy to check the "Fundamental Equality,"

$$[\mathsf{E}:\mathsf{S}] \;=\; [\mathsf{E}_0:\mathsf{S}_0]\,|\Gamma_{\mathsf{E}}:\Gamma_{\mathsf{S}}|, \tag{2.2}$$

where $[\mathsf{E}_0:\mathsf{S}_0]$ is the dimension of E_0 as a left vector space over the division ring S_0 and $|\Gamma_{\mathsf{E}}:\Gamma_{\mathsf{S}}|$ denotes the index in the group Γ_{E} of its subgroup Γ_{S}.

A *graded field* T is a commutative graded division ring. Such a T is an integral domain (as Γ_{T} is torsion free), so it has a quotient field, which we denote $q(\mathsf{T})$. It is known, see [HW$_1$, Cor. 1.3], that T is integrally closed in $q(\mathsf{T})$. An extensive theory of graded algebraic field extensions of graded fields has been developed in [HW$_1$].

If E is a graded division ring, then its center $Z(\mathsf{E})$ is clearly a graded field. *The graded division rings considered in this paper will always be assumed finite-dimensional over their centers.* The finite-dimensionality assures that E has a quotient division ring $q(\mathsf{E})$ obtained by central localization, i.e., $q(\mathsf{E}) = \mathsf{E}\otimes_{\mathsf{T}} q(\mathsf{T})$, where $\mathsf{T} = Z(\mathsf{E})$. Clearly, $Z(q(\mathsf{E})) = q(\mathsf{T})$ and $\operatorname{ind}(\mathsf{E}) = \operatorname{ind}(q(\mathsf{E}))$, where the index of E is defined by $\operatorname{ind}(\mathsf{E})^2 = [\mathsf{E}:\mathsf{T}]$ (see [HW$_2$, p. 89]). If S is a graded field which is a graded subring of $Z(\mathsf{E})$ and $[\mathsf{E}:\mathsf{S}] < \infty$, then E is said to be a *graded division algebra* over S. We recall a fundamental connection between Γ_{E} and $Z(\mathsf{E}_0)$: The field $Z(\mathsf{E}_0)$ is Galois over T_0, and there is a well-defined group epimorphism

$$\Theta_{\mathsf{E}}\colon \Gamma_{\mathsf{E}} \to \operatorname{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0) \quad \text{given by} \quad \deg(e) \mapsto (z \mapsto eze^{-1}), \tag{2.3}$$

for any $e \in \mathsf{E}^*$. (See [HW$_2$, Prop. 2.3] for a proof).

Let $\mathsf{E} = \bigoplus_{\alpha\in\Gamma_{\mathsf{E}}} \mathsf{E}_\alpha$ be a graded division algebra with a graded center T (with, as always, Γ_{E} a torsion-free abelian group). After fixing some total ordering on Γ_{E}, define a function

$$\lambda\colon \mathsf{E}\setminus\{0\} \to \mathsf{E}^* \quad \text{by} \quad \lambda(\textstyle\sum c_\gamma) = c_\delta, \text{ where } \delta \text{ is minimal among the } \gamma\in\Gamma_{\mathsf{E}} \text{ with } c_\gamma \neq 0.$$

Note that $\lambda(a) = a$ for $a \in \mathsf{E}^*$, and

$$\lambda(ab) = \lambda(a)\lambda(b) \text{ for all } a, b \in \mathsf{E}\setminus\{0\}. \tag{2.4}$$

Let $Q = q(\mathsf{E}) = \mathsf{E}\otimes_{\mathsf{T}} q(\mathsf{T})$, which is a division ring as E has no zero divisors and is finite-dimensional over T. We can extend λ to a map defined on all of $Q^* = Q\setminus\{0\}$ as follows: for $q \in Q^*$, write $q = ac^{-1}$ with $a \in \mathsf{E}\setminus\{0\}$, $c \in Z(\mathsf{E})\setminus\{0\}$, and set $\lambda(q) = \lambda(a)\lambda(c)^{-1}$. It follows from (2.4) that $\lambda\colon Q^* \to \mathsf{E}^*$ is well-defined and is a group homomorphism. Since the composition

$$\mathsf{E}^* \hookrightarrow Q^* \xrightarrow{\lambda} \mathsf{E}^* \tag{2.5}$$

is the identity, λ is a splitting map for the injection $\mathsf{E}^* \hookrightarrow Q^*$.

For a graded division algebra E over its center T, there is a reduced norm map $\operatorname{Nrd}_{\mathsf{E}}\colon \mathsf{E}^* \to \mathsf{T}^*$ (see [HaW, §3]) such that for $a \in \mathsf{E}$ one has $\operatorname{Nrd}_{\mathsf{E}}(a) = \operatorname{Nrd}_{q(\mathsf{E})}(a)$. The *reduced Whitehead group*, $\operatorname{SK}_1(\mathsf{E})$, is defined as $\mathsf{E}^{(1)}/\mathsf{E}'$, where $\mathsf{E}^{(1)}$ denotes the set of elements of E^* with reduced norm 1, and E' is the commutator subgroup $[\mathsf{E}^*,\mathsf{E}^*]$ of E^*. This group was studied in detail in [HaW]. We will be using the following facts which were established in that paper:

Remarks 2.1. Let $n = \operatorname{ind}(\mathsf{E})$.

(i) For $\gamma \in \Gamma_{\mathsf{E}}$, if $a \in \mathsf{E}_\gamma$ then $\operatorname{Nrd}_{\mathsf{E}}(a) \in \mathsf{E}_{n\gamma}$. In particular, $\mathsf{E}^{(1)}$ is a subset of E_0.

(ii) If S is any graded subfield of E containing T and $a \in \mathsf{S}$, then $\operatorname{Nrd}_{\mathsf{E}}(a) = N_{\mathsf{S}/\mathsf{T}}(a)^{n/[\mathsf{S}:\mathsf{T}]}$.

(iii) Set

$$\partial \;=\; \operatorname{ind}(\mathsf{E})\big/\big(\operatorname{ind}(\mathsf{E}_0)\,[Z(\mathsf{E}_0){:}\mathsf{T}_0]\big). \tag{2.6}$$

If $a \in \mathsf{E}_0$, then,

$$\operatorname{Nrd}_{\mathsf{E}}(a) \;=\; N_{Z(\mathsf{E}_0)/\mathsf{T}_0}\operatorname{Nrd}_{\mathsf{E}_0}(a)^{\partial} \in \mathsf{T}_0. \tag{2.7}$$

(iv) If N is a normal subgroup of E^*, then $N^n \subseteq \mathrm{Nrd}_{\mathsf{E}}(N)[\mathsf{E}^*, N]$.
For proofs of (i)-(iv) see [HaW, Prop. 3.2 and 3.3].

(v) $\mathrm{SK}_1(\mathsf{E})$ is n-torsion.

Proof. By taking $N = \mathsf{E}^{(1)}$, the assertion follows from (iv). □

A graded division algebra E with center T is said to be *inertial* (or *unramified*) if $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{T}}$. From (2.2), it then follows that $[\mathsf{E} : \mathsf{T}] = [\mathsf{E}_0 : \mathsf{T}_0]$; indeed, E_0 is central simple over T_0 and $\mathsf{E} \cong_{\mathsf{gr}} \mathsf{E}_0 \otimes_{\mathsf{T}_0} \mathsf{T}$. At the other extreme, E is said to be *totally ramified* if $\mathsf{E}_0 = \mathsf{T}_0$. In an intermediate case E is said to be *semiramified* if E_0 is a field and $[\mathsf{E}_0 : \mathsf{T}_0] = |\Gamma_{\mathsf{E}} : \Gamma_{\mathsf{T}}| = \mathrm{ind}(\mathsf{E})$. These definitions are motivated by analogous definitions for valued division algebras ([W$_2$]). Indeed, if a tame valued division algebra is unramified, semiramified, or totally ramified, then so is its associated graded division algebra. Likewise, a graded field extension L of T is said to be *inertial* (or *unramified*) if $\mathsf{L} \cong_{\mathsf{gr}} \mathsf{L}_0 \otimes_{\mathsf{T}_0} \mathsf{T}$ and the field L_0 is separable over T_0. At the other extreme, L is *totally ramified* over T if $[\mathsf{L} : \mathsf{T}] = |\Gamma_{\mathsf{L}} : \Gamma_{\mathsf{T}}|$. A graded division algebra E is said to be *inertially split* if E has a maximal graded subfield L with L inertial over T. When this occurs, $\mathsf{E}_0 = \mathsf{L}_0$, and $\mathrm{ind}(\mathsf{E}) = \mathrm{ind}(\mathsf{E}_0)\,[Z(\mathsf{E}_0) : \mathsf{T}_0]$ by Lemma 2.2 below. In particular, if E is semiramified then E is inertially split, E_0 is abelian Galois over T_0, and the canonical map $\Theta_{\mathsf{E}} \colon \Gamma_{\mathsf{E}} \to \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ has kernel Γ_{T} (see (2.3) above and [HW$_2$, Prop. 2.3]).

Lemma 2.2. *Let E be a graded division algebra with center T. For the ∂ of (2.6), $\partial^2 = |\ker(\Theta_{\mathsf{E}})/\Gamma_{\mathsf{T}}|$. Also, $\partial = 1$ iff E is inertially split.*

Proof. Since Θ_{E} is surjective, $\Gamma_{\mathsf{T}} \subseteq \ker(\Theta_{\mathsf{E}})$, and $Z(\mathsf{E}_0)$ is Galois over T_0, we have

$$\begin{aligned}\partial^2 \ &= \ \mathrm{ind}(\mathsf{E})^2 \big/ \big(\mathrm{ind}(\mathsf{E}_0)^2\,[Z(\mathsf{E}_0) : \mathsf{T}_0]^2\big) \ = \ [\mathsf{E} : \mathsf{T}] \big/ \big([\mathsf{E}_0 : Z(\mathsf{T}_0)]\,[Z(\mathsf{E}_0) : \mathsf{T}_0]\,|\,\mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)|\big)\\ &= \ [\mathsf{E}_0 : \mathsf{T}_0]\,|\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}| \big/ \big([\mathsf{E}_0 : \mathsf{T}_0]\,|\,\mathrm{im}(\Theta_{\mathsf{E}})|\big) \ = \ |\ker(\Theta_{\mathsf{E}})/\Gamma_{\mathsf{T}}|.\end{aligned}$$

Now, suppose M is a maximal subfield of E_0 with M separable over T_0. Then, $M \supseteq Z(\mathsf{E}_0)$ and $[M : Z(\mathsf{E}_0)] = \mathrm{ind}(\mathsf{E}_0)$. Let $\mathsf{L} = M \otimes_{\mathsf{T}_0} \mathsf{T}$ which is a graded subfield of E inertial over T, with $\mathsf{L}_0 = M$. Then,

$$[\mathsf{L} : \mathsf{T}] \ = \ [\mathsf{L}_0 : \mathsf{T}_0] \ = \ [\mathsf{L}_0 : Z(\mathsf{E}_0)]\,[Z(\mathsf{E}_0) : \mathsf{T}_0] \ = \ \mathrm{ind}(\mathsf{E})/\partial.$$

Thus, if $\partial = 1$, then E is inertially split, since L is a maximal graded subfield of E which is inertial over T. Conversely, suppose E is inertially split, say I is a maximal graded subfield of E with I inertial over T. So, $[\mathsf{I}_0 : \mathsf{T}_0] = [\mathsf{I} : \mathsf{T}] = \mathrm{ind}(\mathsf{E})$. Since $\mathsf{I}_0 Z(\mathsf{E}_0)$ is a subfield of E_0, we have

$$\begin{aligned}[\mathsf{I}_0 : \mathsf{T}_0] \ &\leq \ [\mathsf{I}_0 Z(\mathsf{E}_0) : \mathsf{T}_0] \ = \ [\mathsf{I}_0 Z(\mathsf{E}_0) : Z(\mathsf{E}_0)]\,[Z(\mathsf{E}_0) : \mathsf{T}_0]\\ &\leq \ \mathrm{ind}(\mathsf{E}_0)\,[Z(\mathsf{E}_0) : \mathsf{T}_0] \ = \ \mathrm{ind}(\mathsf{E})/\partial \ = \ [\mathsf{I}_0 : \mathsf{T}_0]/\partial.\end{aligned}$$

So, as ∂ is a positive integer, $\partial = 1$. □

2.2. **Unitary SK_1 of division algebras.** We begin with a description of unitary K_1 and SK_1 for a division algebra with an involution. The analogous definitions for graded division algebras will be given in §2.3.

Let D be a division ring finite-dimensional over its center K of index n, and let τ be an involution on D, i.e., τ is an antiautomorphism of D with $\tau^2 = \mathrm{id}$. Let

$$\begin{aligned}S_\tau(D) \ &= \ \{d \in D \mid \tau(d) = d\};\\ \Sigma_\tau(D) \ &= \ \langle S_\tau(D) \cap D^* \rangle\,.\end{aligned}$$

Note that $\Sigma_\tau(D)$ is a normal subgroup of D^*. For, if $a \in S_\tau(D)$, $a \neq 0$, and $b \in D^*$, then $bab^{-1} = [ba\tau(b)][b\tau(b)]^{-1} \in \Sigma_\tau(D)$, as $ba\tau(b), b\tau(b) \in S_\tau(D)$.

Let φ be an isotropic m-dimensional, nondegenerate skew-Hermitian form over D with respect to an involution τ on D. Let ρ be the involution on $M_m(D)$ adjoint to φ, let $U_m(D) = \{a \in M_m(D) \mid a\rho(a) = 1\}$

be the unitary group associated to φ, and let $\operatorname{EU}_m(D)$ denote the normal subgroup of $U_m(D)$ generated by the unitary transvections. For $m > 2$, the Wall spinor norm map $\Theta\colon U_m(D) \to D^*/\Sigma_\tau(D)D'$ was developed in [Wa], where it was shown that $\ker(\Theta) = \operatorname{EU}_m(D)$. Here, D' denotes the multiplicative commutator group $[D^*, D^*]$. Combining this with [D, Cor. 1 of §22] one obtains the commutative diagram:

$$\begin{tikzcd}
U_m(D)/\operatorname{EU}_m(D) \arrow[r, "\Theta", "\cong"'] \arrow[d] & D^*/\big(\Sigma_\tau(D)D'\big) \arrow[d, "1-\tau"] \\
\operatorname{GL}_m(D)/E_m(D) \arrow[r, "\det"] \arrow[d, "\mathrm{Nrd}"'] & D^*/D' \arrow[d, "\mathrm{Nrd}"] \\
K^* \arrow[r, "\mathrm{id}"] & K^*
\end{tikzcd} \tag{2.8}$$

where the map det is the Dieudonné determinant and $1-\tau\colon D^*/\big(\Sigma_\tau(D)D'\big) \longrightarrow D^*/D'$ is defined as $x\Sigma_\tau(D)D' \mapsto x^{1-\tau}D'$, where $x^{1-\tau}$ means $x\tau(x)^{-1}$ (see [HM, 6.4.3]).

From the diagram, and parallel to the "absolute" case, one defines the *unitary Whitehead group*,

$$K_1(D,\tau) \;=\; D^*/\big(\Sigma_\tau(D)D'\big).$$

For any involution τ on D, recall that

$$\operatorname{Nrd}_D(\tau(d)) \;=\; \tau(\operatorname{Nrd}_D(d)), \tag{2.9}$$

for any $d \in D$. For, if $p \in K[x]$ is the minimal polynomial of d over K, then $\tau(p)$ is the minimal polynomial of $\tau(d)$ over K (see also [D, §22, Lemma 5]).

We consider two cases:

2.2.1. *Involutions of the first kind.* In this case the center K of D is elementwise invariant under the involution, i.e., $K \subseteq S_\tau(D)$. Then $S_\tau(D)$ is a K-vector space. The involutions of this kind are further subdivided into two types: *orthogonal* and *symplectic* involutions (see [KMRT, Def. 2.5]). By ([KMRT, Prop. 2.6]), if $\operatorname{char}(K) \neq 2$ and τ is orthogonal then, $\dim_K(S_\tau(D)) = n(n+1)/2$, while if τ is symplectic then $\dim_K(S_\tau(D)) = n(n-1)/2$. However, if $\operatorname{char}(K) = 2$, then $\dim_K(S_\tau(D)) = n(n+1)/2$ for each type.

If $\dim_K(S_\tau(D)) = n(n+1)/2$, then for any $x \in D^*$, we have $xS_\tau(D) \cap S_\tau(D) \neq 0$ by dimension count; it then follows that $D^* = \Sigma_\tau(D)$, and thus $K_1(D,\tau) = 1$. However, in the case $\dim_K(S_\tau(D)) = n(n-1)/2$, Platonov showed that $K_1(D,\tau)$ is not in general trivial, settling Dieudonné's conjecture in negative [P_1]. Note that whenever τ is of the first kind we have $\operatorname{Nrd}_D(\tau(d)) = \operatorname{Nrd}_D(d)$ for all $d \in D$, by (2.9). Thus, $K_1(D,\tau)$ is sent to the identity under the composition $\operatorname{Nrd}\circ(1-\tau)$. This explains why one does not consider the kernel of this map, i.e., the unitary SK_1, for involutions of the first kind. If $\operatorname{char}(K) \neq 2$ and τ is symplectic, then as the m-dimensional form φ over D is skew-Hermitian, its associated adjoint involution ρ on $M_m(D)$ is of orthogonal type, so there is an associated spin group $\operatorname{Spin}(M_m(D),\rho)$. For any $a \in S(D)$ one then has $\operatorname{Nrd}_D(a) \in K^{*2}$ ([KMRT, Lemma 2.9]). One defines $K_1\operatorname{Spin}(D,\tau) = R(D)/\big(\Sigma_\tau(D)D'\big)$, where $R(D) = \{d \in D^* \mid \operatorname{Nrd}_D(d) \in K^{*2}\}$. This group is related to $\operatorname{Spin}(M_m(D),\rho)$, and has been studied in [MY], parallel to the work on absolute SK_1 groups and unitary SK_1 groups for unitary involutions.

