EXERCISES MATH 202B - Sixth Assignment

1. Let V and W be G-modules. Calculate the dimension of $\mathrm{Hom}_G(V,W)$ in terms of the multiplicities of irreducible modules in V and W. You can assume the ground field to be algebraically closed (see also the first problem in the third homework assignment). Conclude that $\dim \mathrm{Hom}_G(V,W) = \dim \mathrm{Hom}_G(W,V)$ (this was the missing part in our proof of Frobenius reciprocity).

2. Let A be an algebra over $\mathbf{C}$. An idempotent $p \in A$ is called minimal if $pAp = \mathbf{C}p$, i.e. pap is a multiple of p for any $a \in A$.
(a) Let A be semisimple and $p \in A$ be a minimal idempotent. Show that Ap is a simple A-module.
(b) Let t be a Young tableau, and P, p, Q, q be as in the lecture. Show that $(qp)^2$ is a multiple of qp.
(c) Calculate $\chi_{reg}(qp)$ (this is the character of the left regular representation).
(d) Show that the multiple in (b) is not equal to 0. (Hint: What is the trace of a nilpotent element?)
(e) Give a different proof to the one in the lecture that $S_t = \mathbf{C}S_n qp$ is a simple S_n-module.

3. Let $p = \sum_{\pi \in S_n} \pi$. Moreover, let $V = \mathbf{C}^N$ with its standard basis $\{e_1, \ ..., \ e_N\}$ and let $d \in Gl(N)$ (all invertible $N \times N$ matrices) be a diagonal matrix with diagonal entries $x_1, x_2, \ ..., \ x_N$.
(a) The vector space $V^{\otimes n}$ is a $Gl(N)$-module, with the action of $Gl(N)$ on $V^{\otimes n}$ defined, as usual, by $g.(v_1 \otimes \ ... \ \otimes v_n) = g.v_1 \otimes \ ... \ \otimes g.v_n$ for any $g \in Gl(V)$. Moreover, we have a representation of S_n on $V^{\otimes n}$ via permutation of the tensor factors. Show that this representation commutes with the action of $Gl(N)$.
(b) Calculate the eigenvalues and eigenspaces of the action of d on $V^{\otimes n}$ (see also (c)).
(c) Calculate the character $\chi_{V^{\otimes n}}(d)$. Use this and the results in (b) to give a proof of the multinomial formula

$$(x_1 + \ ... + x_N)^n = \sum_{m_1 + \ ... \ + m_N = n} \frac{n!}{m_1! m_2! \ ... \ m_N!} x_1^{m_1} x_2^{m_2} \ ... \ x_N^{m_N}.$$

(d) Let $V_\lambda \subset V^{\otimes n}$ be an eigenspace of d. Calculate the dimension of pV_λ (with action of p as defined in (a)).
(e) Calculate the trace of $d_{|pV^{\otimes n}}$, the restriction of d to the subspace $pV^{\otimes n}$. *Remark*: The calculation in (e) can be interpreted as calculating the character of d in the $Gl(N)$-module $pV^{\otimes n}$.