EXERCISES MATH 202B - Eighth Assignment

1. In the following we consider symmetric polynomials in N variables with all Young diagrams having at most N rows. Let E_j be the j-th elementary symmetric function, i.e.
$$E_j = \sum_{1 \le i_1 < \ \dots \ < i_j \le k} x_{i_1} x_{i_2} \ \dots \ x_{i_j}.$$
Define for any Young diagram λ the polynomial E_λ to be the product of the E_j's corresponding to the columns of λ (e.g. $E_{[3,3,1]} = E_3(E_2)^2$).
(a) Show that $E_\lambda = x^\lambda + lower\ terms$.
(b) Show that the E_λ form a basis for the symmetric polynomials in N variables.
(c) Prove the 'fundamental theorem of symmetric functions': The symmetric polynomials over $\mathbf{Z}$ are isomorphic to the polynomial ring $\mathbf{Z}[y_1, \ \dots, \ y_k]$ in k variables. ($Hint$: Show that the map $y_i \mapsto E_i$ induces this isomorphism).

2. Let d be the $N \times N$ diagonal matrix with diagonal entries $x_1, x_2, \dots, \ x_N$. We have calculated $Tr_{V^{\otimes n}}(\pi d)$ for any permutation π in the lecture. You can use the following theorem which we will prove later:
Frobenius' Theorem: The character $\chi_\lambda(\pi)$ of the permutation π in the simple representation labeled by the Young diagram λ with $\le N$ rows is equal to the coefficient of $x^{\lambda+\rho}$ in $Tr_{V^{\otimes n}}(\pi d)\Delta$, where $\rho_i = N - i$, and where $\Delta = \prod_{1 \le i < j \le N}(x_i - x_j)$.
(a) Calculate $\chi_{[2,2,1]}((123)(45))$.
(b) Calculate the S_n character $\chi_\lambda(\pi)$ for all Young diagrams λ, where π is a full n-cycle. ($Hint$: Show first that if λ is not a hook diagram (hook diagrams means it only has boxes in the the first row or the first column), then $\chi_\lambda(\pi) = 0$).
3. Let $\dim V = N$, with $\{v_1, v_2, \ \dots, \ v_N\}$ a basis for V, and let $\alpha \in \mathbf{N}^N$ and V^α be as defined in the lecture. Moreover, let $[1^n]$ denote the Young diagram with all of its n boxes in one column. Let $q = q_{[1^n]} = \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma$.
(a) Show that $q(w_1 \otimes w_2 \otimes \ \dots \ w_n) = 0$ if $w_1, w_2, \ \dots \ w_n$ are linearly dependent. (Hint: It is enough to show this assuming that two of the vectors are equal, by linearity).
(b) Calculate the dimension of qV^α for all possible α. Prove that $Tr_{V^{\otimes n}}(qd) = E_n(x_1, x_2, \ \dots, \ x_N)$, where $d = diag(x_1, \ \dots \ x_N)$.
(c) Let t be a tableau of shape λ and let $q_t = \sum_{\sigma \in Q_t} \varepsilon(\sigma)\sigma$, where Q_t is the column stablizer of t. Show that $q_t V^{\otimes n} = 0$ if the number of rows of λ is greater than N.

Remark It is possible to reprove the combinatorial lemmas about the action of q_t on M^μ by looking at the action of q_t on $V^{\otimes n}$.