2.2.2. *Involutions of the second kind (unitary involutions).* In this case $K \not\subseteq S_\tau(D)$. Then, let $F = K^\tau$ $(= K \cap S_\tau(D)\,)$, which is a subfield of K with $[K:F] = 2$. It was already observed by Dieudonné that $U_m(D) \neq \operatorname{EU}_m(D)$. An important property proved by Platonov and Yanchevskiĭ, which we will use frequently, is that

$$D' \subseteq \Sigma_\tau(D). \tag{2.10}$$

(For a proof, see [KMRT, Prop. 17.26].) Thus $K_1(D,\tau) = D^*/\Sigma_\tau(D)$, which is not trivial in general. The kernel of the map $\mathrm{Nrd}\circ(1-\tau)$ in diagram (2.8), is called the *reduced unitary Whitehead group*, and denoted by $\mathrm{SK}_1(D,\tau)$. Using (2.9), it is straightforward to see that

$$\mathrm{SK}_1(D,\tau) \;=\; \Sigma'_\tau(D)/\Sigma_\tau(D), \quad \text{where} \quad \Sigma'_\tau(D) \;=\; \{a \in D^* \mid \mathrm{Nrd}_D(a) \in F^*\}.$$

Note that we use the notation $\mathrm{SK}_1(D,\tau)$ for the reduced unitary Whitehead group as opposed to Draxl's notation $\mathrm{USK}_1(D,\tau)$ in [D, p. 172] and Yanchevskiĭ's notation $\mathrm{SUK}_1(D,\tau)$ [Y_2] and the notation $\mathrm{USK}_1(D)$ in [KMRT].

Before we define the corresponding groups in the graded setting, let us recall that all the groups above fit in Tits' framework [T] of the *Whitehead group* $W(G,K) = G_K/G_K^+$ where G is an almost simple, simply connected linear algebraic group defined over an infinite field K, with $\mathrm{char}(K) \neq 2$, and G is isotropic over K. Here, G_K is the set of K-rational points of G and G_K^+, is the subgroup of G_K, generated by the unipotent radicals of the minimal K-parabolic subgroups of G. In this setting, for $G_K = \mathrm{SL}_n(D)$, $n > 1$, we have $W(G,K) = \mathrm{SK}_1(D)$; for τ an involution of first or second kind on D and $F = K^\tau$, for $G_F = \mathrm{SL}_n(D,\tau) := \mathrm{SL}_n(D) \cap U_n(D)$ we have $W(G,F) = \mathrm{SK}_1(D,\tau)$; and for τ a symplectic involution on D and ρ the adjoint involution of an m-dimensional isotropic skew-Hermitian form over D with $m \geq 3$, for the spinor group $G_K = \mathrm{Spin}(M_m(D),\rho)$ we have $W(G,K)$ is a double cover of $K_1\,\mathrm{Spin}(D,\tau)$ (see [MY]).

2.3. **Unitary SK_1 of graded division algebras.** We will now introduce the unitary K_1 and SK_1 in the graded setting. Let $\mathsf{E} = \bigoplus_{\gamma\in\Gamma_\mathsf{E}} \mathsf{E}_\gamma$ be a graded division ring (with Γ_E a torsion-free abelian group) such that E has finite dimension n^2 over its center T, a graded field. Let τ be a graded involution of E, i.e., τ is an antiautomorphism of E with $\tau^2 = \mathrm{id}$ and $\tau(\mathsf{E}_\gamma) = \mathsf{E}_\gamma$ for each $\gamma \in \Gamma_\mathsf{E}$. We define $S_\tau(\mathsf{E})$ and $\Sigma_\tau(\mathsf{E})$, analogously to the non-graded cases, as the set of elements of E which are invariant under τ, and the multiplicative group generated by the nonzero homogenous elements of $S_\tau(\mathsf{E})$, respectively. We say the involution of the *first kind* if all the elements of the center T are invariant under τ; it is of the *second kind* (or *unitary*) otherwise. If τ is of the first kind then, parallel to the non-graded case, either $\dim_\mathsf{T}(S_\tau(\mathsf{E})) = n(n{+}1)/2$ or $\dim_\mathsf{T}(S_\tau(\mathsf{E})) = n(n{-}1)/2$. Indeed, one can show these equalities by arguments analogous to the nongraded case as in the proof of [KMRT, Prop. 2.6(1)], as E is split by a graded maximal subfield and the Skolem–Noether theorem is available in the graded setting ([HW_2, Prop. 1.6]). (These equalities can also be obtained by passing to the quotient division algebra as is done in Lemma 2.3(i) below.)

Define the *unitary Whitehead group*

$$K_1(\mathsf{E},\tau) \;=\; \mathsf{E}^*/\big(\Sigma_\tau(\mathsf{E})\mathsf{E}'\big),$$

where $\mathsf{E}' = [\mathsf{E}^*,\mathsf{E}^*]$. If τ is of the first kind, $\mathrm{char}(\mathsf{T}) \neq 2$, and $\dim_\mathsf{T}(S_\tau(\mathsf{E})) = n(n-1)/2$, a proof similar to [KMRT, Prop. 2.9], shows that if $a \in S_\tau(\mathsf{E})$ is homogeneous, then $\mathrm{Nrd}_\mathsf{E}(a) \in \mathsf{T}^{*2}$ (This can also be verified by passing to the quotient division algebra, then using Lemma 2.3(i) below and invoking the corresponding result for ungraded division algebras.) For this type of involution, define the *spinor Whitehead group*

$$K_1\,\mathrm{Spin}(\mathsf{E},\tau) \;=\; \{a \in \mathsf{E}^* \mid \mathrm{Nrd}_\mathsf{E}(a) \in \mathsf{T}^{*^2}\}\,/\big(\Sigma_\tau(\mathsf{E})\mathsf{E}'\big).$$

When the graded involution τ on E is unitary, i.e., $\tau|_\mathsf{T} \neq \mathrm{id}$, let $\mathsf{R} = \mathsf{T}^\tau$, which is a graded subfield of T with $[\mathsf{T}:\mathsf{R}] = 2$. Furthermore, T is Galois over R, with $\mathrm{Gal}(\mathsf{T}/\mathsf{R}) = \{\mathrm{id}, \tau|_\mathsf{T}\}$. (See [$\mathrm{HW}_1$] for Galois theory for graded field extensions.) Define the *reduced unitary Whitehead group*

$$\mathrm{SK}_1(\mathsf{E},\tau) \;=\; \Sigma'_\tau(\mathsf{E})\,/\,(\Sigma_\tau(\mathsf{E})\,\mathsf{E}') \;=\; \Sigma'_\tau(\mathsf{E})\,/\,\Sigma_\tau(\mathsf{E}), \tag{2.11}$$

where

$$\Sigma'_\tau(\mathsf{E}) \;=\; \big\{a \in \mathsf{E}^* \mid \mathrm{Nrd}_\mathsf{E}(a^{1-\tau}) = 1\big\} \;=\; \{a \in \mathsf{E}^* \mid \mathrm{Nrd}_\mathsf{E}(a) \in \mathsf{R}^*\}$$

and

$$\Sigma_\tau(\mathsf{E}) \;=\; \big\langle a \in \mathsf{E}^* \mid a^{1-\tau} = 1\big\rangle \;=\; \langle S_\tau(\mathsf{E}) \cap \mathsf{E}^*\rangle.$$

Here, $a^{1-\tau}$ means $a\tau(a)^{-1}$. See Lemma 2.3(iv) below for the second equality in (2.11). The group $\mathrm{SK}_1(\mathsf{E},\tau)$ will be the main focus of the rest of the paper.

We will use the following facts repeatedly:

Lemma 2.3.

(i) *Any graded involution on* E *extends uniquely to an involution of the same kind (and type) on* $Q = q(\mathsf{E})$.

(ii) *For any graded involution* τ *on* E*, and its extension to* $Q = q(\mathsf{E})$*, we have* $\Sigma_\tau(Q) \cap \mathsf{E}^* \subseteq \Sigma_\tau(\mathsf{E})$.

(iii) *If* τ *is a graded involution of the first kind on* E *with* $\dim_{\mathsf{T}}(S_\tau(\mathsf{E})) = n(n+1)/2$*, then* $\Sigma_\tau(\mathsf{E}) = \mathsf{E}^*$.

(iv) *If* τ *is a unitary graded involution on* E*, then* $\mathsf{E}' \subseteq \Sigma_\tau(\mathsf{E})$.

(v) *If* τ *is a unitary graded involution on* E*, then* $\mathrm{SK}_1(\mathsf{E},\tau)$ *is a torsion group of bounded exponent dividing* $n = \mathrm{ind}(\mathsf{E})$.

Proof.

(i) Let τ be a graded involution on E. Then $q(\mathsf{E}) = \mathsf{E} \otimes_{\mathsf{T}} q(\mathsf{T}) = \mathsf{E} \otimes_{\mathsf{T}} (\mathsf{T} \otimes_{\mathsf{T}^\tau} q(\mathsf{T}^\tau)) = \mathsf{E} \otimes_{\mathsf{T}^\tau} q(\mathsf{T}^\tau)$. The unique extension of τ to $q(\mathsf{E})$ is $\tau \otimes \mathrm{id}_{q(\mathsf{T}^\tau)}$, which we denote simply as τ. It then follows that $S_\tau(q(\mathsf{E})) = S_\tau(\mathsf{E}) \otimes_{\mathsf{T}^\tau} q(\mathsf{T}^\tau)$. Since $q(\mathsf{T}^\tau) = q(\mathsf{T})^\tau$, the assertion follows.

(ii) Note that for the map λ in the sequence (2.5) we have $\tau(\lambda(a)) = \lambda(\tau(a))$ for all $a \in Q^*$. Hence, $\lambda(\Sigma_\tau(Q)) \subseteq \Sigma_\tau(\mathsf{E})$. Since $\lambda|_{\mathsf{E}^*}$ is the identity, we have $\Sigma_\tau(Q) \cap \mathsf{E}^* \subseteq \Sigma_\tau(\mathsf{E})$.

(iii) The extension of the graded involution τ to $Q = q(\mathsf{E})$, also denoted τ, is of the first kind with $\dim_Q(S_\tau(Q)) = n(n+1)/2$ by (i). Therefore $\Sigma_\tau(Q) = Q^*$ (see §2.2.1). Using (ii) now, the assertion follows.

(iv) Since τ is a unitary graded involution, its extension to $Q = q(\mathsf{E})$ is also unitary, by (i). But $Q' \subseteq \Sigma_\tau(Q)$, as noted in (2.10). From (2.5) it follows that $Q' \cap \mathsf{E}^* = \mathsf{E}'$. Hence, using (ii), $\mathsf{E}' \subseteq \mathsf{E}^* \cap Q' \subseteq \mathsf{E}^* \cap \Sigma_\tau(Q) \subseteq \Sigma_\tau(\mathsf{E})$.

(v) Setting $N = \Sigma'_\tau(\mathsf{E})$, Remark 2.1(iv) above, coupled with the fact that $\mathsf{E}' \subseteq \Sigma_\tau(\mathsf{E})$ (iv), implies that $\mathrm{SK}_1(\mathsf{E},\tau)$ is an n-torsion group. This assertion also follows by using (ii) which implies the natural map $\mathrm{SK}_1(\mathsf{E},\tau) \to \mathrm{SK}_1(Q,\tau)$ is injective and the fact that unitary SK_1 of a division algebra of index n is n-torsion ([Y_2, Cor. to 2.5]). □

2.4. **Generalized dihedral groups and field extensions.** The nontrivial case of $\mathrm{SK}_1(\mathsf{E},\tau)$ for τ a unitary graded involution turns out to be when $\mathsf{T} = Z(\mathsf{E})$ is unramified over $\mathsf{R} = \mathsf{T}^\tau$ (see §4.2). When that occurs, we will see in Lemma 4.6(ii) below that $Z(\mathsf{E}_0)$ is a so-called generalized dihedral extension over R_0. We now give the definition and observe a few easy facts about generalized dihedral groups and extensions.

Definition 2.4.

(i) A group G is said to be *generalized dihedral* if G has a subgroup H such that $[G : H] = 2$ and every $\tau \in G\backslash H$ satisfies $\tau^2 = \mathrm{id}$.

Note that if G is generalized dihedral and H the distinguished subgroup, then H is abelian and $(h\tau)^2 = \mathrm{id}$, for all $\tau \in G\backslash H$ and $h \in H$. Thus, $\tau^2 = \mathrm{id}$ and $\tau h \tau^{-1} = h^{-1}$ for all $\tau \in G\backslash H, h \in H$. Furthermore, every subgroup of H is normal in G. Clearly every dihedral group is generalized dihedral, as is every elementary abelian 2-group. More generally, if H is any abelian group and $\chi \in \mathrm{Aut}(H)$ is the map $h \mapsto h^{-1}$, then the semi-direct product $H \rtimes_i \langle\chi\rangle$ is a generalized dihedral group, where $i\colon \langle\chi\rangle \to \mathrm{Aut}(H)$ is the inclusion map. It is easy to check that every generalized dihedral group is isomorphic to such a semi-direct product.

(ii) Let $F \subseteq K \subseteq L$ be fields with $[L : F] < \infty$ and $[K : F] = 2$. We say that L is *generalized dihedral for* K/F if L is Galois over F and every element of $\mathrm{Gal}(L/F) \setminus \mathrm{Gal}(L/K)$ has order 2, i.e., $\mathrm{Gal}(L/F)$ is a generalized dihedral group. Note that when this occurs, L is compositum of fields L_i containing K with each L_i generalized dihedral for K/F with $\mathrm{Gal}(L_i/K)$ cyclic, i.e., L_i is Galois over F with

$\mathrm{Gal}(L_i/F)$ dihedral (or a Klein 4-group). Conversely, if L and M are generalized dihedral for K/F then so is their compositum.

Example 2.5. Let $n \in \mathbb{N}$, $n \geq 3$, and let $F \subseteq K$ be fields with $[K : F] = 2$ and $K = F(\omega)$, where ω is a primitive n-th root of unity (so $\mathrm{char}(F) \nmid n$). Suppose the non-identity element of $\mathrm{Gal}(K/F)$ maps ω to ω^{-1}. For any $c_1, \ldots, c_k \in F^*$, if $\omega \notin F(\sqrt[n]{c_1}, \ldots, \sqrt[n]{c_k})$, then $K(\sqrt[n]{c_1}, \ldots, \sqrt[n]{c_k})$ is generalized dihedral for K/F.

3. Henselian to graded reduction

The main goal of this section is to prove an isomorphism between the unitary SK_1 of a valued division algebra with involution over a henselian field and the graded SK_1 of its associated graded division algebra. We first recall how to associate a graded division algebra to a valued division algebra.

Let D be a division algebra finite dimensional over its center K, with a valuation $v \colon D^* \to \Gamma$. So, Γ is a totally ordered abelian group, and v satisifies the conditions that for all $a, b \in D^*$,

$$\begin{aligned} &(1) \qquad v(ab) \;=\; v(a) + v(b); \\ &(2) \qquad v(a+b) \;\geq\; \min\{v(a), v(b)\} \qquad (b \neq -a). \end{aligned}$$

Let

$$\begin{aligned} V_D &= \{a \in D^* \mid v(a) \geq 0\} \cup \{0\}, \text{ the valuation ring of } v; \\ M_D &= \{a \in D^* \mid v(a) > 0\} \cup \{0\}, \text{ the unique maximal left (and right) ideal of } V_D; \\ \overline{D} &= V_D/M_D, \text{ the residue division ring of } v \text{ on } D; \text{ and} \\ \Gamma_D &= \mathrm{im}(v), \text{ the value group of the valuation.} \end{aligned}$$

Now let K be a field with a valuation v, and suppose v is *henselian*; that is, v has a unique extension to every algebraic field extension of K. Recall that a field extension L of K of degree $n < \infty$ is said to be *tamely ramified* or *tame* over K if, with respect to the unique extension of v to L, the residue field $\overline{L}$ is separable over $\overline{K}$ and $\mathrm{char}(\overline{K}) \nmid \big(n/[\overline{L} : \overline{K}]\big)$. Such an L is necessarily *defectless* over K, i.e., $[L : K] = [\overline{L} : \overline{K}]\,|\Gamma_L : \Gamma_K|$, by [EP, Th. 3.3.3] (applied to N/K and N/L, where N is a normal closure of L over K). Along the same lines, let D be a division algebra with center K (so, by convention, $[D : K] < \infty$); then the henselian valuation v on K extends uniquely to a valuation on D ([W_1]). With respect to this valuation, D is said to be *tamely ramified* or *tame* if the center $Z(\overline{D})$ is separable over $\overline{K}$ and $\mathrm{char}(\overline{K}) \nmid \big[\mathrm{ind}(D)/\big(\mathrm{ind}(\overline{D})[Z(\overline{D}) : \overline{K}]\big)\big]$. Recall from [JW, Prop. 1.7], that whenever the field extension $Z(\overline{D})/\overline{K}$ is separable, it is abelian Galois. It is known that D is tame if and only if D is split by the maximal tamely ramified field extension of K, if and only if $\mathrm{char}(\overline{K}) = 0$ or $\mathrm{char}(\overline{K}) = p \neq 0$ and the p-primary component of D is inertially split, i.e., split by the maximal unramified extension of K ([JW, Lemma 6.1]). We say D is *strongly tame* if $\mathrm{char}(\overline{K}) \nmid \mathrm{ind}(D)$. Note that strong tameness implies tameness. This is clear from the last characterization of tameness, or from (3.1) below. Recall also from [Mor, Th. 3], that for a valued division algebra D finite dimensional over its center K (here not necessarily henselian), we have the "Ostrowski theorem"

$$[D : K] \;=\; q^k\,[\overline{D} : \overline{K}]\,|\Gamma_D : \Gamma_K|, \tag{3.1}$$

where $q = \mathrm{char}(\overline{D})$ and $k \in \mathbb{Z}$ with $k \geq 0$ (and $q^k = 1$ if $\mathrm{char}(\overline{D}) = 0$). If $q^k = 1$ in equation (3.1), then D is said to be *defectless* over K. For background on valued division algebras, see [JW] or the survey paper [W_2].

Remark 3.1. If a field K has a henselian valuation v and L is a subfield of K with $[K : L] < \infty$, then the restriction $w = v|_L$ need not be henselian. But it is easy to see that w is then "semihenselian," i.e., w has more than one but only finitely many different extensions to a separable closure L_{sep} of L. See [En] for a thorough analysis of semihenselian valuations. Notably, Engler shows that w is semihenselian iff the residue field $\overline{L}_w$ is algebraically closed but there is a henselian valuation u on L such that u is a proper coarsening of w and the residue field $\overline{L}_u$ is real closed. When this occurs, $\operatorname{char}(L) = 0$, L is formally real, w has exactly two extensions to L_{sep}, the value group $\Gamma_{L,w}$ has a nontrivial divisible subgroup, and the henselization of L re w is $L(\sqrt{-1})$, which lies in K. For example, if we take any prime number p, let w_p be the p-adic discrete valuation on $\mathbb{Q}$, and let $L = \{r \in \mathbb{R} \mid r \text{ is algebraic over } \mathbb{Q}\}$; then any extension of w_p to L is a semihenselian valuation. Note that if v on K is discrete, i.e., $\Gamma_K \cong \mathbb{Z}$, then w on L cannot be semihenselian, since Γ_L has no nontrivial divisible subgroup; so, w on L must be henselian. This preservation of the henselian property for discrete valuations was asserted in [Y_2, Lemma, p. 195], but the proof given there is invalid.

