EXERCISES MATH 202B - 1st ASSIGNMENT

1. Assume A in an $n \times n$ matrix such that $A^m = I$, the identity matrix. Show that A is diagonalizable. More precisely, show that we can find subspaces $V_j \subset \mathbf{C}^n$ such that $\mathbf{C}^n = \bigoplus_j V_j$, and $Av = e^{2\pi ij/m}v$ for all $v \in V_j$, where $i = \sqrt{-1}$ and $1 \leq j \leq m$.

2. Let A be an $n \times n$ matrix and let $\mathcal{A}$ be the algebra generated by A. Show:
(a) The dimension of $\mathcal{A}$ is at most n.
(b) Find for each $1 \leq j \leq n$ a matrix A such that the algebra $\mathcal{A}$ has dimension j.

3. Let G be a finite group, and let $p = \sum_{g \in G} g \in FG$, where F is a field. Show:
(a) $hp = p$ for all $h \in G$,
(b) Let $a = \sum_g \alpha_g g$, with $\alpha_q \in F$. Show that $ap = (\sum_g \alpha_g)p$.
(c) $p^2 = |G|p$, where $|G|$ is the number of elements in G.
(d) Let V be a G-module, and let $w \in p.V$ (i.e. there exists a vector $v \in V$ such that $p.v = w$). Show that $g.w = w$ for all $g \in G$.

4. Let V be a 2-dimensional vector space (say $V = F^2$). Let $n \in \mathbf{Z}$ act on V via the matrix
$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$
(a) Show that this action of $\mathbf{Z}$ makes V into a $\mathbf{Z}$-module.
(b) Determine all possible submodules of V.
(c) Show that V is not semisimple.

5. Let $G = S_3$, the group of permutations of 3 elements, and let V be a 3-dimensional vector space, with basis $\{e_1, e_2, e_3\}$. Define $\pi.e_i = e_{\pi(i)}$ for $\pi \in S_3$, $i = 1, 2, 3$. Show that
(a) V is a G-module
(b) $V = W \oplus W'$, where $W = F(e_1 + e_2 + e_3)$, and where W' is spanned by the vectors $e_1 - e_2$ and $e_2 - e_3$.
(c) (optional: this will be easier to do later): Show that W' is a simple G-module.
(d) Conclude that V is semisimple.