EXERCISES MATH 202C - 6th Assignment

1. Let e_r be the r-th elementary symmetric function, and let $p_r = \sum_{i=1}^n x_i^r$.

(a) Prove the Newton identity $\sum_{i=1}^k (-1)^{k-i} p_i e_{k-i} = (-1)^{k-1} k p_k$ for all $k \geq 1$; here $e_0 = 1$ and $e_i = 0$ for $i > n$ or $i < 0$.

(b) Prove that every symmetric function can be written as a polynomial in the p_i, $1 \leq i \leq n$.

2. Find the points in $\mathbf{C}^3$ on the variety defined by $x^2y - z^3$, $2xy - 4z - 1$, $z - y^2$ and $x^3 - 4zy$, using Gröbner bases.

3. (a) Let $f, g \in k[x_1, \ldots, x_n]$. Show that there exist well-defined (up to scalars) polynomials which are gcd and lcm of f and g (hint: use unique factorization property in $k[x_1, \ldots, x_n]$). Moreover, show that $fg = gcd(f,g)lcm(f,g)$.
(b) Show that if $\langle h \rangle = \langle f \rangle \cap \langle g \rangle$, then $h = lcm(f,g)$.
(c) Calculate $gcd(f,g)$ for

$$f = x^4 + x^3y + x^3z^2 - x^2y^2 + x^2yz^2 - xy^3 - xy^2z^2 - y^3z^2,$$

$$g = x^4 + 2x^3z^2 - x^2y^2 + x^2z^4 - 2xy^2z^2 - y^2z^4.$$

(d) Let $p = x^2 + xy + xz + yz$ and let $q = x^2 - xy - xz + yz$. Calculate a Gröbner basis for $\langle f, g \rangle \cap \langle p, q \rangle$.

4. Show that if f_m and f_{m+1} are homogeneous polynomials of degree m and $m+1$ respectively, and if $\gcd(f_m, f_{m+1}) = 1$, then $h = f_m + f_{m+1}$ is irreducible.