One associates to a valued division algebra D a graded division algebra as follows: For each $\gamma \in \Gamma_D$, let

$$\begin{aligned} D^{\geq\gamma} &= \{d \in D^* \mid v(d) \geq \gamma\} \cup \{0\}, \text{ an additive subgroup of } D; \\ D^{>\gamma} &= \{d \in D^* \mid v(d) > \gamma\} \cup \{0\}, \text{ a subgroup of } D^{\geq\gamma}; \text{ and} \\ \mathsf{gr}(D)_\gamma &= D^{\geq\gamma}/D^{>\gamma}. \end{aligned}$$

Then define

$$\mathsf{gr}(D) \;=\; \bigoplus_{\gamma\in\Gamma_D} \mathsf{gr}(D)_\gamma.$$

Because $D^{>\gamma}D^{\geq\delta} + D^{\geq\gamma}D^{>\delta} \subseteq D^{>(\gamma+\delta)}$ for all $\gamma,\delta \in \Gamma_D$, the multiplication on $\mathsf{gr}(D)$ induced by multiplication on D is well-defined, giving that $\mathsf{gr}(D)$ is a graded ring, called the *associated graded ring* of D. The multiplicative property (1) of the valuation v implies that $\mathsf{gr}(D)$ is a graded division ring. Clearly, we have $\mathsf{gr}(D)_0 = \overline{D}$ and $\Gamma_{\mathsf{gr}(D)} = \Gamma_D$. For $d \in D^*$, we write $\widetilde{d}$ for the image $d + D^{>v(d)}$ of d in $\mathsf{gr}(D)_{v(d)}$. Thus, the map given by $d \mapsto \widetilde{d}$ is a group epimorphism $\rho\colon D^* \to \mathsf{gr}(D)^*$ with kernel $1 + M_D$, giving us the short exact sequence

$$1 \longrightarrow 1 + M_D \longrightarrow D^* \longrightarrow \mathsf{gr}(D)^* \longrightarrow 1, \tag{3.2}$$

which will be used throughout. For a detailed study of the associated graded algebra of a valued division algebra refer to [HW_2, §4]. As shown in [HaW, Cor. 4.4], the reduced norm maps for D and $\mathsf{gr}(D)$ are related by

$$\widetilde{\mathrm{Nrd}_D(a)} \;=\; \mathrm{Nrd}_{\mathsf{gr}(D)}(\widetilde{a}) \quad \text{for all } a \in D^*. \tag{3.3}$$

Now let K be a field with a henselian valuation v and, as before, let D be a division algebra with center K. Then v extends uniquely to a valuation on D, also denoted v, and one obtains associated to D the graded division algebra $\mathsf{gr}(D) = \bigoplus_{\gamma\in\Gamma_D} D_\gamma$. Further, suppose D is tame with respect to v. This implies that $[\mathsf{gr}(D) : \mathsf{gr}(K)] = [D : K]$, $\mathsf{gr}(K) = Z(\mathsf{gr}(D))$ and D has a maximal subfield L with L tamely ramified over K ([HW_2, Prop. 4.3]). We can then associate to an involution τ on D, a graded involution $\widetilde{\tau}$ on $\mathsf{gr}(D)$. First, suppose τ is of the first kind on D. Then $v \circ \tau$ is also a valuation on D which restricts to v on K; then, $v \circ \tau = v$ since v has a unique extension to D. So, τ induces a well-defined map $\widetilde{\tau}\colon \mathsf{gr}(D) \to \mathsf{gr}(D)$, defined on homogeneous elements by $\widetilde{\tau}(\widetilde{a}) = \widetilde{\tau(a)}$ for all $a \in D^*$. Clearly, $\widetilde{\tau}$ is a well-defined graded involution on $\mathsf{gr}(D)$; it is of the first kind, as it leaves $Z(\mathsf{gr}(D)) = \mathsf{gr}(K)$ invariant.

If τ is a unitary involution on D, let $F = K^\tau$. In this case, we need to assume that the restriction of the valuation v from K to F induces a henselian valuation on F, and that K is tamely ramified over F. Since $(v \circ \tau)|_F = v|_F$, an argument similar to the one above shows that $v \circ \tau$ coincides with v on K and thus on D, and the induced map $\widetilde{\tau}$ on $\mathsf{gr}(D)$ as above is a graded involution. That K is tamely ramified over F means that $[K : F] = [\mathsf{gr}(K) : \mathsf{gr}(F)]$, $\overline{K}$ is separable over $\overline{F}$, and $\operatorname{char}(\overline{F}) \nmid |\Gamma_K : \Gamma_F|$. Since $[K : F] = 2$, K is always tamely ramified over F if $\operatorname{char}(\overline{F}) \neq 2$. But if $\operatorname{char}(\overline{F}) = 2$, K is tamely ramified over F if and only if $[\overline{K} : \overline{F}] = 2$, $\Gamma_K = \Gamma_F$, and $\overline{K}$ is separable (so Galois) over $\overline{F}$. Since K is Galois over F, the canonical map $\mathrm{Gal}(K/F) \to \mathrm{Gal}(\overline{K}/\overline{F})$ is surjective, by [EP, pp. 123–124, proof of Lemma 5.2.6(1)]. Hence,

τ induces the nonidentity $\overline{F}$-automorphism $\overline{\tau}$ of $\overline{K}$. Also $\widetilde{\tau}$ is unitary, i.e., $\widetilde{\tau}|_{\mathsf{gr}(K)} \neq \mathrm{id}$. This is obvious if $\mathrm{char}(\overline{F}) \neq 2$, since then $K = F(\sqrt{c})$ for some $c \in F^*$, and $\widetilde{\tau}(\widetilde{\sqrt{c}}) = \widetilde{\tau(\sqrt{c})} = -\widetilde{\sqrt{c}} \neq \widetilde{\sqrt{c}}$. If $\mathrm{char}(\overline{F}) = 2$, then K is unramified over F and $\widetilde{\tau}|_{\mathsf{gr}(K)_0} = \overline{\tau}$ (the automorphism of $\overline{K}$ induced by $\tau|_K$) which is nontrivial as $\mathrm{Gal}(K/F)$ maps onto $\mathrm{Gal}(\overline{K}/\overline{F})$; so again $\widetilde{\tau}|_{\mathsf{gr}(K)} \neq \mathrm{id}$. Thus, $\widetilde{\tau}$ is a unitary graded involution in any characteristic. Moreover, for the graded fixed field $\mathsf{gr}(K)^{\widetilde{\tau}}$ we have $\mathsf{gr}(F) \subseteq \mathsf{gr}(K)^{\widetilde{\tau}} \subsetneqq \mathsf{gr}(K)$ and $[\mathsf{gr}(K) : \mathsf{gr}(F)] = 2$, so $\mathsf{gr}(K)^{\widetilde{\tau}} = \mathsf{gr}(F)$.

Theorem 3.2. *Let (D, v) be a tame valued division algebra over a henselian field K, with $\mathrm{char}(\overline{K}) \neq 2$. If τ is an involution of the first kind on D, then*

$$K_1(D, \tau) \;\cong\; K_1(\mathsf{gr}(D), \widetilde{\tau}),$$

and if τ is symplectic, then

$$K_1 \operatorname{Spin}(D, \tau) \;\cong\; K_1 \operatorname{Spin}(\mathsf{gr}(D), \widetilde{\tau}).$$

Proof. Let $\rho\colon D^* \to \mathsf{gr}(D)^*$ be the group epimorphism given in (3.2). Clearly $\rho(S_\tau(D)) \subseteq S_{\widetilde{\tau}}(\mathsf{gr}(D))$, so $\rho(\Sigma_\tau(D)) \subseteq \Sigma_{\widetilde{\tau}}(\mathsf{gr}(D))$. Consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & (1+M_D)\cap\Sigma_\tau(D)D' \arrow[r] \arrow[d] & \Sigma_\tau(D)D' \arrow[r, "\rho"] \arrow[d] & \Sigma_{\widetilde{\tau}}(\mathsf{gr}(D))\,\mathsf{gr}(D)' \arrow[r, dotted] \arrow[d] & 1 \\
1 \arrow[r] & (1+M_D) \arrow[r] & D^* \arrow[r, "\rho"] & \mathsf{gr}(D)^* \arrow[r] & 1.
\end{tikzcd} \tag{3.4}$$

The top row of the diagram is exact. To see this, note that $\rho(D') = \mathsf{gr}(D)'$. Thus, it suffices to show that ρ maps $S_\tau(D) \cap D^*$ onto $S_{\widetilde{\tau}}(\mathsf{gr}(D)) \cap \mathsf{gr}(D)^*$. For this, take any $d \in D^*$ with $\widetilde{d} = \widetilde{\tau}(\widetilde{d})$. Let $b = \frac{1}{2}(d + \tau(d)) \in S_\tau(D)$. Since $v(b) = v(\tau(b))$ and $\widetilde{d} + \widetilde{\tau(d)} = 2\widetilde{d} \neq 0$, $\widetilde{b} = \frac{1}{2}\widetilde{(d + \tau(d))} = \frac{1}{2}(\widetilde{d} + \widetilde{\tau(d)}) = \widetilde{d}$. Since τ on D is an involution of the first kind, the index of D is a power of 2 ([D, Th. 1, §16]). As $\mathrm{char}(\overline{K}) \neq 2$, it follows that the valuation is strongly tame, and by [Ha, Lemma 2.1],

$$1 + M_D \;=\; (1 + M_K)[D^*, 1 + M_D] \;\subseteq\; \Sigma_\tau(D)D'.$$

Therefore, the left vertical map is the identity map. It follows (for example using the snake lemma) that $K_1(D, \tau) \cong K_1(\mathsf{gr}(D), \widetilde{\tau})$. The proof for $K_1 \operatorname{Spin}$ when τ is of symplectic type is similar. □

The key to proving the corresponding result for unitary involutions is the Congruence Theorem:

Theorem 3.3 (Congruence Theorem)**.** *Let D be a tame division algebra over a field K with henselian valuation v. Let $D^{(1)} = \{a \in D^* \mid \mathrm{Nrd}_D(a) = 1\}$. Then,*

$$D^{(1)} \cap (1 + M_D) \;\subseteq\; [D^*, D^*].$$

This theorem was proved by Platonov in [P_2] for v a complete discrete valuation, and it was an essential tool in all his calculations of SK_1 for division rings. The Congruence Theorem was asserted by Ershov in [E] in the generality given here. A full proof is given in [HaW, Th. B.1].

Proposition 3.4 (Unitary Congruence Theorem)**.** *Let D be a tame division algebra over a field K with henselian valuation v, and let τ be a unitary involution on D. Let $F = K^\tau$. If F is henselian with respect to $v|_F$ and K is tamely ramified over F, then*

$$(1 + M_D) \cap \Sigma'_\tau(D) \;\subseteq\; \Sigma_\tau(D).$$

Proof. The only published proof of this we know is [Y$_2$, Th. 4.9], which is just for the case v discrete rank 1; that proof is rather hard to follow, and appears to apply for other valuations only if D is inertially split. Here we provide another proof, in full generality.

We use the well-known facts that

$$\mathrm{Nrd}_D(1+M_D) \;=\; 1+M_K \quad\text{and}\quad N_{K/F}(1+M_K) \;=\; 1+M_F. \tag{3.5}$$

(The second equation holds as K is tamely ramified over F.) See [E, Prop. 2] or [HaW, Prop. 4.6, Cor. 4.7] for a proof.

Now, take $m \in M_D$ with $\mathrm{Nrd}_D(1+m) \in F$. Then $\mathrm{Nrd}_D(1+m) \in F \cap (1+M_K) = 1+M_F$. By (3.5) there is $c \in 1+M_K$ with $\mathrm{Nrd}_D(1+m) = N_{K/F}(c) = c\tau(c)$, and there is $b \in 1+M_D$ with $\mathrm{Nrd}_D(b) = c$. Then,

$$\mathrm{Nrd}_D(b\tau(b)) \;=\; c\tau(c) \;=\; N_{K/F}(c) \;=\; \mathrm{Nrd}_D(1+m).$$

Let $s = (1+m)(b\tau(b))^{-1} \in 1+M_D$. Since $\mathrm{Nrd}_D(s) = 1$, by the Congruence Theorem for SK_1, Th. 3.3 above, $s \in [D^*, D^*] \subseteq \Sigma_\tau(D)$, (recall (2.10)) . Since $b\tau(b) \in S_\tau(D)$, we have $1+m = s(b\tau(b)) \in \Sigma_\tau(D)$. $\square$

Theorem 3.5. *Let D be a tame division algebra over a field K with henselian valuation v. Let τ be a unitary involution on D, and let $F = K^\tau$. If F is henselian with respect to $v|_F$ and K is tamely ramified over F, then τ induces a unitary graded involution $\widetilde{\tau}$ of $\mathsf{gr}(D)$ with $\mathsf{gr}(F) = \mathsf{gr}(K)^{\widetilde{\tau}}$, and*

$$\mathrm{SK}_1(D,\tau) \;\cong\; \mathrm{SK}_1(\mathsf{gr}(D),\widetilde{\tau}).$$

Proof. That $\widetilde{\tau}$ is a unitary graded involution on $\mathsf{gr}(D)$ and $\mathsf{gr}(F) = \mathsf{gr}(K)^{\widetilde{\tau}}$ was already observed (see the discussion before Th. 3.2). For the canonical epimorphism $\rho\colon D^* \to \mathsf{gr}(D)^*$, $a \mapsto \widetilde{a}$, it follows from (3.3) that $\rho(\Sigma'_\tau(D) \subseteq \Sigma'_{\widetilde{\tau}}(\mathsf{gr}(D))$. Also, clearly $\rho(S_\tau(D)) \subseteq S_{\widetilde{\tau}}(\mathsf{gr}(D))$, so $\rho(\Sigma_\tau(D)) \subseteq \Sigma_{\widetilde{\tau}}(\mathsf{gr}(D))$. Thus, there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & (1+M_D)\cap\Sigma_\tau(D) \arrow[r] \arrow[d] & \Sigma_\tau(D) \arrow[r, "\rho"] \arrow[d] & \Sigma_{\widetilde{\tau}}(\mathsf{gr}(D)) \arrow[r, dotted] \arrow[d] & 1 \\
1 \arrow[r] & (1+M_D)\cap\Sigma'_\tau(D) \arrow[r] & \Sigma'_\tau(D) \arrow[r, "\rho"] & \Sigma'_{\widetilde{\tau}}(\mathsf{gr}(D)) \arrow[r, dotted] & 1,
\end{tikzcd} \tag{3.6}$$

where the vertical maps are inclusions, and the left vertical map is bijective, by Prop. 3.4 above.

To see that the bottom row of diagram (3.6) is exact at $\Sigma'_{\widetilde{\tau}}(\mathsf{gr}(D))$, take $b \in D$ with $\mathrm{Nrd}_{\mathsf{gr}(D)}(\widetilde{b}) \in \mathsf{gr}(F)$. Let $c = \mathrm{Nrd}_D(b) \in K^*$. Then $\widetilde{c} = \mathrm{Nrd}_{\mathsf{gr}(D)}(\widetilde{b}) \in \mathsf{gr}(F)$, so $\widetilde{c} = \widetilde{t}$ for some $t \in F^*$. Let $u = c^{-1}t \in 1+M_K$. By (3.5) above, there is $d \in 1+M_D$ with $\mathrm{Nrd}_D(d) = u$. So, $\mathrm{Nrd}_D(bd) = cu = t \in F^*$. Thus, $bd \in \Sigma'_\tau(D)$ and $\rho(bd) = \widetilde{bd} = \widetilde{b}$. This gives the claimed exactness, and shows that the bottom row of diagram (3.6) is exact.

To see that the top row of diagram (3.6) is exact at $\Sigma_{\widetilde{\tau}}(\mathsf{gr}(D))$, it suffices to show that ρ maps $S_\tau(D) \cap D^*$ onto $S_{\widetilde{\tau}}(\mathsf{gr}(D)) \cap \mathsf{gr}(D)^*$. For this, take any $d \in D^*$ with $\widetilde{d} = \widetilde{\tau}(\widetilde{d})$. If $\mathrm{char}(\overline{F}) \neq 2$, as in the proof of Th. 3.2, let $b = \frac{1}{2}(d+\tau(d)) \in S_\tau(D)$. Since $v(b) = v(\tau(b))$ and $\widetilde{d} + \widetilde{\tau(d)} = 2\widetilde{d} \neq 0$, we have $\widetilde{b} = \frac{1}{2}\widetilde{(d+\tau(d))} = \frac{1}{2}(\widetilde{d} + \widetilde{\tau(d)}) = \widetilde{d}$. If $\mathrm{char}(\overline{F}) = 2$, then K is unramified over F, so $\overline{K}$ is Galois over $\overline{F}$ with $[\overline{K} : \overline{F}] = 2$, and the map $\overline{\tau}\colon \overline{K} \to \overline{K}$ induced by τ is the nonidentity $\overline{F}$-automorphism of $\overline{K}$. Of course, $\overline{K} = \mathsf{gr}(K)_0$ and $\overline{\tau} = \widetilde{\tau}|_{\mathsf{gr}(K)_0}$. Because $\overline{K}$ is separable over $\overline{F}$, the trace $\mathrm{tr}_{\overline{K}/\overline{F}}$ is surjective, so there is $r \in V_K$ with $\widetilde{r} + \widetilde{\tau}(\widetilde{r}) = 1 \in \mathsf{gr}(F)_0$. Let $c = rd + \tau(rd) \in S_\tau(D)$. We have $\widetilde{rd} = \widetilde{r}\widetilde{d}$ and

$$\widetilde{\tau(rd)} \;=\; \widetilde{\tau}(\widetilde{rd}) \;=\; \widetilde{\tau}(\widetilde{r}\widetilde{d}) \;=\; \widetilde{\tau}(\widetilde{d})\widetilde{\tau}(\widetilde{r}) \;=\; \widetilde{\tau}(\widetilde{r})\widetilde{d}.$$

Since $v(rd) = v(\tau(rd))$ and $\widetilde{rd} + \widetilde{\tau(rd)} = \widetilde{r}\widetilde{d} + \widetilde{\tau}(\widetilde{r})\widetilde{d} = \widetilde{d} \neq 0$, we have $\widetilde{c} = \widetilde{rd} + \widetilde{\tau(rd)} = \widetilde{d}$. So, in all cases $\rho(S_\tau(D) \cap D^*) = S_{\widetilde{\tau}}(\mathsf{gr}(D)) \cap \mathsf{gr}(D)^*$, from which it follows that the bottom row of diagram (3.6) is exact.

Since each row of (3.6) is exact, we have a right exact sequence of cokernels of the vertical maps, which yields the isomorphism of the theorem. □

Having established the bridge between the unitary K-groups in the graded setting and the non-graded henselian case (Th. 3.2, Th. 3.5), we can deduce known formulas in the literature for the unitary Whitehead group of certain valued division algebras, by passing to the graded setting. The proofs are much easier than those previously available. We will do this systematically for unitary involutions in Section 4. Before we turn to that, here is an example with an involution of the first kind:

Example 3.6. Let E be a graded division algebra over its center T with an involution τ of the first kind. If E is unramified over T, then, by using $\mathsf{E}^* = \mathsf{E}_0^*\mathsf{T}^*$, it follows easily that

$$K_1(\mathsf{E},\tau) \;\cong\; K_1(\mathsf{E}_0, \tau|_{\mathsf{E}_0}), \tag{3.7}$$

and, if $\mathrm{char}(\mathsf{E}) \neq 2$ and τ is symplectic,

$$K_1\,\mathrm{Spin}(\mathsf{E},\tau) \;\cong\; K_1\,\mathrm{Spin}(\mathsf{E}_0, \tau|_{\mathsf{E}_0}). \tag{3.8}$$

Now if D is a tame and unramified division algebra over a henselian valued field and D has an involution τ of the first kind, then the associated graded division ring $\mathsf{gr}(D)$ is also unramified with the corresponding graded involution $\widetilde{\tau}$ of the first kind; then Th. 3.2 and (3.7) above show that

$$K_1(D,\tau) \;\cong\; K_1(\mathsf{gr}(D),\widetilde{\tau}) \;\cong\; K_1(\mathsf{gr}(D)_0, \tau|_{\mathsf{gr}(D)_0}) \;=\; K_1(\overline{D},\overline{\tau}),$$

yielding a theorem of Platonov-Yanchevskiĭ [PY, Th. 5.11] (that $K_1(D,\tau) \cong K_1(\overline{D},\overline{\tau})$ when D is unramified over K and the valuation is henselian and discrete rank 1.) Similarly, when $\mathrm{char}(\overline{D}) \neq 2$ and τ is symplectic,

$$K_1\,\mathrm{Spin}(D,\tau) \;\cong\; K_1\,\mathrm{Spin}(\mathsf{gr}(D),\widetilde{\tau}) \;\cong\; K_1\,\mathrm{Spin}(\mathsf{gr}(D)_0, \tau|_{\mathsf{gr}(D)_0}) \;=\; K_1\,\mathrm{Spin}(\overline{D},\overline{\tau}).$$

Remark 3.7. We have the following commutative diagram connecting unitary SK_1 to non-unitary SK_1, where $\mathrm{SH}^0(D,\tau)$ and $\mathrm{SH}^0(D)$ are the cokernels of $\mathrm{Nrd}\circ(1-\tau)$ and Nrd respectively (see diagram (2.8)).

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{SK}_1(D,\tau) \arrow[r] \arrow[d] & D^*/\Sigma(D) \arrow[r, "\mathrm{Nrd}\circ(1-\tau)"] \arrow[d, "1-\tau"] & K^* \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{SH}^0(D,\tau) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{SK}_1(D) \arrow[r] & D^*/D' \arrow[r, "\mathrm{Nrd}"] & K^* \arrow[r] & \mathrm{SH}^0(D) \arrow[r] & 1.
\end{tikzcd} \tag{3.9}$$

Now, let D be a tame valued division algebra with center K and with a unitary involution τ, such that the valuation restricts to a henselian valuation on $F = K^\tau$. By Th. 3.5, $\mathrm{SK}_1(D,\tau) \cong \mathrm{SK}_1(\mathsf{gr}(D),\widetilde{\tau})$ and by [HaW, Th. 4.8, Th. 4.12], $\mathrm{SK}_1(D) \cong \mathrm{SK}_1(\mathsf{gr}(D))$ and $\mathrm{SH}^0(D) \cong \mathrm{SH}^0(\mathsf{gr}(D))$. However, $\mathrm{SH}^0(D,\tau)$ is not stable under "valued filtration", i.e., $\mathrm{SH}^0(D,\tau) \ncong \mathrm{SH}^0(\mathsf{gr}(D),\widetilde{\tau})$. In fact using (3.2), we can build a commutative diagram with exact rows,

$$\begin{tikzcd}
1 \arrow[r] & (1+M_K)\cap \mathrm{Nrd}(D^*)^{1-\tau} \arrow[r] \arrow[d, hook] & \mathrm{Nrd}(D^*)^{1-\tau} \arrow[r] \arrow[d, hook] & \mathrm{Nrd}\big(\mathsf{gr}(D)^*\big)^{1-\widetilde{\tau}} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & 1+M_K \arrow[r] & K^* \arrow[r] & \mathsf{gr}(K)^* \arrow[r] & 1,
\end{tikzcd}$$

which induces the exact sequence

$$1 \longrightarrow (1+M_K)/\big((1+M_K)\cap \mathrm{Nrd}(D^*)^{1-\tau}\big) \longrightarrow \mathrm{SH}^0(D,\tau) \longrightarrow \mathrm{SH}^0(\mathsf{gr}(D),\widetilde{\tau}) \longrightarrow 1.$$

By considering the norm $N_{K/F}\colon K^* \to F^*$, we clearly have $\mathrm{Nrd}(D^*)^{1-\tau} \subseteq \ker N_{K/F}$. However, by (3.5), $N_{K/F}\colon 1+M_K \to 1+M_F$ is surjective, which shows that $1+M_K/\big((1+M_K)\cap\mathrm{Nrd}(D^*)^{1-\tau}\big)$ is not trivial and thus $\mathrm{SH}^0(D,\tau) \ncong \mathrm{SH}^0(\mathsf{gr}(D),\widetilde{\tau})$.

4. Graded Unitary SK_1 Calculus

Let E be a graded division algebra over its center T with a unitary graded involution τ, and let $\mathsf{R} = \mathsf{T}^\tau$. Since $[\mathsf{T} : \mathsf{R}] = 2 = [\mathsf{T}_0 : \mathsf{R}_0]\, |\Gamma_{\mathsf{T}} : \Gamma_{\mathsf{R}}|$, there are just two possible cases:

- T is totally ramified over R, i.e., $|\Gamma_{\mathsf{T}} : \Gamma_{\mathsf{R}}| = 2$
- T is unramfied over R, i.e., $|\Gamma_{\mathsf{T}} : \Gamma_{\mathsf{R}}| = 1$.

We will consider $\mathrm{SK}_1(\mathsf{E}, \tau)$ in these two cases separately in §4.1 and §4.2.

The following notation will be used throughout this section and the next: Let τ' be another involution on E. We write $\tau' \sim \tau$ if $\tau'|_{Z(\mathsf{E})} = \tau|_{Z(\mathsf{E})}$. For $t \in \mathsf{E}^*$, let φ_t denote the map from E to E given by conjugation by t, i.e., $\varphi_t(x) = txt^{-1}$. Let $\Sigma_0 = \Sigma_\tau \cap \mathsf{E}_0^*$ and $\Sigma_0' = \Sigma_\tau' \cap \mathsf{E}_0^*$.

We first collect some facts which will be used below. They all follow by easy calculations.

Remarks 4.1.

(i) We have $\tau' \sim \tau$ if and only if there is a $t \in \mathsf{E}^*$ with $\tau(t) = t$ and $\tau' = \tau\varphi_t$. (The proof is analogous to the ungraded version given, e.g. in [KMRT, Prop. 2.18].)

(ii) If $\tau' \sim \tau$, then $\Sigma_{\tau'} = \Sigma_\tau$ and $\Sigma_{\tau'}' = \Sigma_\tau'$; thus $\mathrm{SK}_1(\mathsf{E}, \tau') = \mathrm{SK}_1(\mathsf{E}, \tau)$. (See [$\mathrm{Y}_1$, Lemma 1] for the analogous ungraded result.)

(iii) For any $s \in \mathsf{E}^*$, we have $\tau\varphi_s = \varphi_{\tau(s)^{-1}}\tau$. Hence, $\tau\varphi_s$ is an involution (necessarily $\sim \tau$) if and only if $\tau\varphi_s = \varphi_{s^{-1}}\tau$ if and only if $\tau(s)/s \in \mathsf{T}$.

(iv) If $s \in \mathsf{E}_\gamma^*$ and $\tau(s) = s$, then $\Sigma_\tau' \cap \mathsf{E}_\gamma = s\Sigma_0'$ and $S_\tau \cap \mathsf{E}_\gamma = s(S_{\tau_s} \cap \mathsf{E}_0)$ where $\tau_s = \tau\varphi_s$.

4.1. T/R **totally ramified.** Let E be a graded division algebra with a unitary graded involution τ such that $\mathsf{T} = Z(\mathsf{E})$ is totally ramified over $\mathsf{R} = \mathsf{T}^\tau$. In this section we will show that $\mathrm{SK}_1(\mathsf{E}, \tau) = 1$. Note that the assumption that T/R is totally ramified implies that $\mathrm{char}(\mathsf{T}) \neq 2$. For, if $\mathrm{char}(\mathsf{T}) = 2$ and T is totally ramified over a graded subfield R with $[\mathsf{T} : \mathsf{R}] = 2$, then for any $x \in \mathsf{T}^* \backslash \mathsf{R}^*$, we have $\deg(x^2) \in \Gamma_R$, so $x^2 \in \mathsf{R}$; thus, T is purely inseparable over R. That cannot happen here, as $\tau|_{\mathsf{T}}$ is a nontrivial R-automorphism of T.

Lemma 4.2. *If T is totally ramified over R, then $\tau \sim \tau'$ for some graded involution τ', where $\tau'|_{\mathsf{E}_0}$ is of the first kind.*

Proof. Let $Z_0 = Z(\mathsf{E}_0)$. Since T is totally ramified over R, $\mathsf{T}_0 = \mathsf{R}_0$, so $\tau|_{Z_0} \in \mathrm{Gal}(Z_0/\mathsf{T}_0)$. Since the map $\Theta_{\mathsf{E}} \colon \Gamma_{\mathsf{E}} \to \mathrm{Gal}(Z_0/\mathsf{T}_0)$ is surjective (see (2.3)), there is $\gamma \in \Gamma_{\mathsf{E}}$ with $\Theta_{\mathsf{E}}(\gamma) = \tau|_{Z_0}$. Choose $y \in \mathsf{E}_\gamma^*$ with $\tau(y) = \pm y$. Then set $\tau' = \tau\varphi_{y^{-1}}$. □

Example 4.3. Here is a construction of examples of graded division algebras E with unitary graded involution τ with E totally ramified over $Z(\mathsf{E})^\tau$. We will see below that these are all such examples. Let R be any graded field with $\mathrm{char}(\mathsf{R}) \neq 2$, and let A be a graded division algebra with center R, such that A is totally ramified over R with $\exp(\Gamma_{\mathsf{A}}/\Gamma_{\mathsf{R}}) = 2$. Let T be a graded field extension of R with $[\mathsf{T} : \mathsf{R}] = 2$, T totally ramified over R, and $\Gamma_{\mathsf{T}} \cap \Gamma_{\mathsf{A}} = \Gamma_{\mathsf{R}}$. Let $\mathsf{E} = \mathsf{A} \otimes_{\mathsf{R}} \mathsf{T}$, which is a graded central simple algebra over T, as A is graded central simple over R, by [HW_2, Prop. 1.1]. But because $\Gamma_{\mathsf{T}} \cap \Gamma_{\mathsf{A}} = \Gamma_{\mathsf{R}}$, we have $\mathsf{E}_0 = \mathsf{A}_0 \otimes_{\mathsf{R}_0} \mathsf{T}_0 = \mathsf{R}_0 \otimes_{\mathsf{R}_0} \mathsf{R}_0 = \mathsf{R}_0$. Since E_0 is a division ring, E must be a graded division ring, which is totally ramified over R, as $\mathsf{E}_0 = \mathsf{R}_0$. Now, because A is totally ramified over R, we have $\exp(\mathsf{A}) = \exp(\Gamma_{\mathsf{A}}/\Gamma_{\mathsf{R}}) = 2$, and $\mathsf{A} = \mathsf{Q}_1 \otimes_{\mathsf{R}} \ldots \otimes_{\mathsf{R}} \mathsf{Q}_m$, where each Q_i is a graded symbol algebra of degree at most 2, i.e., a graded quaternion algebra. Let σ_i be a graded involution of the first kind on Q_i (e.g., the canonical symplectic graded involution), and let ρ be the nonidentity R-automorphism of T. Then, $\sigma = \sigma_1 \otimes \ldots \otimes \sigma_m$ is a graded involution of the first kind on A, so $\sigma \otimes \rho$ is a unitary graded involution on E, with $\mathsf{T}^\tau = \mathsf{R}$.

Proposition 4.4. *If* E *is totally ramified over* R*, and* $\mathsf{E} \neq \mathsf{T}$*, then* $\Sigma_\tau = \mathsf{E}^*$*, so* $\mathrm{SK}_1(\mathsf{E},\tau) = 1$*. Furthermore,* E *and* τ *are as described in Ex. 4.3.*

Proof. We have $\mathsf{E}_0 = \mathsf{T}_0 = \mathsf{R}_0$. For any $\gamma \in \Gamma_\mathsf{E}$, there is a nonzero $a \in \mathsf{E}_\gamma$ with $\tau(a) = \epsilon a$ where $\epsilon = \pm 1$. Then, for any $b \in \mathsf{E}_\gamma$, $b = ra$ for some $r \in \mathsf{E}_0 = \mathsf{R}_0$. Since r is central and symmetric, $\tau(b) = \epsilon b$. Thus, every element of E^* is symmetric or skew-symmetric. Indeed, fix any $t \in \mathsf{T}^* \setminus \mathsf{R}^*$. Then $\tau(t) \neq t$, as $t \notin \mathsf{R}^*$. Hence, $\tau(t) = -t$. Since t is central and skew-symmetric, every $a \in \mathsf{E}^*$ is symmetric iff ta is skew-symmetric. Thus, $\mathsf{E}^* = S_\tau^* \cup tS_\tau^*$. To see that $\Sigma_\tau = \mathsf{E}^*$, it suffices to show that $t \in \Sigma_\tau$. To see this, take any $c, d \in \mathsf{E}^*$ with $dc \neq cd$. (They exist, as $\mathsf{E} \neq \mathsf{T}$.) By replacing c (resp. d) if necessary by tc (resp. td), we may assume that $\tau(c) = c$ and $\tau(d) = d$. Then, $dc = \tau(cd) = \epsilon cd$, where $\epsilon = \pm 1$; since $dc \neq cd$, $\epsilon = -1$; hence $\tau(tcd) = tcd$. Thus, $t = (tcd)c^{-1}d^{-1} \in \Sigma_\tau(\mathsf{E})$, completing the proof that $\Sigma_\tau(\mathsf{E}) = \mathsf{E}^*$.

For $\gamma \in \Gamma_\mathsf{E}$, let $\overline{\gamma} = \gamma + \Gamma_\mathsf{T} \in \Gamma_\mathsf{E}/\Gamma_\mathsf{T}$. To see the structure of E, recall that as E is totally ramified over T there is a well-defined nondegenerate $\mathbb{Z}$-bilinear symplectic pairing $\beta\colon (\Gamma_\mathsf{E}/\Gamma_\mathsf{T}) \times \Gamma_\mathsf{E}/\Gamma_\mathsf{T}) \to \mathsf{E}_0^*$ given by $\beta(\overline{\gamma}, \overline{\delta}) = y_\gamma y_\delta y_\gamma^{-1} y_\delta^{-1}$ for any nonzero $y_\gamma \in \mathsf{E}_\gamma$, $y_\delta \in \mathsf{E}_\delta$. The computation above for c and d shows that $\mathrm{im}(\beta) = \{\pm 1\}$. Since the pairing β is nondegenerate by [HW$_2$, Prop. 2.1] there is a symplectic base of $\Gamma_\mathsf{E}/\Gamma_\mathsf{T}$, i.e., a subset $\{\overline{\gamma}_1, \overline{\delta}_1, \ldots, \overline{\gamma}_m, \overline{\delta}_m\}$ of $\Gamma_\mathsf{E}/\Gamma_\mathsf{T}$ such that $\beta(\overline{\gamma}_i, \overline{\delta}_i) = -1$ while $\beta(\overline{\gamma}_i, \overline{\gamma}_j) = \beta(\overline{\delta}_i, \overline{\delta}_j) = 1$ for all i, j, and $\beta(\overline{\gamma}_i, \overline{\delta}_j) = 1$ whenever $i \neq j$, and $\Gamma_\mathsf{E} = \langle \gamma_1, \delta_1, \ldots, \gamma_m, \delta_m \rangle + \Gamma_\mathsf{T}$. Choose any nonzero $\mathbf{i}_i \in \mathsf{E}_{\gamma_i}$ and $\mathbf{j}_i \in \mathsf{E}_{\delta_i}$. The properties of the $\overline{\gamma}_i, \overline{\delta}_i$ under β translate to: $\mathbf{i}_i\mathbf{j}_i = -\mathbf{j}_i\mathbf{i}_i$ while $\mathbf{i}_i\mathbf{i}_j = \mathbf{i}_j\mathbf{i}_i$ and $\mathbf{j}_i\mathbf{j}_j = \mathbf{j}_j\mathbf{j}_i$ for all i, j, and $\mathbf{i}_i\mathbf{j}_j = \mathbf{j}_j\mathbf{i}_i$ whenever $i \neq j$. Since $\beta(\overline{2\gamma_i}, \overline{\eta}) = 1$ for all i and all $\eta \in \Gamma_\mathsf{E}$, each $\mathbf{i}_i^2$ is central in E. But also $\tau(\mathbf{i}_i^2) = \mathbf{i}_i^2$, as $\tau(\mathbf{i}_i) = \pm\mathbf{i}_i$. So, each $\mathbf{i}_i^2 \in \mathsf{R}^*$, and likewise each $\mathbf{j}_i^2 \in \mathsf{R}^*$. Let $\mathsf{Q}_i = \mathsf{R}\text{-span}\{1, \mathbf{i}_i, \mathbf{j}_i, \mathbf{i}_i\mathbf{j}_i\}$ in E. The relations on the $\mathbf{i}_i, \mathbf{j}_i$ show that each Q_i is a graded quaternion algebra over R, and the distinct Q_i centralize each other in E. Since each Q_i is graded central simple over R, $\mathsf{Q}_1 \otimes_\mathsf{R} \ldots \otimes_\mathsf{R} \mathsf{Q}_m$ is graded central simple over R by [HW$_2$, Prop. 1.1]. Let $\mathsf{A} = \mathsf{Q}_1 \ldots \mathsf{Q}_m \subseteq \mathsf{E}$. The graded R-algebra epimorphism $\mathsf{Q}_1 \otimes_\mathsf{R} \ldots \otimes_\mathsf{R} \mathsf{Q}_m \to \mathsf{A}$ must be an isomorphism, as the domain is graded simple. If $\Gamma_\mathsf{T} \subseteq \Gamma_\mathsf{A}$, then $\mathsf{T} \subseteq \mathsf{A}$, since E is totally ramified over R. But this cannot occur, as T centralizes A but $\mathsf{T} \supsetneqq \mathsf{R} = Z(\mathsf{A})$. Hence, as $|\Gamma_\mathsf{T} : \Gamma_\mathsf{R}| = 2$, we must have $\Gamma_\mathsf{T} \cap \Gamma_\mathsf{A} = \Gamma_\mathsf{R}$. The graded R-algebra homomorphism $\mathsf{A} \otimes_\mathsf{R} \mathsf{T} \to \mathsf{E}$ is injective since its domain is graded simple, by [HW$_2$, Prop. 1.1]; it is also surjective, since $\mathsf{E}_0 = \mathsf{R}_0 \subseteq \mathsf{A} \otimes_\mathsf{R} \mathsf{T}$ and $\Gamma_{\mathsf{A}\otimes_\mathsf{R}\mathsf{T}} \supseteq \langle \gamma_1, \delta_1, \ldots, \gamma_m, \delta_m \rangle + \Gamma_\mathsf{T} = \Gamma_\mathsf{E}$. Clearly, $\tau = \tau|_\mathsf{A} \otimes \tau|_\mathsf{T}$. □

Proposition 4.5. *If* $\mathsf{E} \neq \mathsf{T}$ *and* T *is totally ramified over* R*, then* $\Sigma_\tau = \mathsf{E}^*$*, so* $\mathrm{SK}_1(\mathsf{E},\tau) = 1$.

Proof. The case where $\mathsf{E}_0 = \mathsf{T}_0$ was covered by Prop. 4.4. Thus, we may assume that $\mathsf{E}_0 \supsetneqq \mathsf{T}_0$. By Lemma 4.2 and Remark 4.1(ii), we can assume that $\tau|_{\mathsf{E}_0}$ is of the first kind. Further, we can assume that $\mathsf{E}_0^* = \Sigma_{\tau|_{\mathsf{E}_0}}(\mathsf{E}_0)$. For, if $\tau|_{\mathsf{E}_0}$ is symplectic, take any $a \in \mathsf{E}_0^*$ with $\tau(a) = -a$, and let $\tau' = \tau\varphi_a$. Then, $\tau' \sim \tau$ (see Remark 4.1(iii)). Also, $\tau'|_{Z(\mathsf{E}_0)} = \tau|_{Z(\mathsf{E}_0)}$, as $a \in \mathsf{E}_0$ and so $\varphi_a|_{Z(\mathsf{E}_0)} = \mathrm{id}$. Therefore, $\tau'|_{\mathsf{E}_0}$ is of the first kind. But as $\tau(a) = -a$, $\tau'|_{\mathsf{E}_0}$ is orthogonal. Thus $\mathsf{E}_0^* = \Sigma_{\tau'|_{\mathsf{E}_0}}(\mathsf{E}_0)$, as noted at the beginning of §2.2.1. Now replace τ by τ'.

We consider two cases.

Case I. Suppose for each $\gamma \in \Gamma_\mathsf{E}$ there is $x_\gamma \in \mathsf{E}_\gamma^*$ such that $\tau(x_\gamma) = x_\gamma$. Then, $\mathsf{E}^* = \bigcup_{\gamma\in\Gamma_\mathsf{E}} \mathsf{E}_0^* x_\gamma \subseteq \Sigma_\tau(\mathsf{E})$, as desired.

Case II. Suppose there is $\gamma \in \Gamma_\mathsf{E}$ with $\mathsf{E}_\gamma \cap S_\tau = 0$. Then $\tau(d) = -d$ for each $d \in \mathsf{E}_\gamma$. Fix $t \in \mathsf{E}_\gamma^*$. For any $a \in \mathsf{E}_0$, we have $ta \in \mathsf{E}_\gamma$; so, $-ta = \tau(ta) = \tau(a)\tau(t) = -\tau(a)t$. That is,

$$\tau(a) \;=\; \varphi_t(a) \quad \text{for all } a \in \mathsf{E}_0. \tag{4.1}$$

Let $\tau'' = \tau\varphi_t$, which is a unitary involution on E with $\tau'' \sim \tau$ (see Remark 4.1(iii)). But, $\tau''(a) = a$ for all $a \in \mathsf{E}_0$, i.e., $\tau''|_{\mathsf{E}_0} = \mathrm{id}$. This implies that E_0 is a field. Replace τ by τ''. The rest of the argument uses this new τ. So $\tau|_{\mathsf{E}_0} = \mathrm{id}$. If we are now in Case I for this τ, then we are done by Case I. So, assume we are in Case II. Take any $\gamma \in \Gamma_\mathsf{E}$ with $\mathsf{E}_\gamma \cap S_\tau = 0$. For any nonzero $t \in \mathsf{E}_\gamma$, equation (4.1) applies to t, showing $\varphi_t(a) = \tau(a) = a$ for all $a \in \mathsf{E}_0$; hence for the map Θ_E of (2.3), $\Theta_\mathsf{E}(\gamma) = \mathrm{id}_{\mathsf{E}_0}$. But recall that E_0 is

Galois over T_0 and $\Theta_{\mathsf{E}}\colon \Gamma_{\mathsf{E}} \to \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ is surjective. Since $\mathsf{E}_0 \neq \mathsf{T}_0$, there is $\delta \in \Gamma_{\mathsf{E}}$ with $\Theta_{\mathsf{E}}(\delta) \neq \mathrm{id}$. Hence, there must be some $s \in \mathsf{E}_\delta^* \cap S_\tau$. Likewise, since $\Theta_{\mathsf{E}}(\gamma-\delta) = \Theta_{\mathsf{E}}(\gamma)\Theta_{\mathsf{E}}(\delta)^{-1} \neq \mathrm{id}$, there is some $r \in \mathsf{E}_{\gamma-\delta}^* \cap S_\tau$. Then, as $rs \in \mathsf{E}_\gamma^*$, we have $\mathsf{E}_\gamma^* = \mathsf{E}_0^* rs \subseteq \Sigma_\tau$. This is true for every γ with $\mathsf{E}_\gamma \cap S_\tau = 0$. But for any other $\gamma \in \Gamma_{\mathsf{E}}$, there is an x_γ in $\mathsf{E}_\gamma^* \cap S_\tau$; then $\mathsf{E}_\gamma^* = \mathsf{E}_0^* x_\gamma \subseteq \Sigma_\tau$. Thus, $\mathsf{E}^* = \bigcup_{\gamma\in\Gamma_{\mathsf{E}}} \mathsf{E}_\gamma^* \subseteq \Sigma_\tau$. □

4.2. T/R **unramified.** Let E be a graded division algebra with a unitary involution τ such that $\mathsf{T} = Z(\mathsf{E})$ is unramified over $\mathsf{R} = \mathsf{T}^\tau$. In this subsection, we will give a general formula for $\mathrm{SK}_1(\mathsf{E},\tau)$ in terms of data in E_0.

Lemma 4.6. *Suppose* T *is unramified over* R. *Then,*

(i) *Every* E_γ *contains both nonzero symmetric and skew symmetric elements.*
(ii) $Z(\mathsf{E}_0)$ *is a generalized dihedral extension for* T_0 *over* R_0 *(see Def. 2.4).*
(iii) *If* T *is unramified over* R, *then* $\mathrm{SK}_1(\mathsf{E},\tau) = \Sigma_0'/\Sigma_0$.

Proof.

(i) If $\mathrm{char}(\mathsf{E}) = 2$, it is easy to see that every E_γ contains a symmetric element (which is also skew symmetric) regardless of any assumption on T/R. Let $\mathrm{char}(\mathsf{E}) \neq 2$. Since $[\mathsf{T}_0 : \mathsf{R}_0] = 2$ and $\mathsf{R}_0 = \mathsf{T}_0^\tau$, there is $c \in \mathsf{T}_0$ with $\tau(c) = -c$. Now there is $t \in \mathsf{E}_\gamma$, $t \neq 0$, with $\tau(t) = \epsilon t$ where $\epsilon = \pm 1$. Then $\tau(ct) = -\epsilon ct$.

(ii) Let $G = \mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{R}_0)$ and $H = \mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)$. Note that $[G : H] = 2$. Since τ is unitary, $\tau|_{Z(\mathsf{E}_0)} \in G \setminus H$. We will denote $\tau|_{Z(\mathsf{E}_0)}$ by $\overline{\tau}$ and will show that for any $h \in H$, $(\overline{\tau}h)^2 = 1$. By (2.3), $\Theta_{\mathsf{E}}\colon \Gamma_{\mathsf{E}} \to \mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)$ is onto, so there is $\gamma \in \Gamma_{\mathsf{E}}$, such that $\Theta_{\mathsf{E}}(\gamma) = h$. Also by (i), there is an $x \in \mathsf{E}_\gamma^*$ with $\tau(x) = x$. Then $\tau\varphi_x$ is an involution, where φ_x is conjugation by x; therefore, $\tau\varphi_x|_{Z(\mathsf{E}_0)} \in G$ has order 2. But $\varphi_x|_{Z(\mathsf{E}_0)} = \Theta_{\mathsf{E}}(\gamma) = h$. Thus $(\overline{\tau}h)^2 = 1$.

(iii) By (i), for each $\gamma \in \Gamma_{\mathsf{E}}$, there is $s_\gamma \in \mathsf{E}_\gamma$, $s_\gamma \neq 0$, with $\tau(s_\gamma) = s_\gamma$. By Remark 4.1(iv), $\Sigma_\tau' = \bigcup_{\gamma\in\Gamma_{\mathsf{E}}} s_\gamma \Sigma_0'$. Since each $s_\gamma \in S_\tau \subseteq \Sigma_\tau$, the injective map $\Sigma_0'/\Sigma_0 \to \Sigma_\tau'/\Sigma_\tau$ is an isomorphism. □

To simplify notation in the next theorem, let $\overline{\tau} = \tau|_{Z(\mathsf{E}_0)} \in \mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{R}_0)$, and for any $h \in \mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)$, write $\Sigma_{h\overline{\tau}}(\mathsf{E}_0)$ for $\Sigma_\rho(\mathsf{E}_0)$ for any unitary involution ρ on E_0 such that $\rho|_{Z(\mathsf{E}_0)} = h\overline{\tau}$. This is well-defined, independent of the choice of ρ, by the ungraded analogue of Remark(4.1)(ii).

Theorem 4.7. *Let* E *be a graded division algebra with center* T, *with a unitary graded involution* τ, *such that* T *is unramified over* $\mathsf{R} = \mathsf{T}^\tau$. *For each* $\gamma \in \Gamma_{\mathsf{E}}$ *choose a nonzero* $x_\gamma \in S_\tau \cap \mathsf{E}_\gamma$. *Let* $H = \mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)$. *Then,*

$$\mathrm{SK}_1(\mathsf{E},\tau) \;\cong\; (\Sigma_\tau' \cap \mathsf{E}_0)/(\Sigma_\tau \cap \mathsf{E}_0),$$

with

$$\Sigma_\tau' \cap \mathsf{E}_0 \;=\; \big\{a \in \mathsf{E}_0^* \mid N_{Z(\mathsf{E}_0)/\mathsf{T}_0}\mathrm{Nrd}_{\mathsf{E}_0}(a)^{\partial} \in \mathsf{R}_0\big\}, \quad \textit{where} \quad \partial \;=\; \mathrm{ind}(\mathsf{E})/\big(\mathrm{ind}(\mathsf{E}_0)\,[Z(\mathsf{E}_0):\mathsf{T}_0]\big) \tag{4.2}$$

and

$$\Sigma_\tau \cap \mathsf{E}_0 \;=\; P\cdot X, \quad \textit{where} \quad P \;=\; \textstyle\prod_{h\in H}\Sigma_{h\overline{\tau}}(\mathsf{E}_0) \quad \textit{and} \quad X \;=\; \langle x_\gamma x_\delta x_{\gamma+\delta}^{-1} \mid \gamma,\delta \in \Gamma_{\mathsf{E}}\rangle \;\subseteq\; \mathsf{E}_0^*. \tag{4.3}$$

Furthermore, if $H = \langle h_1,\ldots,h_m\rangle$, *then* $P \;=\; \prod_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{0,1\}^m} \Sigma_{h_1^{\varepsilon_1}\ldots h_m^{\varepsilon_m}\overline{\tau}}(\mathsf{E}_0)$.

Before proving the theorem, we record the following:

Lemma 4.8. *Let* A *be a central simple algebra over a field* K, *with an involution* τ *and an automorphism or anti-automorphism* σ. *Then,*

(i) $\sigma\tau\sigma^{-1}$ *is an involution of* A *of the same kind as* τ, *and*

$$S_{\sigma\tau\sigma^{-1}} \;=\; \sigma(S_\tau), \quad \textit{so} \quad \Sigma_{\sigma\tau\sigma^{-1}} \;=\; \sigma(\Sigma_\tau).$$

(ii) *Suppose A is a division ring. If σ and τ are each unitary involutions, then (writing $S_\tau^* = S_\tau \cap A^*$),*
$$S_\tau^* \ \subseteq\ S_\sigma^* \cdot \sigma(S_\tau^*) \ =\ S_\sigma^* \cdot S_{\sigma\tau\sigma^{-1}}^*, \quad \textit{so} \quad \Sigma_\tau \ \subseteq\ \Sigma_\sigma \cdot \Sigma_{\sigma\tau\sigma^{-1}}.$$

Proof.

(i) This follows by easy calculations.

(ii) Observe that if $a \in S_\tau^*$, then $a = \big(a\sigma(a)\big)\sigma(a^{-1})$ with $a\sigma(a) \in S_\sigma^*$ and $\sigma(a^{-1}) \in \sigma(S_\tau^*) = S_{\sigma\tau\sigma^{-1}}^*$ by (i). Thus, (ii) follows from (i) and the fact that $A' \subseteq \Sigma_\tau \cap \Sigma_\sigma$ (see (2.10)). □

Proof of Theorem 4.7. First note that by Lemma 4.6(iii) the canonical map
$$(\Sigma_\tau' \cap \mathsf{E}_0) \,/\, (\Sigma_\tau \cap \mathsf{E}_0) \ \longrightarrow\ \Sigma_\tau'/\Sigma_\tau \ =\ \mathrm{SK}_1(\mathsf{E}, \tau)$$
is an isomorphism. The description of $\Sigma_\tau' \cap \mathsf{E}_0$ in (4.2) is immediate from the fact that for $a \in \mathsf{E}_0$, $\mathrm{Nrd}_{\mathsf{E}}(a) = N_{Z(\mathsf{E}_0)/\mathsf{T}_0}\mathrm{Nrd}_{\mathsf{E}_0}(a)^\partial \in \mathsf{T}_0$ (see Remark 2.1(iii)).

For $\Sigma_\tau \cap \mathsf{E}_0$, note that for each $\gamma \in \Gamma_{\mathsf{E}}$, if $a \in \mathsf{E}_0$, then $ax_\gamma \in S_\tau$ if and only if $x_\gamma\tau(a)x_\gamma^{-1} = a$. That is, $S_\tau \cap \mathsf{E}_\gamma = S(\varphi_{x_\gamma}\tau; \mathsf{E}_0)x_\gamma$, where $S(\varphi_{x_\gamma}\tau; \mathsf{E}_0)$ denotes the set of symmetric elements in E_0 for the unitary involution $\varphi_{x_\gamma}\tau|_{\mathsf{E}_0}$. Therefore,
$$\Sigma_\tau \cap \mathsf{E}_0 \ =\ \big\langle S(\varphi_{x_\gamma}\tau; \mathsf{E}_0)^* x_\gamma \mid \gamma \in \Gamma_{\mathsf{E}} \big\rangle \ \cap\ \mathsf{E}_0.$$
Take a product $a_1x_1 \dots a_kx_k$ in $\Sigma_\tau \cap \mathsf{E}_0$ where each $x_i = x_{\gamma_i}$ for some $\gamma_i \in \Gamma_{\mathsf{E}}$ and $a_i \in S(\varphi_{x_i}\tau; \mathsf{E}_0)^*$. Then,
$$a_1x_1 \dots a_kx_k \ =\ a_1\varphi_{x_1}(a_2)\dots\varphi_{x_1\dots x_{i-1}}(a_i)\dots\varphi_{x_1\dots x_{k-1}}(a_k)x_1\dots x_k \ \in\ \mathsf{E}_{\gamma_1+\dots+\gamma_k}. \tag{4.4}$$
So, $\gamma_1 + \dots + \gamma_k = 0$. Now, as $a_i \in S(\varphi_{x_i}\tau; \mathsf{E}_0)$ and $\tau\varphi_{x_j}^{-1} = \varphi_{x_j}\tau$ for all j, by Lemma 4.8(i) we obtain
$$\varphi_{x_1\dots x_{i-1}}(a_i) \ \in\ S(\varphi_{x_1}\dots\varphi_{x_{i-1}}(\varphi_{x_i}\tau)\varphi_{x_{i-1}}^{-1}\dots\varphi_{x_1}^{-1}; \mathsf{E}_0)^* \ =\ S(\varphi_{x_1\dots x_{i-1}x_ix_{i-1}\dots x_1}\tau; \mathsf{E}_0)^* \ \subseteq\ \Sigma_{h\overline{\tau}}(\mathsf{E}_0) \ \subseteq P, \tag{4.5}$$
where $h = \varphi_{x_1\dots x_{i-1}x_ix_{i-1}\dots x_1}|_{Z(\mathsf{E}_0)} \in H$. Note also that if $k = 1$, then $x_1 \in S_\tau \cap \mathsf{E}_0^* \subseteq \Sigma_{\overline{\tau}}(\mathsf{E}_0) \subseteq P$.

If $k > 1$, then
$$x_1 \dots x_k \ =\ x_{\gamma_1}\dots x_{\gamma_k} \ =\ (x_{\gamma_1}x_{\gamma_2}x_{\gamma_1+\gamma_2}^{-1})(x_{\gamma_1+\gamma_2}x_{\gamma_3}\dots x_{\gamma_k}),$$
with $(\gamma_1 + \gamma_2) + \gamma_3 + \dots + \gamma_k = 0$. It follows by induction on k that $x_1 \dots x_k \in X$. With this and (4.4) and (4.5), we have $a_1x_1\dots a_kx_k \in P \cdot X$ (which is a group, as $\mathsf{E}_0' \subseteq \Sigma_{\overline{\tau}}(\mathsf{E}_0) \subseteq P$ by (2.10)), showing that $\Sigma_\tau \cap \mathsf{E}_0 \subseteq P \cdot X$. For the reverse inclusion, take any $h \in H$ and choose $\gamma \in \Gamma_{\mathsf{E}}$ with $\varphi_{x_\gamma}|_{Z(\mathsf{E}_0)} = h$. Then, $x_\gamma \in S_\tau^* \subseteq \Sigma_\tau$ and $S(\varphi_{x_\gamma}\tau; \mathsf{E}_0)^*x_\gamma = S_\tau^* \cap \mathsf{E}_\gamma \subseteq \Sigma_\tau$, so $\Sigma_{h\overline{\tau}}(\mathsf{E}_0) = \Sigma_{\varphi_{x_\gamma}\tau}(\mathsf{E}_0) = \langle S(\varphi_{x_\gamma}\tau; \mathsf{E}_0)^*\rangle \subseteq \Sigma_\tau \cap \mathsf{E}_0$. Thus, $P \subseteq \Sigma_\tau \cap \mathsf{E}_0$, and clearly also $X \subseteq \Sigma_\tau \cap \mathsf{E}_0$. Hence, $\Sigma_\tau \cap \mathsf{E}_0 = P \cdot X$.

The final equality for P in the Theorem follows from Lemma 4.9 below by taking $U = \mathsf{E}_0^*$, $A = H$, and $W_h = \Sigma_{h\overline{\tau}}(\mathsf{E}_0)$ for $h \in H$. To see that the lemma applies, note that each $\Sigma_{h\overline{\tau}}(\mathsf{E}_0)$ contains E_0' by (2.10). Furthermore, take any $h, \ell \in H$, and choose $x, y \in E^* \cap S_\tau$ with $\varphi_x|_{Z(\mathsf{E}_0)} = h$ and $\varphi_y|_{Z(\mathsf{E}_0)} = \ell$. Then,
$$(\varphi_y\tau)(\varphi_x\tau)(\varphi_y\tau)^{-1} \ =\ \varphi_y\tau\varphi_x\varphi_y^{-1} \ =\ \varphi_{yx^{-1}y}\tau,$$
and $\varphi_{yx^{-1}y}|_{Z(\mathsf{E}_0)} = \ell h^{-1}\ell = \ell^2h^{-1}$. Hence, by Lemma 4.8(ii), $\Sigma_{h\overline{\tau}}(\mathsf{E}_0) \subseteq \Sigma_{\ell\overline{\tau}}(\mathsf{E}_0)\Sigma_{\ell^2h^{-1}\overline{\tau}}(\mathsf{E}_0)$. This shows that hypothesis (4.6) of Lemma 4.9 below is satisfied here. □

Lemma 4.9. *Let U be a group, A an abelian group, and $\{W_a \mid a \in A\}$ a family of subgroups of U with each $W_a \supseteq [U, U]$. Suppose*
$$W_a \ \subseteq\ W_bW_{2b-a} \quad \textit{for all } a, b \in A. \tag{4.6}$$
If $A = \langle a_1, \dots, a_m\rangle$, then
$$\prod_{a\in A} W_a \ =\ \prod_{(\varepsilon_1,\dots,\varepsilon_m)\in\{0,1\}^m} W_{\varepsilon_1a_1+\dots+\varepsilon_ma_m}.$$

Proof. Since each $W_a \supseteq [U,U]$, we have $W_aW_b = W_bW_a$, and this is a subgroup of U, for all $a, b \in A$. Let

$$Q \;=\; \prod_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{0,1\}^m} W_{\varepsilon_1a_1+\ldots+\varepsilon_ma_m}.$$

We prove by induction on m that each $W_a \subseteq Q$. The lemma then follows, as Q is a subgroup of U. Note that condition (4.6) can be conveniently restated,

$$\text{if}\;\; a+b \;=\; 2d \in A, \;\;\text{then}\;\; W_a \;\subseteq\; W_dW_b. \tag{4.7}$$

Take any $c \in A$. Then, (4.7) shows that $W_{-c} \subseteq W_0W_c$. Take any $i \in \mathbb{Z}$, and suppose $W_{ic} \subseteq W_0W_c$. Then by (4.7) $W_{-ic} \subseteq W_0W_{ic} \subseteq W_0W_c$. So, by (4.7) again, $W_{(i+2)c} \subseteq W_cW_{-ic} \subseteq W_0W_c$ and $W_{(i-2)c} \subseteq W_{-c}W_{-ic} \subseteq W_0W_c$. Hence, by induction (starting with $j=0$ and $j=1$), $W_{jc} \subseteq W_0W_c$ for every $j \in \mathbb{Z}$. This proves the lemma when $m=1$.

Now assume $m>1$ and let $B = \langle a_1,\ldots,a_{m-1}\rangle \subseteq A$. By induction, for all $b \in B$,

$$W_b \;\subseteq\; \prod_{(\varepsilon_1,\ldots,\varepsilon_{m-1})\in\{0,1\}^{m-1}} W_{\varepsilon_1a_1+\ldots+\varepsilon_{m-1}a_{m-1}} \;\subseteq\; Q.$$

Also, by the cyclic case done above, $W_{ja_m} \subseteq W_0W_{a_m} \subseteq Q$ for all $j \in \mathbb{Z}$. So, for any $b \in B, j \in \mathbb{Z}$, using (4.7),

$$W_{2b+ja_m} \;\subseteq\; W_bW_{-ja_m} \;\subseteq\; Q, \tag{4.8}$$

and

$$W_{b+2ja_m} \;\subseteq\; W_{ja_m}W_{-b} \;\subseteq\; Q. \tag{4.9}$$

Let $d = a_{i_1} + \ldots + a_{i_\ell}$ for any indices $1 \le i_1 < i_2 < \ldots < i_\ell \le m-1$. Since $W_{d+a_m} \subseteq Q$ by hypothesis, from (4.7) and (4.9) it follows that

$$W_{d+3a_m} \;\subseteq\; W_{d+2a_m}W_{d+a_m} \;\subseteq\; Q. \tag{4.10}$$

Now, take any element of A; it has the form $b + ja_m$ for some $b \in B$ and $j \in \mathbb{Z}$. If $b \in 2B$ or if j is even, then (4.8) and (4.9) show that $W_{b+ja_m} \subseteq Q$. The remaining case is that j is odd and $b \notin 2B$, so $b = 2c+d$, where $c \in B$ and $d = a_{i_1} + \ldots + a_{i_\ell}$ for some indices with $1 \le i_1 < i_2 < \ldots < i_\ell \le m-1$. Set $q=1$ if $j \equiv 3 \pmod 4$ and $q=3$ if $j \equiv 1 \pmod 4$. Then, $W_{d+qa_m} \subseteq Q$ by definition if $q=1$ or by (4.10) if $q=3$. Hence, by (4.7),

$$W_{b+ja_m} \;=\; W_{(2c+d)+ja_m} \;\subseteq\; W_{(c+d)+((j+q)/2)a_m}W_{d+qa_m} \;\subseteq\; Q, \tag{4.11}$$

using (4.9) as $c+d \in B$ and $(j+q)/2$ is even. Thus, $W_a \subseteq Q$, for all $a \in A$. □

Corollary 4.10. *If* E *is unramified over* R*, then* $\mathrm{SK}_1(\mathsf{E},\tau) \cong \mathrm{SK}_1(\mathsf{E}_0,\tau|_{\mathsf{E}_0})$.

Proof. Since E is unramified over R, we have T is unramified over R, $Z(\mathsf{E}_0) = \mathsf{T}_0$, and $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{R}}$, so we can choose all the x_γ's to lie in R. The assertion thus follows immediately from Th. 4.7, as $P = \Sigma_{\tau|_{\mathsf{E}_0}}(\mathsf{E}_0)$ and $X \subseteq R_0^* \subseteq P$. (Alternatively, more directly, one can observe that $\Sigma'_0 = \Sigma'_{\tau|_{\mathsf{E}_0}}(\mathsf{E}_0)$ and $\Sigma_0 = \Sigma_{\tau|_{\mathsf{E}_0}}(\mathsf{E}_0)$ and so deduce the Corollary by Lemma 4.6(iii).) □

Corollary 4.11. *If* T *is unramified over* R *and* E *has a maximal graded subfield* M *unramified over* T *and another maximal graded subfield* L *totally ramified over* T *with* $\tau(\mathsf{L}) = \mathsf{L}$*, then* E *is semiramified with* $\mathsf{E}_0 = \mathsf{M}_0$ *(a field) and* $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{L}}$*, and*

$$\mathrm{SK}_1(\mathsf{E},\tau) \;\cong\; \big\{a \in \mathsf{E}_0 \mid N_{\mathsf{E}_0/\mathsf{T}_0}(a) \in \mathsf{R}_0\big\} \big/ \prod_{h\in\mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)} \mathsf{E}_0^{*h\overline{\tau}}. \tag{4.12}$$

Proof. Let $n = \mathrm{ind}(\mathsf{E})$. Since $[\mathsf{M}_0 : \mathsf{T}_0] = n$ and $|\Gamma_{\mathsf{L}} : \Gamma_{\mathsf{T}}| = n$, it follows from the Fundamental Equality (2.2) for E/T, M/T, and L/T that $[\mathsf{E}_0 : \mathsf{T}_0] = [\Gamma_{\mathsf{E}} : \Gamma_{\mathsf{T}}] = n$, $\mathsf{E}_0 = \mathsf{M}_0$, which is is a field, and $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{L}}$. Thus E is semiramified, so for the ∂ of (2.6), $\partial = 1$. Now, L^τ is a graded subfield of L with $[\mathsf{L} : \mathsf{L}^\tau] = 2$. Since $\mathsf{L}_0 = \mathsf{T}_0$ while $(\mathsf{L}^\tau)_0 = (\mathsf{L}_0)^\tau = \mathsf{R}_0$, L must be unramified over L^τ; hence, $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{L}} = \Gamma_{\mathsf{L}^\tau}$. Therefore, one can choose all the x_γ's in Th. 4.7 to lie in L^τ. Then each $x_\gamma x_\delta x_{\gamma+\delta}^{-1} \in (\mathsf{L}^\tau)_0^* = \mathsf{R}_0^* = \mathsf{E}_0^{*\overline{\tau}}$. Hence, $X \subseteq P = \prod_{h\in \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)} \mathsf{E}_0^{*h\overline{\tau}}$, so the formula for $\mathrm{SK}_1(\mathsf{E},\tau)$ in Th. 4.7 reduces to (4.12). □

Remark 4.12. In a sequel to this paper [W3], the following will be shown: With the hypotheses of Th. 4.7, suppose E is semiramified with a graded maximal subfield L totally ramified over T such that $\tau(\mathsf{L}) = \mathsf{L}$, and suppose $\mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ is bicyclic, say $\mathsf{E}_0 = N \otimes_{T_0} N'$ with N and N' each cyclic Galois over T_0. Then,

$$\mathrm{SK}_1(\mathsf{E},\tau) \ \cong\ \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0;\mathsf{R}_0)\big/\big(\mathrm{Br}(N/\mathsf{T}_0;\mathsf{R}_0) + \mathrm{Br}(N'/\mathsf{T}_0;\mathsf{R}_0)\big),$$

where $\mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0)$ is the relative Brauer group $\ker\big(\mathrm{Br}(\mathsf{T}_0) \to \mathrm{Br}(\mathsf{E}_0)\big)$ and $\mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0;\mathsf{R}_0)$ is the kernel of $\mathrm{cor}_{\mathsf{E}_0\to\mathsf{R}_0}\colon \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0) \to \mathrm{Br}(\mathsf{E}_0/\mathsf{R}_0)$. (Compare this with [Y2, Th. 5.6].) A further formula will be given assuming only that E is semiramified over T with $\mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ bicyclic.

In his construction of division algebras D with nontrivial SK_1, Platonov worked originally in [P2, §4] with a division algebra D where $Z(D)$ is a Laurent power series field; he gave an exact sequence relating $\mathrm{SK}_1(D)$ with $\mathrm{SK}_1(\overline{D})$ and what he called the "group of projective conorms." Yanchevskiĭ gave in [Y2, 4.11] an analogous exact sequence in the unitary case. Their results and proofs are valid whenever $Z(D)$ has a henselian discrete (rank 1) valuation. We show here that their results hold more generally whenever $Z(D)$ has a henselian valuation with $\Gamma_D/\Gamma_{Z(D)}$ cyclic. We work in the equivalent graded setting where the arguments are more transparent.

As before, let E be a graded division algebra finite dimensional over its center T with a unitary graded involution τ, and let $\mathsf{R} = \mathsf{T}^\tau$. Assume that T is unramified over R and that $\Gamma_\mathsf{E}/\Gamma_\mathsf{T}$ is cyclic group. (This cyclicity holds, e.g., whenever $\Gamma_\mathsf{T} \cong \mathbb{Z}$.) It follows that the surjective map $\Theta_\mathsf{E}\colon \Gamma_\mathsf{E} \to \mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)$ has kernel Γ_T. (For, by [HW2, Prop. 2.1, (2.3), Remark 2.4(i)], $\ker(\Theta_\mathsf{E})/\Gamma_\mathsf{T}$ has a nondegenerate symplectic pairing, and hence has even rank as a finite abelian group. But here $\ker(\Theta_\mathsf{E})/\Gamma_\mathsf{T}$ is a cyclic group.) Hence, $\partial = 1$ by Lemma 2.2, so E is inertially split. Invoking Lemma 4.6(i), choose any $s \in \mathsf{E}^*$ with $\deg(s) + \Gamma_\mathsf{T}$ a generator of $\Gamma_\mathsf{E}/\Gamma_\mathsf{T}$, such that $\tau(s) = s$. Let $\sigma = \varphi_s \in \mathrm{Aut}_\mathsf{T}(\mathsf{E})$; so $\sigma\tau$ is another T/R-graded involution of E, and $\tau\sigma = \sigma^{-1}\tau$ (see Remark 4.1(iii)). By the choice of s, $\sigma|_{Z(\mathsf{E}_0)}$ is a generator of the cyclic group $\mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)$. Note that $\mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{R}_0) = \langle \sigma|_{Z(\mathsf{E}_0)}, \tau|_{Z(\mathsf{E}_0)}\rangle$ is a dihedral group. Recall our convention that $c^{\sigma\tau}$ means $\sigma(\tau(c))$. Let

$$\begin{aligned}
\mathcal{S} &= \big\{(\beta, b) \in Z(\mathsf{E}_0)^* \times \mathsf{E}_0^* \mid \beta^{\sigma-1} = \mathrm{Nrd}_{\mathsf{E}_0}(b)\big\};\\
\mathcal{N} &= \pi_1(\mathcal{S}) \text{ (projection into the first component)}\\
&= \big\{\beta \in Z(\mathsf{E}_0)^* \mid \beta^{\sigma-1} = \mathrm{Nrd}_{\mathsf{E}_0}(b) \text{ for some } b \in \mathsf{E}_0^*\big\};\\
\mathcal{W} &= \mathsf{T}_0^* \cdot \mathrm{Nrd}_{\mathsf{E}_0}(\mathsf{E}_0^*) \ \subseteq\ Z(\mathsf{E}_0)^*;\\
\mathcal{P} &= \mathcal{N}/\mathcal{W}, \text{ which is Platonov's group of projective conorms for } \mathsf{E} \text{ [P}_2\text{, §4]}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_\tau &= \big\{(\alpha, a) \in Z(\mathsf{E}_0)^* \times \mathsf{E}_0^* \mid \alpha^{\sigma-1} = \mathrm{Nrd}_{\mathsf{E}_0}(a)^{1-\sigma\tau}\big\};\\
\mathcal{N}_\tau &= \pi_1(\mathcal{S}_\tau) \text{ (projection into the first component)}\\
&= \big\{\alpha \in Z(\mathsf{E}_0)^* \mid \alpha^{\sigma-1} = \mathrm{Nrd}_{\mathsf{E}_0}(a)^{1-\sigma\tau} \text{ for some } a \in \mathsf{E}_0^*\big\};\\
\mathcal{W}_\tau &= \mathsf{T}_0^* \cdot \mathrm{Nrd}_{\mathsf{E}_0}(\Sigma_\tau(\mathsf{E}_0)) \ \subseteq\ Z(\mathsf{E}_0)^*;\\
\mathcal{PU}_\tau &= \mathcal{N}_\tau/\mathcal{W}_\tau, \text{ which is Yanchevskiĭ's group of unitary projective conorms for } (\mathsf{E},\tau) \text{ [Y}_2\text{, 4.11]}.
\end{aligned}$$

Proposition 4.13. *If* T *is unramified over* R *and* $\Gamma_\mathsf{E}/\Gamma_\mathsf{T}$ *is cyclic, then for any generator* σ *of the cyclic group* $\mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)$*, we have*

(i) $\mathrm{SK}_1(\mathsf{E}) \ \cong\ \{a \in E_0^* \mid N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathsf{E}_0}(a)) = 1\} \big/ \big([\mathsf{E}_0^*, \mathsf{E}_0^*] \cdot \{c^{\sigma-1} \mid c \in \mathsf{E}_0^*\}\big)$.

(ii) $\mathrm{SK}_1(\mathsf{E},\tau) \ \cong\ \{a \in E_0^* \mid N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathsf{E}_0}(a)) \in \mathsf{R}_0\} \big/ \big(\Sigma_\tau(\mathsf{E}_0) \cdot \Sigma_{\sigma\tau}(\mathsf{E}_0)\big)$.

(iii) *The following sequence is exact:*

$$\mathrm{SK}_1(\mathsf{E}_0,\sigma\tau)\longrightarrow \mathrm{SK}_1(\mathsf{E},\tau)\xrightarrow{f}\mathcal{P}U_\tau\longrightarrow 1, \tag{4.13}$$

where the map $f\colon \mathrm{SK}_1(\mathsf{E},\tau)\to\mathcal{P}U_\tau$ *is the composition of* $a\,\Sigma_\tau(\mathsf{E}_0)\cdot\Sigma_{\sigma\tau}(\mathsf{E}_0)\mapsto(\alpha,a)\in\mathcal{S}_\tau$ *and* $(\alpha,a)\mapsto\alpha\mathcal{W}_\tau\in\mathcal{P}U_\tau$.

(iv) *There is a commutative diagram with exact rows:*

$$\begin{tikzcd}
 & \mathrm{SK}_1(\mathsf{E}_0,\sigma\tau) \arrow[r] \arrow[d, "{a\mapsto a^{1-\sigma\tau}}"'] & \mathrm{SK}_1(\mathsf{E},\tau) \arrow[r, "f"] \arrow[d, "{a\mapsto a^{1-\sigma\tau}}"'] & \mathcal{P}U_\tau \arrow[r] \arrow[d, "{\alpha\mapsto\alpha}"'] & 1 \\
1 \arrow[r] & \mathrm{SK}_1(\mathsf{E}_0) \arrow[r] \arrow[d, "{b\mapsto b}"'] & \mathrm{SK}_1(\mathsf{E}) \arrow[r, "g"] \arrow[d, "{b\mapsto b}"'] & \mathcal{P} \arrow[r] \arrow[d, "{\beta\mapsto\beta^{1+\tau}}"'] & 1 \\
 & \mathrm{SK}_1(\mathsf{E}_0,\sigma\tau) \arrow[r] & \mathrm{SK}_1(\mathsf{E},\tau) \arrow[r] & \mathcal{P}U_\tau \arrow[r] & 1
\end{tikzcd}$$

where the map $g\colon \mathrm{SK}_1(\mathsf{E})\to\mathcal{P}$ *is the composition of* $b\,[\mathsf{E}_0^*,\mathsf{E}_0^*]\langle c^{\sigma-1}\rangle\mapsto(\beta,b)\in\mathcal{S}$ *and* $(\beta,b)\mapsto\beta\mathcal{W}\in\mathcal{P}$.

(v) *If* E_0 *is a field, then* $\mathrm{SK}_1(\mathsf{E},\tau)=1$.

Proof.

(i) This formula was given by Suslin [S1, Prop. 1.7] for a division algebra over a field with a complete discrete valuation. In order to prove it in the graded setting we need two exact sequences which were given in [HaW, Th. 3.4]:

$$\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}\wedge\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}\longrightarrow \mathsf{E}^{(1)}/[\mathsf{E}_0^*,\mathsf{E}^*]\longrightarrow\mathrm{SK}_1(\mathsf{E})\longrightarrow 1,$$
$$1\longrightarrow\ker\widetilde{N}/[\mathsf{E}_0^*,\mathsf{E}^*]\longrightarrow \mathsf{E}^{(1)}/[\mathsf{E}_0^*,\mathsf{E}^*]\longrightarrow \mu_\partial(\mathsf{T}_0)\cap\widetilde{N}(\mathsf{E}_0^*)\longrightarrow 1,$$

where $\mathsf{E}^{(1)}=\{a\in\mathsf{E}^*\mid \mathrm{Nrd}_{\mathsf{E}}(a)=1\}\subseteq\mathsf{E}_0$ and $\widetilde{N}=N_{Z(\mathsf{E}_0)/\mathsf{T}_0}\circ\mathrm{Nrd}_{\mathsf{E}_0}\colon\mathsf{E}_0^*\to\mathsf{T}_0^*$. Since $\partial=1$ (see the paragraph prior to the Proposition) and the wedge product of a cyclic group with itself is trivial, these exact sequences yield

$$\mathrm{SK}_1(\mathsf{E})\ \cong\ \{a\in\mathsf{E}_0^*\mid N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathsf{E}_0}(a))=1\}\big/[\mathsf{E}_0^*,\mathsf{E}^*].$$

We are left to show that $[\mathsf{E}_0^*,\mathsf{E}^*]=[\mathsf{E}_0^*,\mathsf{E}_0^*]\cdot\{c^{\sigma-1}\mid c\in\mathsf{E}_0^*\}$. This follows from the fact that $\mathsf{E}^*/\mathsf{T}^*\mathsf{E}_0^*\cong\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ is cyclic together with the following observation, which is easily verified using the standard commutator identities: If G is a group and N is a normal subgroup of G such that $G/Z(G)N$ is a cyclic group generated by, say, $xZ(G)N$, then $[N,G]=[N,N][x,N]$ where $[x,N]=\{[x,n]\mid n\in N\}$. (Here, take $G=\mathsf{E}^*$, $N=\mathsf{E}_0^*$, and for x take any $s\in\mathsf{E}_\gamma^*$ for any $\gamma\in\Gamma_{\mathsf{E}}$ such that $\Theta_{\mathsf{E}}(\gamma+\Gamma_{\mathsf{T}})=\sigma$.)

(ii) By Th. 4.7, taking into account that $\partial=1$ and $\mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)=\langle\sigma\rangle$, we have,

$$\begin{aligned}\mathrm{SK}_1(\mathsf{E},\tau)\ &\cong\ (\Sigma'_\tau\cap\mathsf{E}_0)/(\Sigma_\tau\cap\mathsf{E}_0)\\ &=\ \{a\in\mathsf{E}_0^*\mid N_{Z(\mathsf{E}_0)/\mathsf{T}_0}\mathrm{Nrd}_{\mathsf{E}_0}(a)\in\mathsf{R}_0\}\big/\big(\Sigma_\tau(\mathsf{E}_0)\cdot\Sigma_{\sigma\tau}(\mathsf{E}_0)\cdot\langle x_\gamma x_\delta x_{\gamma+\delta}^{-1}\mid\gamma,\delta\in\Gamma_{\mathsf{E}}\rangle\big),\end{aligned} \tag{4.14}$$

where for each $\gamma\in\Gamma_{\mathsf{E}}$, x_γ is chosen in E_γ^* with $x_\gamma=\tau(x_\gamma)$ and $x_\gamma\neq 0$, using Lemma 4.6(i). Let $\mathsf{L}=\mathsf{R}[s]$, where s is chosen in E^* with $\varphi(s)|_{\mathsf{E}_0}=\sigma$, which is possible as $\Theta_{\mathsf{E}}\colon\Gamma_{\mathsf{E}}\to\mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0)$ is surjective (see (2.3)). Moreover, s can be chosen with $\tau(s)=s$. Since $\ker(\Theta_{\mathsf{E}})=\Gamma_{\mathsf{T}}=\Gamma_{\mathsf{R}}$, we have $\Gamma_{\mathsf{E}}=\langle\deg(s)\rangle+\Gamma_{\mathsf{R}}$. Thus, L is a graded subfield of E with $\Gamma_{\mathsf{L}}=\Gamma_{\mathsf{E}}$ and $\tau|_{\mathsf{L}}=\mathrm{id}$. For each $\gamma\in\Gamma_{\mathsf{E}}$ we can choose $x_\gamma\in\mathsf{L}_\gamma^*$; then for all $\gamma,\delta\in\Gamma_{\mathsf{E}}$, we have $x_\gamma x_\delta x_{\gamma+\delta}^{-1}\in\mathsf{L}_0^*\subseteq\Sigma_\tau(\mathsf{E}_0)$. Thus, the $\langle x_\gamma x_\delta x_{\gamma+\delta}^{-1}\rangle$ term in (4.14) is redundant, yielding the formula in (ii).

(iii) We first check that f is well-defined: Take any $a\in\mathsf{E}_0^*$ with $N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathsf{E}_0}(a))\in\mathsf{R}_0^*$. Let $c=\mathrm{Nrd}_{\mathsf{E}_0}(a)$. Then, as $\mathsf{R}_0=\mathsf{T}_0^\tau$, $1=N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(c)^{1-\tau}=N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(c)^{1-\sigma\tau}=N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(c^{1-\sigma\tau})$. By Hilbert 90, there is $\alpha\in Z(\mathsf{E}_0)^*$ with $\alpha^{\sigma-1}=c^{1-\sigma\tau}=\mathrm{Nrd}_{\mathsf{E}_0}(a)^{1-\sigma\tau}$. Hence $(\alpha,a)\in\mathcal{S}_\tau$, so $\alpha\in\mathcal{N}_\tau$, and the choice of α is unique up to $\mathsf{T}_0^* \subseteq \mathcal{W}_\tau$. Thus, the image of a in $\mathcal{P}U_\tau$ is independent of the choice of α. Suppose further that $a = pq$ for some $p \in \Sigma_\tau(\mathsf{E}_0)$, $q \in \Sigma_{\sigma\tau}(\mathsf{E}_0)$, say, $p = s_1 \dots s_k$ with each $s_i \in S_\tau(\mathsf{E}_0)$. Then,

$$\begin{aligned} \mathrm{Nrd}_{\mathsf{E}_0}(p)^\tau &= \mathrm{Nrd}_{\mathsf{E}_0}(s_1)^\tau \dots \mathrm{Nrd}_{\mathsf{E}_0}(s_k)^\tau = \mathrm{Nrd}_{\mathsf{E}_0}(s_1^\tau) \dots \mathrm{Nrd}_{\mathsf{E}_0}(s_k^\tau) \\ &= \mathrm{Nrd}_{\mathsf{E}_0}(s_1) \dots \mathrm{Nrd}_{\mathsf{E}_0}(s_k) = \mathrm{Nrd}_{\mathsf{E}_0}(p); \end{aligned} \tag{4.15}$$

likewise, $\mathrm{Nrd}_{\mathsf{E}_0}(q)^{\sigma\tau} = \mathrm{Nrd}_{\mathsf{E}_0}(q)$. So,

$$\alpha^{\sigma-1} = \mathrm{Nrd}_{\mathsf{E}_0}(pq)^{1-\sigma\tau} = \mathrm{Nrd}_{\mathsf{E}_0}(p)^{1-\sigma\tau}\mathrm{Nrd}_{\mathsf{E}_0}(q)^{1-\sigma\tau} = \mathrm{Nrd}_{\mathsf{E}_0}(p)^{1-\sigma}.$$

Hence, $\big(\alpha\mathrm{Nrd}_{\mathsf{E}_0}(p)\big)^{\sigma-1} = 1$, showing that $\alpha\mathrm{Nrd}_{\mathsf{E}_0}(p) \in \mathsf{T}_0$; Thus $\alpha \in \mathcal{W}_\tau$. This proves that f is well-defined.

For the subjectivity of f, take any $\alpha \in \mathcal{N}_\tau$. Then, there is $a \in \mathsf{E}_0^*$ with $\alpha^{\sigma-1} = \mathrm{Nrd}_{\mathsf{E}_0}(a)^{1-\sigma\tau}$. So, $N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathsf{E}_0}(a))^{1-\sigma\tau} = N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\alpha^{\sigma-1}) = 1$, which shows that $N_{Z(\mathsf{E}_0)/\mathsf{T}_0}(\mathrm{Nrd}_{\mathsf{E}_0}(a)) \in \mathsf{T}_0^{\sigma\tau} = \mathsf{T}_0^\tau = \mathsf{R}_0$, and hence $a \in \Sigma'_\tau(\mathsf{E}) \cap \mathsf{E}_0^*$. Since $f\big(a\Sigma_\tau(\mathsf{E}_0)\Sigma_{\sigma\tau}(\mathsf{E}_0)\big) = \alpha\mathcal{W}_\tau$, f is surjective.

Finally, we determine $\ker(f)$: The image of $\mathrm{SK}_1(\mathsf{E}_0, \sigma\tau)$ in $\mathrm{SK}_1(\mathsf{E}, \tau)$ is $\Sigma'_{\sigma\tau}(\mathsf{E}_0)\Sigma_\tau(\mathsf{E}_0)\big/\Sigma_{\sigma\tau}(\mathsf{E}_0)\Sigma_\tau(\mathsf{E}_0)$. An element in this image is represented by some $a \in \Sigma'_{\sigma\tau}(\mathsf{E}_0)$. For such an a, $\mathrm{Nrd}_{\mathsf{E}_0}(a)^{1-\sigma\tau} = 1$. Then $(1, a) \in \mathcal{S}_\tau$, so that f maps the image of a to 1 in $\mathcal{P}U_\tau$. Conversely, suppose $a\Sigma_\tau(\mathsf{E}_0)\Sigma_{\sigma\tau}(\mathsf{E}_0) \in \ker(f)$. That is, $\mathrm{Nrd}_{\mathsf{E}_0}(a)^{1-\sigma\tau} = \alpha^{\sigma-1}$, where $\alpha \in \mathcal{W}_\tau$, so $\alpha = c\mathrm{Nrd}_{\mathsf{E}_0}(d)$ with $c \in \mathsf{T}_0^*$ and $d \in \Sigma_\tau(\mathsf{E}_0)$. So, $\mathrm{Nrd}_{\mathsf{E}_0}(d) = \mathrm{Nrd}_{\mathsf{E}_0}(d)^\tau$ by the argument of (4.15) above, and hence

$$\mathrm{Nrd}_{\mathsf{E}_0}(a)^{1-\sigma\tau} = \alpha^{\sigma-1} = \big(c\mathrm{Nrd}_{\mathsf{E}_0}(d)\big)^{\sigma-1} = \mathrm{Nrd}_{\mathsf{E}_0}(d)^{\sigma-1} = \mathrm{Nrd}_{\mathsf{E}_0}(d)^{\sigma\tau-1}.$$

Thus, $\mathrm{Nrd}_{\mathsf{E}_0}(ad)^{1-\sigma\tau} = 1$, i.e., $ad \in \Sigma'_{\sigma\tau}(\mathsf{E}_0)$. Hence, $a = (ad)d^{-1} \in \Sigma'_{\sigma\tau}(\mathsf{E}_0)\Sigma_\tau(\mathsf{E}_0)$. This shows that $\ker(f)$ coincides with the image of $\mathrm{SK}_1(\mathsf{E}_0, \sigma\tau)$ in $\mathrm{SK}_1(\mathsf{E}, \tau)$, completing the proof of exactness of the sequence.

(iv) Exactness of the middle row is proved by an analogous but easier argument to that for (iii). Commutativity of the left rectangles of the diagram is evident. Commutativity of the top right rectangle is clear from the definitions. Commutativity of the bottom right rectangle is easy to check using the identity

$$(1 - \sigma\tau) \circ (\sigma - 1) = (\sigma - 1) \circ (1 + \tau), \tag{4.16}$$

which follows from $(\sigma\tau)^2 = \mathrm{id}$. Note that for each column of the diagram, the composition of the two maps is the squaring map.

(v) For this part, the proof follows closely Yanchevskiĭ's proof in [Y_2, 4.13]. (But our notational convention for products of functions is $fg = f \circ g$, whereas his appears to be $fg = g \circ f$.) Suppose E_0 is a field. For simplicity we denote $\overline{\tau} = \tau|_{\mathsf{E}_0}$ by τ. Take $a \in \Sigma'_\tau(\mathsf{E}) \cap \mathsf{E}_0$. So, $N_{\mathsf{E}_0/\mathsf{T}_0}(a^{1-\tau}) = 1$. We will show that $a \in \mathsf{E}_0^\tau\mathsf{E}_0^{\sigma\tau}$. It then follows by (ii) above that $\mathrm{SK}_1(\mathsf{E}, \tau) = 1$. But since E_0 is cyclic over T_0, by Hilbert 90 there is a $b \in \mathsf{E}_0^*$ such that $a^{\tau-1} = b^{\sigma-1}$ where $\langle\sigma\rangle = \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$. So, $1 = a^{(\tau+1)(\tau-1)} = b^{(\tau+1)(\sigma-1)}$. Analogously to (4.16), we have $(\tau + 1)(\sigma - 1) = (\sigma - 1)(1 - \tau\sigma)$. So $b^{(\sigma-1)(1-\tau\sigma)} = 1$. Setting $c = b^{(1-\tau\sigma)}$, we have $c^{\sigma-1} = 1$, so $c \in \mathsf{T}_0$. But, $N_{\mathsf{T}_0/\mathsf{R}_0}(c) = c^{1+\tau\sigma} = b^{(1+\tau\sigma)(1-\tau\sigma)} = 1$. By Hilbert 90 we have $c = d^{\tau\sigma-1}$ for some $d \in \mathsf{T}_0^*$. Let $t = bd \in \mathsf{E}_0^*$. Then, $t^{1-\tau\sigma} = b^{(1-\tau\sigma)}d^{(1-\tau\sigma)} = d^{(\tau\sigma-1)}d^{(1-\tau\sigma)} = 1$, i.e., $t \in \mathsf{E}_0^{\tau\sigma}$. So, $\sigma(t) = \tau(t) \in \mathsf{E}_0^{\sigma\tau}$. Thus, $a^{\tau-1} = b^{\sigma-1} = (t/d)^{\sigma-1} = t^{\sigma-1} = t^{\tau-1}$, as $d \in \mathsf{T}_0$. This shows that $(a\tau(t))^{\tau-1} = 1$, i.e., $a\tau(t) \in \mathsf{E}^\tau$; hence $a = (a\tau(t))\tau(t)^{-1} \in \mathsf{E}_0^\tau\mathsf{E}_0^{\sigma\tau}$. □

5. Totally ramified algebras

For a graded division algebra E totally ramified over its center T with a unitary graded involution τ, two possible cases can arise: either T is totally ramified over $\mathsf{R} = \mathsf{T}^\tau$, or T is unramified over R. In the first case, we showed in Prop. 4.4 that $\mathrm{SK}_1(\mathsf{E}, \tau)$ is trivial. We now obtain an easily computable explicit formula for $\mathrm{SK}_1(\mathsf{E},\tau)$ in the second case. For a field K and for $n \in \mathbb{N}$, we write μ_n for the group of all n-th roots of unity in an algebraic closure of K. Then set $\mu_n(K) = \mu_n \cap K^*$.

Theorem 5.1. *If* E *is totally ramified over* T *of index* n *and* T *is unramified over* R*, then*

$$\begin{aligned}\mathrm{SK}_1(\mathsf{E},\tau) \ &\cong\ \big\{a \in \mathsf{T}_0^* \mid a^n \in \mathsf{R}_0^*\big\} \big/ \big\{a \in \mathsf{T}_0^* \mid a^e \in \mathsf{R}_0^*\big\} &(5.1)\\ &\cong\ \big\{\omega \in \mu_n(\mathsf{T}_0) \mid \tau(\omega) = \omega^{-1}\big\}/\mu_e, &(5.2)\end{aligned}$$

where e *is the exponent of* $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$*. In particular,*

(i) *The restriction of the map* $K_1(\mathsf{E},\tau) \to K_1(\mathsf{E})$ *given by* $a\Sigma_\tau \mapsto a^{1-\tau}\mathsf{E}'$*, induces an injective map*

$$\alpha\colon\ \mathrm{SK}_1(\mathsf{E},\tau) \longrightarrow \mathrm{SK}_1(\mathsf{E}) \ \cong\ \mu_n(\mathsf{T}_0)/\mu_e.$$

(ii) *If the exponent* e *of* E *is odd, then* α *is an isomorphism.*

(iii) *If* $e > 2$ *then* $\mathsf{T}_0 = \mathsf{R}_0(\mu_e)$*, and* τ *acts on* μ_e *by* $\omega \mapsto \omega^{-1}$*.*

Proof. Since T is unramified over R and $\mathsf{E}_0 = \mathsf{T}_0$, the formulas of Th. 4.7 for $\mathrm{SK}_1(\mathsf{E},\tau)$ reduce to $\partial = n$ and

$$\mathrm{SK}_1(\mathsf{E},\tau) \ \cong\ \{a \in \mathsf{T}_0^* \mid a^n \in \mathsf{R}_0^*\} \big/ \big(\mathsf{R}_0^*\, \langle x_\gamma x_\delta x_{\gamma+\delta}^{-1} \mid \gamma,\delta \in \Gamma_{\mathsf{E}}\rangle\big), \tag{5.3}$$

where each $x_\gamma \in \mathsf{E}_\gamma^*$ with $\tau(x_\gamma) = x_\gamma$. Recall that as E/T is totally ramified, the canonical pairing $\mathsf{E}^* \times \mathsf{E}^* \to \mu_e(\mathsf{T}_0)$ given by $(s,t) \mapsto [s,t]$ is surjective ([HW$_2$, Prop. 2.1]), and $\mu_e(\mathsf{T}_0) = \mu_e$, i.e., T_0 contains all e-th roots of unity. Since each $\mathsf{E}_\gamma = \mathsf{T}_0 x_\gamma$ with T_0 central, it follows that $\{[x_\delta, x_\gamma] \mid \gamma,\delta \in \Gamma_{\mathsf{E}}\} = \mu_e$. Now consider $c = x_\gamma x_\delta x_{\gamma+\delta}^{-1}$ for any $\gamma,\delta \in \Gamma_{\mathsf{E}}$. Then, $\tau(c) = x_{\gamma+\delta}^{-1} x_\delta x_\gamma$. Note that $x_\delta x_\gamma$ and $x_{\gamma+\delta}$ each lie in $\mathsf{E}_{\gamma+\delta} = \mathsf{T}_0 x_{\gamma+\delta}$, so they commute. Hence,

$$\tau(c)c^{-1} \ =\ x_{\gamma+\delta}^{-1}(x_\delta x_\gamma) x_{\gamma+\delta} x_\delta^{-1} x_\gamma^{-1} \ =\ [x_\delta, x_\gamma]. \tag{5.4}$$

Since $[x_\delta, x_\gamma] \in \mu_e$, this shows that $c \in \big\{a \in \mathsf{T}_0^* \mid a^e \in \mathsf{R}_0^*\big\}$. For the reverse inclusion, take any d in T_0^* such that $d^e \in \mathsf{R}_0^*$. So $\tau(d)d^{-1} \in \mu_e$. Thus, $\tau(d)d^{-1} = [x_\delta, x_\gamma]$, for some $\gamma,\delta \in \Gamma_{\mathsf{E}}$. Taking $c = x_\gamma x_\delta x_{\gamma+\delta}^{-1}$, we have $\tau(d)d^{-1} = \tau(c)c^{-1}$ by (5.4), which implies that dc^{-1} is τ-stable, so lies in R_0^*; thus, $d \in \mathsf{R}_0^*\, \langle x_\gamma x_\delta x_{\gamma+\delta}^{-1} \mid \gamma,\delta \in \Gamma_{\mathsf{E}}\rangle$. Therefore, $\mathsf{R}_0^*\, \langle x_\gamma x_\delta x_{\gamma+\delta}^{-1} \mid \gamma,\delta \in \Gamma_{\mathsf{E}}\rangle = \{a \in \mathsf{T}_0^* \mid a^e \in \mathsf{R}_0^*\}$. Inserting this in (5.3) we obtain (5.1).

(i) Consider the well-defined map $\alpha\colon \mathrm{SK}_1(\mathsf{E},\tau) \to \mathrm{SK}_1(\mathsf{E})$ given by $a\Sigma_\tau \mapsto a^{1-\tau}\mathsf{E}'$ (see diagram (3.9) for the non-graded version). By [HaW, Cor. 3.6(ii)], $\mathrm{SK}_1(\mathsf{E}) \cong \mu_n(\mathsf{T}_0)/\mu_e$. Taking into account formula (5.1) for $\mathrm{SK}_1(\mathsf{E},\tau)$, it is easy to see that α is injective.

We now verify that

$$\mathrm{im}(\alpha) \ =\ \big\{\omega \in \mu_n(\mathsf{T}_0) \mid \tau(\omega) = \omega^{-1}\big\}\big/\mu_e, \tag{5.5}$$

and thus obtain (5.2). Indeed, since $\mu_e = \{[x_\delta, x_\gamma] \mid \gamma,\delta \in \Gamma_{\mathsf{E}}\}$, by setting $c = x_\gamma x_\delta x_{\gamma+\delta}^{-1}$ we have $[x_\delta, x_\gamma] = \tau(c)c^{-1}$ by (5.4). This shows that $\mu_e \subseteq \big\{\omega \in \mu_n(\mathsf{T}_0) \mid \tau(\omega) = \omega^{-1}\big\}$. Now for any $\omega \in \mu_n(\mathsf{T}_0)$ with $\tau(\omega) = \omega^{-1}$, we have $N_{\mathsf{T}_0/\mathsf{R}_0}(\omega) = 1$, so Hilbert 90 guarantees that $\omega = c^{1-\tau}$ for some $c \in \mathsf{T}_0^*$. Then, $(c^n)^{1-\tau} = \omega^n = 1$, so $c^n \in \mathsf{R}_0^*$. Thus, $c \in \Sigma_\tau'$, and clearly $\alpha(c\Sigma_\tau) = \omega\mu_e$. This shows $\supseteq$ in (5.5); the reverse inclusion is clear from the definition of α.

(ii) Suppose e is odd. Let $m = |\mu_n(\mathsf{T}_0)|$. So, $\mu_n(\mathsf{T}_0) = \mu_m$, with $m \,|\, n$. Also, $e \,|\, m$, as $\mu_e \subseteq \mathsf{T}_0$. Since e and n have the same prime factors, this is also true for e and m. Recall that $\mathrm{Aut}(\mu_m) \cong (\mathbb{Z}/m\mathbb{Z})^*$, the multiplicative group of units of the ring $\mathbb{Z}/m\mathbb{Z}$; so, $|\,\mathrm{Aut}(\mu_m)| = \varphi(m)$, where φ is Euler's φ-function. Since $e \,|\, m$ and e and m have the same prime factors (all odd), the canonical map $\psi\colon\ \mathrm{Aut}(\mu_m) \to \mathrm{Aut}(\mu_e)$ given by restriction is surjective with kernel of order $\varphi(m)/\varphi(e) = m/e$, which is odd. Therefore, ψ induces an isomorphism on the 2-torsion subgroups, ${}_2\mathrm{Aut}(\mu_m) \cong {}_2\mathrm{Aut}(\mu_e)$. Now, $\tau|_{\mu_m} \in {}_2\mathrm{Aut}(\mu_m)$ and we saw for (i) that $\tau|_{\mu_e}$ is the inverse map $\omega \mapsto \omega^{-1}$. The inverse map on μ_m also lies in ${}_2\mathrm{Aut}(\mu_m)$ and has the same restriction to μ_e as τ. Hence, $\tau|_{\mu_m}$ must be the inverse map. That is, $\{\omega \in \mu_n(\mathsf{T}_0) \mid \tau(\omega) = \omega^{-1}\} = \mu_n(\mathsf{T}_0)$. Therefore, (5.5) above shows that $\mathrm{im}(\alpha) = \mu_n(\mathsf{T}_0)/\mu_e$, which we noted above is isomorphic to $\mathrm{SK}_1(\mathsf{E})$.

(iii) We saw in the proof of part (i) that τ acts on μ_e by the inverse map. So, if $e > 2$, then $\mu_e \not\subseteq \mathsf{R}_0$. Since $[\mathsf{T}_0 : \mathsf{R}_0] = 2$, it then follows that $\mathsf{T}_0 = \mathsf{R}_0(\mu_e)$. □

Remark 5.2. The isomorphism $\mathrm{SK}_1(\mathsf{E}, \tau) \cong \mathrm{SK}_1(\mathsf{E})$ of part (ii) of the above theorem can be obtained under the milder condition that $\mathsf{E}_0 = \mathsf{T}_0\mathsf{E}'$ provided that the exponent of E is a prime power. The proof is similar.

Example 5.3. Let $r_1, \ldots, r_m$ be integers with each $r_i \geq 2$. Let $e = \mathrm{lcm}(r_1, \ldots, r_m)$, and let $n = r_1 \ldots r_m$. Let C be any field such that $\mu_e \subseteq C$ and C has an automorphism θ of order 2 such that $\theta(\omega) = \omega^{-1}$ for all $\omega \in \mu_e$. Let R be the fixed field C^θ. Let $x_1, \ldots, x_{2m}$ be $2m$ independent indeterminates, and let K be the iterated Laurent power series field $C((x_1)) \ldots ((x_{2m}))$. This K is equipped with its standard valuation $v \colon K^* \to \mathbb{Z}^{2m}$ where $\mathbb{Z}^{2m}$ is given the right-to-left lexicographical ordering. With this valuation K is henselian (see [W$_2$, p. 397]). Consider the tensor product of symbol algebras

$$D \;=\; \Big(\frac{x_1, x_2}{K}\Big)_{\omega_1} \otimes_K \ldots \otimes_K \Big(\frac{x_{2m-1}, x_{2m}}{K}\Big)_{\omega_m},$$

where for $1 \leq i \leq m$, ω_i is a primitive r_i-th root of unity in C. Using the valuation theory developed for division algebras, it is known that D is a division algebra, the valuation v extends to D, and D is totally ramified over K (see [W$_2$, Ex. 4.4(ii)] and [TW, Ex. 3.6]) with

$$\Gamma_D/\Gamma_K \;\cong\; \prod_{i=1}^{m} (\mathbb{Z}/r_i\mathbb{Z}) \times (\mathbb{Z}/r_i\mathbb{Z}),$$

and $\overline{D} = \overline{K} \cong C$. Extend θ to an automorphism θ' of order 2 on K in the obvious way, i.e., acting by θ on the coefficients of a Laurent series, and with $\theta'(x_i) = x_i$ for $1 \leq i \leq 2m$. On each of the symbol algebras $\big(\frac{x_{2i-1}, x_{2i}}{K}\big)_{\omega_i}$ with its generators $\mathbf{i}_i$ and $\mathbf{j}_i$ such that $\mathbf{i}_i^{r_i} = x_{2i-1}$, $\mathbf{j}_i^{r_i} = x_{2i}$, and $\mathbf{i}_i\mathbf{j}_i = \omega_i\mathbf{j}_i\mathbf{i}_i$, define an involution τ_i as follows: $\tau_i(c\,\mathbf{i}_i^k\mathbf{j}_i^l) = \theta'(c)\,\mathbf{j}_i^l\mathbf{i}_i^k$, where $c \in K$ and $0 \leq l, k < r_i$. Clearly $K^{\tau_i} = K^{\theta'} = R((x_1)) \ldots ((x_{2m}))$, and therefore τ_i is a unitary involution. Since the τ_i agree on K for $1 \leq i \leq m$, they yield a unitary involution $\tau = \otimes_{i=1}^m \tau_i$ on D. Now by Th. 3.5, $\mathrm{SK}_1(D, \tau) \cong \mathrm{SK}_1(\mathsf{gr}(D), \widetilde{\tau})$. Since D is totally ramified over K, which is unramified over K^τ, we have correspondingly that $\mathsf{gr}(D)$ is totally ramified over $\mathsf{gr}(K)$, which is unramified over $\mathsf{gr}(K)^{\widetilde{\tau}}$. Also, $\mathsf{gr}(K)_0 \cong \overline{K} \cong C$. We have $\exp(\mathsf{gr}(D)) = \exp(D) = \exp(\Gamma_D/\Gamma_K) = \mathrm{lcm}(r_1, \ldots, r_m) = e$ and $\mathrm{ind}(\mathsf{gr}(D)) = \mathrm{ind}(D) = r_1 \ldots r_m = n$. By Th. 5.1,

$$\mathrm{SK}_1(D, \tau) \;\cong\; \mathrm{SK}_1(\mathsf{gr}(D), \widetilde{\tau}) \;\cong\; \{\omega \in \mu_n(C) \mid \theta(\omega) = \omega^{-1}\}/\mu_e,$$

while by [HaW, Th. 4.8, Cor. 3.6(ii)],

$$\mathrm{SK}_1(D) \;\cong\; \mathrm{SK}_1(\mathsf{gr}(D)) \;\cong \mu_n(C)/\mu_e.$$

Here are some more specific examples:

(i) Let $C = \mathbb{C}$, the complex numbers, and let θ be complex conjugation, which maps every root of unity to its inverse. So, $R = C^\theta = \mathbb{R}$. Then, $\mathrm{SK}_1(D, \tau) \cong \mathrm{SK}_1(D) \cong \mu_n/\mu_e \cong \mathbb{Z}/(n/e)\mathbb{Z}$.

(ii) Let $r_1 = r_2 = 4$, so $e = 4$ and $n = 16$. Let ω_{16} be a primitive sixteenth root of unity in $\mathbb{C}$, and let $C = \mathbb{Q}(\omega_{16})$, the sixteenth cyclotomic extension of $\mathbb{Q}$. Recall that $\mathrm{Gal}(C/\mathbb{Q}) \cong \mathrm{Aut}(\mu_{16}) \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, Let $\theta \colon C \to C$ be the automorphism which maps $\omega_{16} \mapsto (\omega_{16})^7$. Then, $\theta^2 = \mathrm{id}_C$, as $7^2 \equiv 1 \pmod{16}$, and $\{\omega \in \mu_{16} \mid \theta(\omega) = \omega^{-1}\} = \mu_8$. Thus, $\mathrm{SK}_1(D, \tau) \cong \mu_8/\mu_4 \cong \mathbb{Z}/2\mathbb{Z}$, while $\mathrm{SK}_1(D) \cong \mu_{16}/\mu_4 \cong \mathbb{Z}/4\mathbb{Z}$. So, here the injection $\mathrm{SK}_1(D, \tau) \to \mathrm{SK}_1(D)$ is not surjective.

(iii) Let $r_1 = \ldots = r_m = 2$, so $e = 2$ and $n = 2^m$. Here, C could be any quadratic extension of any field R with $\mathrm{char}(R) \neq 2$. Take θ to be the unique nonidentity R-automorphism of C. The resulting D is a tensor product of m quaternion algebras over $C((x_1)) \ldots ((x_{2m}))$, and $\mathrm{SK}_1(D, \tau) \cong \{\omega \in \mu_{2^m}(C) \mid \theta(\omega) = \omega^{-1}\}/\mu_2$, while $\mathrm{SK}_1(D) \cong \mu_{2^m}(C)/\mu_2$.

Ex. 5.3 gives an indication how to use the graded approach to recover results in the literature on the unitary SK_1 in a unified manner and to extend them from division algebras with discrete valued groups to arbitrary valued groups. While $\mathrm{SK}_1(D)$ has long been known for the D of Ex. 5.3, the formula for $\mathrm{SK}_1(D,\tau)$ is new.

Here is a more complete statement of what the results in the preceding sections yield for $\mathrm{SK}_1(D,\tau)$ for valued division algebras D.

Theorem 5.4. *Let (D,v) be a tame valued division algebra over a field K with $v|_K$ henselian, with a unitary involution τ; let $F=K^\tau$, and suppose $v|_F$ is henselian and that K is tamely ramified over F. Let $\overline{\tau}$ be the involution on $\overline{D}$ induced by τ. Then,*

(1) *Suppose K is unramified over F.*
  (i) *If D is unramified over K, then $\mathrm{SK}_1(D,\tau)\cong \mathrm{SK}_1(\overline{D},\overline{\tau})$.*
  (ii) *If D is totally ramified over K, let $e=\exp(D)$ and $n=\mathrm{ind}(D)$; then,*
$$\mathrm{SK}_1(D,\tau) \;\cong\; \{\omega\in\mu_n(\overline{K}) \mid \tau(\omega)=\omega^{-1}\}/\mu_e,$$
  while $\mathrm{SK}_1(D)\cong\mu_n(\overline{K})/\mu_e$.
  (iii) *If D has a maximal graded subfield M unramified over K and another maximal graded subfield L totally ramified over K, with $\tau(L)=L$, then D is semiramified and*
$$\mathrm{SK}_1(D,\tau) \;=\; \big\{a\in\overline{D}^* \mid N_{\overline{D}/\overline{K}}(a)\in\overline{F}\big\}\Big/\prod_{h\in\mathrm{Gal}(\overline{D}/\overline{K})}\overline{F}^{*h\overline{\tau}}.$$
  (iv) *Suppose Γ_D/Γ_K is cyclic. Let σ be a generator of $\mathrm{Gal}(Z(\overline{D})/\overline{K})$. Then,*
$$\mathrm{SK}_1(D,\tau) \;\cong\; \{a\in\overline{D}^* \mid N_{Z(\overline{D})/\overline{K}}(\mathrm{Nrd}_{\overline{D}}(a))\in\overline{F}\}\Big/\big(\Sigma_{\overline{\tau}}(\overline{D})\cdot\Sigma_{\sigma\overline{\tau}}(\overline{D})\big).$$
  (v) *If D is inertially split, $\overline{D}$ is a field and $\mathrm{Gal}(\overline{D}/\overline{K})$ is cyclic, then $\mathrm{SK}_1(D,\tau)=1$.*

(2) *If K is totally ramified over F, then $\mathrm{SK}_1(D,\tau)=1$.*

Proof. Let $\mathsf{gr}(D)$ be the associated graded division algebra of D. The tameness assumptions assure that $\mathsf{gr}(K)$ is the center of $\mathsf{gr}(D)$ with $[\mathsf{gr}(D):\mathsf{gr}(K)]=[D:K]$ and that the graded involution $\widetilde{\tau}$ on $\mathsf{gr}(D)$ induced by τ is unitary with $\mathsf{gr}(K)^{\widetilde{\tau}}=\mathsf{gr}(K^\tau)$. In each case of Th. 5.4, the conditions on D yield analogous conditions on $\mathsf{gr}(D)$. Since by Th. 3.5, $\mathrm{SK}_1(D,\tau)\cong\mathrm{SK}_1(\mathsf{gr}(D),\widetilde{\tau})$, (2) and (1)(v) follow immediately from Prop. 4.5 and Prop. 4.13(v), respectively. Part (1)(i), also follows from Th. 3.5, and Cor. 4.10 as follows:
$$\mathrm{SK}_1(D,\tau) \;\cong\; \mathrm{SK}_1(\mathsf{gr}(D),\widetilde{\tau}) \;\cong\; \mathrm{SK}_1(\mathsf{gr}(D)_0,\tau|_{\mathsf{gr}(D)_0}) \;=\; \mathrm{SK}_1(\overline{D},\overline{\tau}).$$
Parts (1)(ii), (1)(iii), and (1)(iv) follow similarly using Th. 5.1, Cor. 4.11, and Prop. 4.13(ii) respectively. $\square$

In the special case that the henselian valuation on K is discrete (rank 1), Th. 5.4 (1)(i), (iii), (iv), (v) and (2) were obtained by Yanchevskiĭ [Y_2]. In this discrete case, the assumption that v on K is henselian already implies that $v|_F$ is henselian (see Remark 3.1).

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